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A NEW REGULARIZATION METHOD FOR THE CAUCHY PROBLEM OF THE HELMHOLTZ EQUATION WITH NONHOMOGENEOUS CAUCHY DATA

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. In this paper, we investigate the Cauchy problem for the Helmholtz equation in the infinite strip $\{(x, y) \mid x \in \mathbb{R},\ 0 < y < 1\}$ with nonhomogeneous Cauchy data given at $y = 0$. The problem is severely ill-posed. We shall use the Fourier transform to get an integral equation and give a regularized solution by directly perturbing this equation in combination with truncating high frequencies. The error estimate between the regularization solution and the exact solution is given. Finally, a numerical example shows the effectiveness of the proposed method.

1. Introduction

The Cauchy problem of the Helmholtz equation is often encountered in many branches of science and engineering. It is used to describe the vibration of a structure [1], the acoustic cavity problem [2], the radiation wave [6], and the heat conduction in fins [14]. The direct problems, i.e., Dirichlet, Neumann or mixed boundary value problems for the Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points of concerned domain. This is called an inverse problem. The Cauchy problem for the Helmholtz equation is known to be severely ill-posed in the sense that a small change in the Cauchy data may result in a dramatic change in the solution [8]. Hence it is impossible to solve that problem by using classical numerical methods and it requires special techniques, for example, regularization methods. In recent years, the Cauchy problems associated with the Helmholtz equation have been studied by using different numerical methods, such as the Landweber method with boundary element method (BEM) [11], the conjugate gradient method [10], the

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method of fundamental solutions (MFS) [15] and so on. In the present paper, we consider the following Cauchy problem for the modified Helmholtz equation with nonhomogeneous Cauchy conditions

$$\Delta u(x,y) + k^2 u(x,y) = 0,\ x \in \mathbb{R},\ 0 < y < 1, \tag{1.1}$$

$$u_y(x,0) = \varphi(x),\ x \in \mathbb{R}, \tag{1.2}$$

$$u(x,0) = \psi(x),\ x \in \mathbb{R}, \tag{1.3}$$

where Δ denotes the Laplace operator, $\varphi(x)$, $\psi(x) \in L^2(\mathbb{R})$ are given data, and k is a real number. We note that if the boundary condition $u_y(x,0) = \varphi(x) = 0$, the problem (1.1)–(1.3) has been considered by many authors, such as [9, 12, 13]. However, their methods cannot be applied easily to solve (1.1)–(1.3) when the boundary condition is replaced by $u_y(x,0) = \varphi(x)$.

Recently, in [6], Chu-Li Fu et al. approximated the problem (1.1)–(1.3) by the Fourier regularization method. Furthermore, in that paper, the error between the regularization solution and the exact solution is given as follows

$$\begin{aligned}\left\|u(\cdot,y) - u_{\delta,\xi_{\max}}(\cdot,y)\right\| \leq\ & (2E_1)^y \delta^{1-y} \left(\ln \frac{2E_1}{\delta}\right)^{-py} (2 + \mathrm{o}(1)) \\ & + (E_2)^y \delta^{1-y} \left(\ln \frac{E_2}{\delta}\right)^{-py} (2 + \mathrm{o}(1)),\end{aligned}$$

where E_1, E_2 are the priori bound, $p \geq 0$. However, it is easy to see that the convergence of the approximate solution is very slow when $p = 0$ and y is in a neighborhood of 1. Moreover, the error in case $p = 0$ and $y = 1$ is not given here. In the present paper, we will improve that result by using a new regularization method.

First, we define

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\xi x} dx,$$

the Fourier transform of function $f(x)$.

Next, applying the Fourier transform with respect to variable $x \in \mathbb{R}$, we transform the problem (1.1)–(1.3) to the following one

$$\widehat{u}_{\xi\xi}(\xi,y) + \widehat{u}_{yy}(\xi,y) + k^2 \widehat{u}(\xi,y) = 0,\ \xi \in \mathbb{R},\ 0 < y < 1, \tag{1.4}$$

$$\widehat{u}_y(\xi,0) = \widehat{\varphi}(\xi),\ \xi \in \mathbb{R}, \tag{1.5}$$

$$\widehat{u}(\xi,0) = \widehat{\psi}(\xi),\ \xi \in \mathbb{R}. \tag{1.6}$$

Without loss of generality, we can assume that $k \geq 0$. If u is the solution of the problem (1.1)–(1.3), then its Fourier transform $\widehat{u}$ is the solution of the problem

(1.4)–(1.6) and has the following form

$$\widehat{u}(\xi,y) = \begin{cases} \frac{\widehat{\psi}(\xi)}{2}\left(e^{\sqrt{\xi^2-k^2}y}+e^{-\sqrt{\xi^2-k^2}y}\right) \\ \quad +\frac{\widehat{\varphi}(\xi)}{2\sqrt{\xi^2-k^2}}\left(e^{\sqrt{\xi^2-k^2}y}-e^{-\sqrt{\xi^2-k^2}y}\right), & |\xi|\geq k, \\ \widehat{\psi}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right)+\dfrac{\widehat{\varphi}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right), & |\xi|<k \end{cases}$$

$$= \begin{cases} \frac{\widehat{\psi}(\xi)}{2}\left(e^{\sqrt{\xi^2-k^2}y}+e^{-\sqrt{\xi^2-k^2}y}\right) \\ \quad +\left(\frac{e^{2\sqrt{\xi^2-k^2}y}-1}{2\sqrt{\xi^2-k^2}e^{2\sqrt{\xi^2-k^2}y}}\right)e^{\sqrt{\xi^2-k^2}y}\widehat{\varphi}(\xi), & |\xi|\geq k, \\ \widehat{\psi}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right)+\dfrac{\widehat{\varphi}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right), & |\xi|<k. \end{cases} \tag{1.7}$$

In the present paper, we shall approximate problem (1.7) by the following problem

$$\begin{aligned} \widehat{u^\epsilon}(\xi,y) = &\left[\frac{\widehat{\psi}(\xi)}{2}\left(\frac{1}{\alpha(\epsilon)+e^{-\sqrt{\xi^2-k^2}y}}+e^{-\sqrt{\xi^2-k^2}y}\right)\right. \\ &\left.+\left(\frac{e^{2\sqrt{\xi^2-k^2}y}-1}{2\sqrt{\xi^2-k^2}e^{2\sqrt{\xi^2-k^2}y}}\right)e^{\sqrt{\xi^2-k^2}y}\widehat{\varphi}(\xi)\right]\chi_{[-\beta(\epsilon),-k]\cup[k,\beta(\epsilon)]}(\xi) \\ &+\left[\widehat{\psi}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right)+\frac{\widehat{\varphi}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right)\right]\chi_{(-k,k)}(\xi) \end{aligned} \tag{1.8}$$

or

$$\begin{aligned} u^\epsilon(x,y) = &\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left[\frac{\widehat{\psi}(\xi)}{2}\left(\frac{1}{\alpha(\epsilon)+e^{-\sqrt{\xi^2-k^2}y}}+e^{-\sqrt{\xi^2-k^2}y}\right)\right. \\ &\left.+\left(\frac{e^{2\sqrt{\xi^2-k^2}y}-1}{2\sqrt{\xi^2-k^2}e^{2\sqrt{\xi^2-k^2}y}}\right)e^{\sqrt{\xi^2-k^2}y}\widehat{\varphi}(\xi)\right]\chi_{[-\beta(\epsilon),-k]\cup[k,\beta(\epsilon)]}(\xi)e^{i\xi x}d\xi \\ &+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left[\widehat{\psi}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right)+\frac{\widehat{\varphi}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right)\right]\times \\ &\times\chi_{(-k,k)}(\xi)e^{i\xi x}d\xi, \end{aligned} \tag{1.9}$$

where $\alpha(\epsilon)$ and $\beta(\epsilon)$ depend on ϵ, $\alpha(\epsilon)\in(0,1)$ is a regularization parameter, and $\beta(\epsilon)>0$ will be chosen later such that $\beta(\epsilon)$ tends to infinity when ϵ tends to zero. For convenience, we denote $\alpha(\epsilon)$ by α, and $\beta(\epsilon)$ by β.

The rest of the article is divided into three sections. In Section 2, we shall give the main results. The proofs will be presented in Section 3. Finally, a numerical experiment will be given in Section 4, which proves the efficiency of our method.

2. The main results

Assume that u_{ex} is the exact solution of (1.1)–(1.3), v_{ex} is the solution of problem (1.9) corresponding to the exact data φ_{ex}, ψ_{ex} and v_ϵ is the solution of problem (1.9) corresponding to the measured data φ_ϵ, ψ_ϵ, where φ_{ex}, ψ_{ex}, and φ_ϵ, ψ_ϵ are in the right-hand side of (1.9) such that $\|\varphi_\epsilon - \varphi_{ex}\|_{L^2(\mathbb{R})} \leq \epsilon$, $\|\psi_\epsilon - \psi_{ex}\|_{L^2(\mathbb{R})} \leq \epsilon$, where $\|\cdot\|_{L^2(\mathbb{R})}$ is the norm on $L^2(\mathbb{R})$. Then, we have

(2.1)
$$\widehat{u}_{ex}(\xi,y) = \begin{cases} \dfrac{\widehat{\psi}_{ex}(\xi)}{2}\left(e^{\sqrt{\xi^2-k^2}y} + e^{-\sqrt{\xi^2-k^2}y}\right) & \\ \quad + \left(\dfrac{e^{2\sqrt{\xi^2-k^2}y}-1}{2\sqrt{\xi^2-k^2}e^{2\sqrt{\xi^2-k^2}y}}\right) e^{\sqrt{\xi^2-k^2}y}\widehat{\varphi}_{ex}(\xi), & |\xi| \geq k, \\ \widehat{\psi}_{ex}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right) + \dfrac{\widehat{\varphi}_{ex}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right), & |\xi| < k, \end{cases}$$

$$\begin{aligned} \widehat{v}_{ex}(\xi,y) = &\left[\frac{\widehat{\psi}_{ex}(\xi)}{2}\left(\frac{1}{\alpha+e^{-\sqrt{\xi^2-k^2}y}} + e^{-\sqrt{\xi^2-k^2}y}\right)\right. \\ &\left. + \left(\frac{e^{2\sqrt{\xi^2-k^2}y}-1}{2\sqrt{\xi^2-k^2}e^{2\sqrt{\xi^2-k^2}y}}\right) e^{\sqrt{\xi^2-k^2}y}\widehat{\varphi}_{ex}(\xi)\right]\chi_{[-\beta,-k]\cup[k,\beta]}(\xi) \\ &+ \left[\widehat{\psi}_{ex}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right) + \frac{\widehat{\varphi}_{ex}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right)\right]\chi_{(-k,k)}(\xi), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \widehat{v}_{\epsilon}(\xi,y) = &\left[\frac{\widehat{\psi}_{\epsilon}(\xi)}{2}\left(\frac{1}{\alpha+e^{-\sqrt{\xi^2-k^2}y}} + e^{-\sqrt{\xi^2-k^2}y}\right)\right. \\ &\left. + \left(\frac{e^{2\sqrt{\xi^2-k^2}y}-1}{2\sqrt{\xi^2-k^2}e^{2\sqrt{\xi^2-k^2}y}}\right) e^{\sqrt{\xi^2-k^2}y}\widehat{\varphi}_{\epsilon}(\xi)\right]\chi_{[-\beta,-k]\cup[k,\beta]}(\xi) \\ &+ \left[\widehat{\psi}_{\epsilon}(\xi)\cos\left(\sqrt{k^2-\xi^2}y\right) + \frac{\widehat{\varphi}_{\epsilon}(\xi)}{\sqrt{k^2-\xi^2}}\sin\left(\sqrt{k^2-\xi^2}y\right)\right]\chi_{(-k,k)}(\xi). \end{aligned} \tag{2.3}$$

We have the estimate

$$\|v_\epsilon - u_{ex}\|_{L^2(\mathbb{R})} = \|\widehat{v}_\epsilon - \widehat{u}_{ex}\|_{L^2(\mathbb{R})} \leq \|\widehat{v}_\epsilon - \widehat{v}_{ex}\|_{L^2(\mathbb{R})} + \|\widehat{v}_{ex} - \widehat{u}_{ex}\|_{L^2(\mathbb{R})}. \tag{2.4}$$

We first have the following lemma.

Lemma 2.1 (The stability of a solution of problem (1.8))**.** *Suppose that* φ_{ex}, ψ_{ex}, φ_ϵ, $\psi_\epsilon \in L^2(\mathbb{R})$ *and* $\|\varphi_\epsilon - \varphi_{ex}\|_{L^2(\mathbb{R})} \leq \epsilon$, $\|\psi_\epsilon - \psi_{ex}\|_{L^2(\mathbb{R})} \leq \epsilon$. *Then we have*

$$\|\widehat{v}_\epsilon(\cdot,y) - \widehat{v}_{ex}(\cdot,y)\|_{L^2(\mathbb{R})} \leq \frac{2}{\alpha}\epsilon + \sqrt{2}(e^\beta + 1)\epsilon$$

for all $y \in (0,1]$.

The main conclusion of the present paper is as follows

Theorem 2.2. *Let φ_{ex}, ψ_{ex}, φ_ϵ, $\psi_\epsilon \in L^2(\mathbb{R})$ and $\|\varphi_\epsilon - \varphi_{ex}\|_{L^2(\mathbb{R})} \le \epsilon$, $\|\psi_\epsilon - \psi_{ex}\|_{L^2(\mathbb{R})} \le \epsilon$, $\epsilon \in (0,1)$. Suppose that there are two non-negative constants E_1 and E_2 such that $\|\psi_{ex}\|_{L^2(\mathbb{R})} \le E_1$ and $\left\|\frac{\partial}{\partial x} u_{ex}(\cdot, y)\right\|_{L^2(\mathbb{R})} \le E_2$. Then, with $\alpha = \epsilon^{\frac{1}{2}}$ and $\beta = \ln \frac{1}{\epsilon^{1/8}}$, we have, for every $0 < y \le 1$,*

$$\|v_\epsilon(\cdot, y) - u_{ex}(\cdot, y)\|_{L^2(\mathbb{R})} \le \frac{C}{\sqrt{\ln \frac{1}{\epsilon}}},$$

where $C = 6 + E_1 + 8E_2$.

3. Proofs of the main results

Proof of Lemma 2.1. First, from (2.2) and (2.3), we have

$$\begin{aligned}
&\|\widehat{v}_\epsilon(\cdot, y) - \widehat{v}_{ex}(\cdot, y)\|^2_{L^2(\mathbb{R})} \\
=&\int_{\mathbb{R}} |\widehat{v}_\epsilon(\xi, y) - \widehat{v}_{ex}(\xi, y)|^2 \, d\xi \\
=&\int_{\mathbb{R}} |\widehat{v}_\epsilon(\xi, y) - \widehat{v}_{ex}(\xi, y)|^2 \, \chi_{[-\beta,-k]\cup[k,\beta]}(\xi) d\xi \\
&\qquad + \int_{\mathbb{R}} |\widehat{v}_\epsilon(\xi, y) - \widehat{v}_{ex}(\xi, y)|^2 \, \chi_{(-k,k)}(\xi) d\xi \\
=&\int_{\mathbb{R}} \left| \frac{1}{2} \left(\tfrac{1}{\alpha + e^{-\sqrt{\xi^2 - k^2} y}} + e^{-\sqrt{\xi^2 - k^2} y} \right) \left[\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi) \right] \right. \\
&\qquad \left. + \left(\tfrac{e^{2\sqrt{\xi^2 - k^2} y} - 1}{2\sqrt{\xi^2 - k^2} e^{2\sqrt{\xi^2 - k^2} y}} \right) e^{\sqrt{\xi^2 - k^2} y} \left[\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi) \right] \right|^2 \chi_{[-\beta,-k]\cup[k,\beta]}(\xi) d\xi \\
&\qquad + \int_{\mathbb{R}} \left| \left[\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi) \right] \cos\left(\sqrt{k^2 - \xi^2} y \right) \right. \\
&\qquad \left. + \tfrac{1}{\sqrt{k^2 - \xi^2}} \left[\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi) \right] \sin\left(\sqrt{k^2 - \xi^2} y \right) \right|^2 \chi_{(-k,\, k)}(\xi) d\xi. \qquad (3.1)
\end{aligned}$$

Using the inequality $\dfrac{e^{|x|} - 1}{|x|} \le e^{|x|}$, we have

$$\left| \frac{e^{2\sqrt{\xi^2 - k^2} y} - 1}{2\sqrt{\xi^2 - k^2} e^{2\sqrt{\xi^2 - k^2} y}} \right| \le 1, \text{ for } 0 < y \le 1. \qquad (3.2)$$

Note that $e^{-\sqrt{\xi^2-k^2}y} \le 1$, $e^{\sqrt{\xi^2-k^2}y} \le e^{|\xi|}$, for $|\xi| \ge k$, and $\left|\cos\left(\sqrt{k^2-\xi^2}y\right)\right| \le 1$, $\left|\sin\left(\sqrt{k^2-\xi^2}y\right)\right| \le \sqrt{k^2-\xi^2}$ for $|\xi| < k$, $0 < y \le 1$, $\alpha \in (0,1)$, then from (3.1), (3.2) we get

$$\begin{aligned}
&\|\widehat{v}_\epsilon(\cdot,y) - \widehat{v}_{ex}(\cdot,y)\|^2_{L^2(\mathbb{R})} \\
\le\ & \int\limits_{\mathbb{R}} \left[\tfrac{1}{2}\left(\tfrac{1}{\alpha}+1\right)\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right| + e^{|\xi|}\left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|\right]^2 \chi_{[-\beta,-k]\cup[k,\beta]}(\xi)d\xi \\
&+\int\limits_{\mathbb{R}} \left(\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right| + \left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|\right)^2 \chi_{(-k,\ k)}(\xi)d\xi \\
\le\ & \int\limits_{\mathbb{R}} \left(\tfrac{1}{\alpha}\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right| + e^{|\xi|}\left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|\right)^2 \chi_{[-\beta,-k]\cup[k,\beta]}(\xi)d\xi \\
&+\int\limits_{\mathbb{R}} \left(\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right| + \left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|\right)^2 \chi_{(-k,\ k)}(\xi)d\xi.
\end{aligned}$$

By the inequality $(a+b)^2 \le 2(a^2+b^2)$, we deduce

$$\begin{aligned}
&\|\widehat{v}_\epsilon(\cdot,y) - \widehat{v}_{ex}(\cdot,y)\|^2_{L^2(\mathbb{R})} \\
\le\ & 2\int\limits_{\mathbb{R}} \left(\frac{1}{\alpha^2}\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right|^2 + e^{2|\xi|}\left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|^2\right) \chi_{[-\beta,-k]\cup[k,\beta]}(\xi)d\xi \\
&+2\int\limits_{\mathbb{R}} \left(\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right|^2 + \left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|^2\right) \chi_{(-k,\ k)}(\xi)d\xi \\
\le\ & \frac{2}{\alpha^2}\int\limits_{\mathbb{R}} \left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right|^2 \chi_{[-\beta,-k]\cup[k,\beta]}(\xi)d\xi \\
&+2e^{2\beta}\int\limits_{\mathbb{R}} \left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|^2 \chi_{[-\beta,-k]\cup[k,\beta]}(\xi)d\xi \\
&+2\int\limits_{\mathbb{R}} \left(\left|\widehat{\psi}_\epsilon(\xi) - \widehat{\psi}_{ex}(\xi)\right|^2 + \left|\widehat{\varphi}_\epsilon(\xi) - \widehat{\varphi}_{ex}(\xi)\right|^2\right) \chi_{(-k,k)}(\xi)d\xi \\
\le\ & \frac{2}{\alpha^2}\left\|\widehat{\psi}_\epsilon - \widehat{\psi}_{ex}\right\|^2_{L^2(\mathbb{R})} + 2e^{2\beta}\left\|\widehat{\varphi}_\epsilon - \widehat{\varphi}_{ex}\right\|^2_{L^2(\mathbb{R})} \\
&+2\left(\left\|\widehat{\psi}_\epsilon - \widehat{\psi}_{ex}\right\|^2_{L^2(\mathbb{R})} + \left\|\widehat{\varphi}_\epsilon - \widehat{\varphi}_{ex}\right\|^2_{L^2(\mathbb{R})}\right) \\
\le\ & \frac{4}{\alpha^2}\left\|\widehat{\psi}_\epsilon - \widehat{\psi}_{ex}\right\|^2_{L^2(\mathbb{R})} + 2(e^{2\beta}+1)\left\|\widehat{\varphi}_\epsilon - \widehat{\varphi}_{ex}\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Applying the inequality $\sqrt{a^2+b^2} \leq a+b$ for $a, b \geq 0$, we obtain

$$\begin{aligned}\|\widehat{v}_\epsilon(\cdot,y) - \widehat{v}_{ex}(\cdot,y)\|_{L^2(\mathbb{R})} &\leq \frac{2}{\alpha}\left\|\widehat{\psi}_\epsilon - \widehat{\psi}_{ex}\right\|_{L^2(\mathbb{R})} + \sqrt{2}(e^\beta+1)\,\|\widehat{\varphi}_\epsilon - \widehat{\varphi}_{ex}\|_{L^2(\mathbb{R})} \\ &\leq \frac{2}{\alpha}\epsilon + \sqrt{2}(e^\beta+1)\epsilon.\end{aligned}$$

This completes the proof of Lemma 2.1. □

Proof of Theorem 2.2. Taking into account (2.1) and (2.2), we have

$$\begin{aligned}&\|\widehat{v}_{ex}(\cdot,y) - \widehat{u}_{ex}(\cdot,y)\|^2_{L^2(\mathbb{R})} \\ =\ & \int_{\mathbb{R}} |\widehat{v}_{ex}(\xi,y) - \widehat{u}_{ex}(\xi,y)|^2\, d\xi \\ =\ & \int_{\mathbb{R}} |\widehat{v}_{ex}(\xi,y) - \widehat{u}_{ex}(\xi,y)|^2\, \chi_{[-\beta,-k]\cup[k,\beta]}(\xi) d\xi \\ & + \int_{\mathbb{R}} |\widehat{v}_{ex}(\xi,y) - \widehat{u}_{ex}(\xi,y)|^2\, \chi_{(-k,k)}(\xi) d\xi \\ & + \int_{\mathbb{R}} |\widehat{v}_{ex}(\xi,y) - \widehat{u}_{ex}(\xi,y)|^2\, \chi_{(-\infty,-\beta)\cup(\beta,+\infty)}(\xi) d\xi \\ =\ & \int_{\mathbb{R}} |\widehat{v}_{ex}(\xi,y) - \widehat{u}_{ex}(\xi,y)|^2\, \chi_{[-\beta,-k]\cup[k,\beta]}(\xi) d\xi \\ & + \int_{\mathbb{R}} |\widehat{u}_{ex}(\xi,y)|^2\, \chi_{(-\infty,-\beta)\cup(\beta,+\infty)}(\xi) d\xi. \qquad (3.3)\end{aligned}$$

Moreover, one has, for $|\xi| \geq k$ and $0 < y \leq 1$,

$$\left|\frac{1}{\alpha + e^{-\sqrt{\xi^2-k^2}y}} - e^{\sqrt{\xi^2-k^2}y}\right| \leq \alpha e^{2|\xi|}. \qquad (3.4)$$

From (2.1), (2.2), (3.3) and (3.4), we deduce

$$\begin{aligned}&\|\widehat{v}_{ex}(\cdot,y) - \widehat{u}_{ex}(\cdot,y)\|^2_{L^2(\mathbb{R})} \\ \leq\ & \frac{1}{4}\int_{\mathbb{R}} \left|\frac{1}{\alpha + e^{-\sqrt{\xi^2-k^2}y}} - e^{\sqrt{\xi^2-k^2}y}\right|^2 \left|\widehat{\psi}_{ex}(\xi)\right|^2 \chi_{[-\beta,-k]\cup[k,\beta]}(\xi) d\xi \\ & + \int_{\mathbb{R}} \frac{1}{|\xi|^2}\, |\xi \widehat{u}_{ex}(\xi,y)|^2\, \chi_{(-\infty,-\beta)\cup(\beta,+\infty)}(\xi) d\xi\end{aligned}$$

$$\leq \frac{1}{4}\alpha^2 e^{4\beta}\left\|\widehat{\psi}_{ex}\right\|^2_{L^2(\mathbb{R})} + \frac{1}{\beta^2}\left\|\frac{\partial}{\partial x}\widehat{u}_{ex}(\cdot,y)\right\|^2_{L^2(\mathbb{R})}.$$

Applying the inequality $\sqrt{a^2+b^2} \leq a+b$ for $a,\ b \geq 0$, we get

$$\|\widehat{v}_{ex}(\cdot,y) - \widehat{u}_{ex}(\cdot,y)\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\alpha e^{2\beta}\left\|\widehat{\psi}_{ex}\right\|_{L^2(\mathbb{R})} + \frac{1}{\beta}\left\|\frac{\partial}{\partial x}\widehat{u}_{ex}(\cdot,y)\right\|_{L^2(\mathbb{R})}. \tag{3.5}$$

From (2.4), using Lemma 2.1 and (3.5), we obtain the error estimate

$$\begin{aligned} &\|v_\epsilon(\cdot,y) - u_{ex}(\cdot,y)\|_{L^2(\mathbb{R})} \\ \leq\ & \frac{2}{\alpha}\epsilon + \sqrt{2}(e^\beta+1)\epsilon + \frac{1}{2}\alpha e^{2\beta}\left\|\widehat{\psi}_{ex}\right\|_{L^2(\mathbb{R})} + \frac{1}{\beta}\left\|\frac{\partial}{\partial x}\widehat{u}_{ex}(\cdot,y)\right\|_{L^2(\mathbb{R})}. \end{aligned}$$

The choice of $\alpha = \epsilon^{\frac{1}{2}}$ and $\beta = \ln\frac{1}{\epsilon^{1/8}}$ leads to

$$\|v_\epsilon(\cdot,y) - u_{ex}(\cdot,y)\|_{L^2(\mathbb{R})} \leq \frac{C}{\sqrt{\ln\dfrac{1}{\epsilon}}},$$

where

$$C = 6 + E_1 + 8E_2.$$

The proof of Theorem 2.2 is complete. □

4. A numerical experiment

In this section, we give a numerical example demonstrating how the suggested method works. We consider the equation

$$\Delta u + u = 0,\ x \in \mathbb{R},\ 0 < y < 1,$$

where u satisfies

$$u_y(x,0) = \varphi(x),$$
$$u(x,0) = \psi(x).$$

Consider the exact data $\varphi_{ex}(x) = \dfrac{12}{x^2+36}$, $\psi_{ex}(x) = e^{-\frac{1}{4}x^2}$. Then

$$\widehat{\varphi}_{ex}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{12}{x^2+36}e^{-i\xi x}dx = \sqrt{2\pi}e^{-6|\xi|} \tag{4.1}$$

and

$$\widehat{\psi}_{ex}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{1}{4}x^2}e^{-i\xi x}dx = \sqrt{2}e^{-\xi^2}. \tag{4.2}$$

From (4.1), (4.2) and (2.1), we have

$$\widehat{u}_{ex}(\xi,y)=\begin{cases}\dfrac{e^{-\xi^2}}{\sqrt{2}}\left(e^{\sqrt{\xi^2-1}y}+e^{-\sqrt{\xi^2-1}y}\right)\\ \quad+\dfrac{\sqrt{2\pi}e^{-6|\xi|}}{2\sqrt{\xi^2-1}}\left(e^{\sqrt{\xi^2-1}y}-e^{-\sqrt{\xi^2-1}y}\right), & |\xi|\geq 1,\\ \sqrt{2}e^{-\xi^2}\cos\left(\sqrt{1-\xi^2}y\right)+\dfrac{\sqrt{2\pi}e^{-6|\xi|}}{\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}y\right), & |\xi|<1.\end{cases}$$

Consider the measured data

$$\varphi_\epsilon(x)=\left(\sqrt{\frac{3}{\pi}}\epsilon+1\right)\varphi_{ex}(x), \tag{4.3}$$

$$\psi_\epsilon(x)=\left(\frac{1}{\sqrt{2}}\sqrt[4]{\frac{2}{\pi}}\epsilon+1\right)\psi_{ex}(x) \tag{4.4}$$

we have

$$\|\varphi_\epsilon-\varphi_{ex}\|_{L^2(\mathbb{R})}=\|\widehat{\varphi}_\epsilon-\widehat{\varphi}_{ex}\|_{L^2(\mathbb{R})}=\left(\int_{-\infty}^{+\infty}6\epsilon^2e^{-12|\xi|}d\xi\right)^{1/2}=\epsilon$$

and

$$\|\psi_\epsilon-\psi_{ex}\|_{L^2(\mathbb{R})}=\left\|\widehat{\psi}_\epsilon-\widehat{\psi}_{ex}\right\|_{L^2(\mathbb{R})}=\left(\int_{-\infty}^{+\infty}\sqrt{\frac{2}{\pi}}\epsilon^2e^{-2\xi^2}d\xi\right)^{1/2}=\epsilon.$$

From (4.1), (4.2), (4.3), (4.4), (2.3) and taking into account that $\alpha=\epsilon^{\frac{1}{2}}$, $\beta=\ln\frac{1}{\sqrt[8]{\epsilon}}$, we obtain the regularized solution

$$\begin{aligned}\widehat{v}_\epsilon(\xi,y) \;=\; &\left[\frac{e^{-\xi^2}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\sqrt[4]{\frac{2}{\pi}}\epsilon+1\right)\left(\frac{1}{\epsilon^{\frac{1}{2}}+e^{-\sqrt{\xi^2-1}y}}+e^{-\sqrt{\xi^2-1}y}\right)\right.\\ &\left.+\left(\frac{\sqrt{2\pi}e^{-6|\xi|}}{2\sqrt{\xi^2-1}}\right)\left(\sqrt{\frac{3}{\pi}}\epsilon+1\right)\left(e^{\sqrt{\xi^2-1}y}-e^{-\sqrt{\xi^2-1}y}\right)\right]\times\\ &\times\chi_{\left[-\ln\frac{1}{\sqrt[8]{\epsilon}},-1\right]\cup\left[1,\ln\frac{1}{\sqrt[8]{\epsilon}}\right]}(\xi)\\ &+\left[e^{-\xi^2}\left(\sqrt[4]{\frac{2}{\pi}}\epsilon+\sqrt{2}\right)\cos\left(\sqrt{1-\xi^2}y\right)\right.\\ &\left.+\frac{\sqrt{2\pi}e^{-6|\xi|}}{\sqrt{1-\xi^2}}\left(\sqrt{\frac{3}{\pi}}\epsilon+1\right)\sin\left(\sqrt{1-\xi^2}y\right)\right]\chi_{(-1,1)}(\xi).\end{aligned}$$

Let ϵ be $\epsilon_1 = 10^{-5}$, $\epsilon_2 = 9 \times 10^{-7}$, $\epsilon_3 = 10^{-10}$ respectively. If we put

$$y = \{0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1\}$$

we get the following tables

$\epsilon_1 = 10^{-5}$	
y	$\|v_\epsilon - u_{ex}\|_{L^2(\mathbb{R})}$
0	1.0012×10^{-1}
0.1	1.0090×10^{-1}
0.2	1.0324×10^{-1}
0.3	1.0718×10^{-1}
0.4	1.1280×10^{-1}
0.5	1.2022×10^{-1}
0.6	1.2959×10^{-1}
0.7	1.4109×10^{-1}
0.8	1.5498×10^{-1}
0.9	1.7152×10^{-1}
1	1.9107×10^{-1}

$\epsilon_2 = 9 \times 10^{-7}$	
y	$\|v_\epsilon - u_{ex}\|_{L^2(\mathbb{R})}$
0	5.3438×10^{-2}
0.1	3.5884×10^{-2}
0.2	3.7231×10^{-2}
0.3	3.9516×10^{-2}
0.4	4.2804×10^{-2}
0.5	4.7188×10^{-2}
0.6	5.2790×10^{-2}
0.7	5.9769×10^{-2}
0.8	6.8319×10^{-2}
0.9	7.8683×10^{-2}
1	9.1155×10^{-2}

$\epsilon_3 = 10^{-10}$	
y	$\|v_\epsilon - u_{ex}\|_{L^2(\mathbb{R})}$
0	1.4491×10^{-4}
0.1	1.5248×10^{-4}
0.2	1.7016×10^{-4}
0.3	2.0117×10^{-4}
0.4	2.4798×10^{-4}
0.5	3.1428×10^{-4}
0.6	4.0532×10^{-4}
0.7	5.2833×10^{-4}
0.8	6.9306×10^{-4}
0.9	9.1259×10^{-4}
1	1.2044×10^{-3}

where $\|v_\epsilon - u_{ex}\|_{L^2(\mathbb{R})} = \|\widehat{v}_\epsilon - \widehat{u}_{ex}\|_{L^2(\mathbb{R})}$ and we have the graphics displayed in figures 2, 3, and 4 on the rectangular domain $[-10, 10] \times [0, 1]$

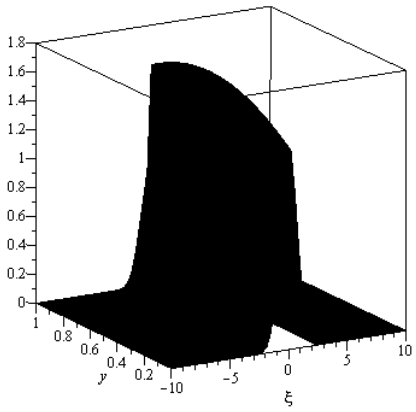

Figure 1: The Fourier transform of the exact solution

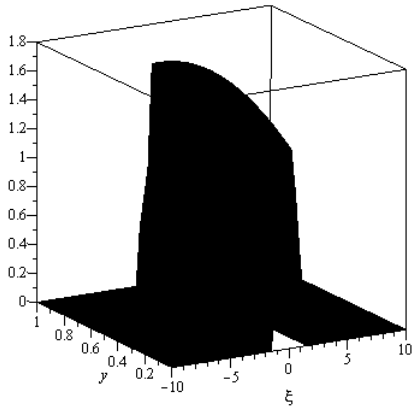

Figure 2: The Fourier transform of the regularized solution with $\epsilon_1 = 10^{-5}$

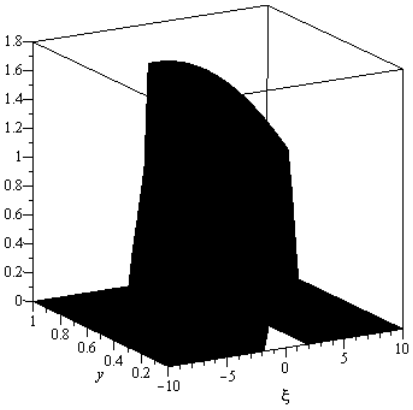

Figure 3: The Fourier transform of the regularized solution with $\epsilon_2 = 9 \times 10^{-7}$

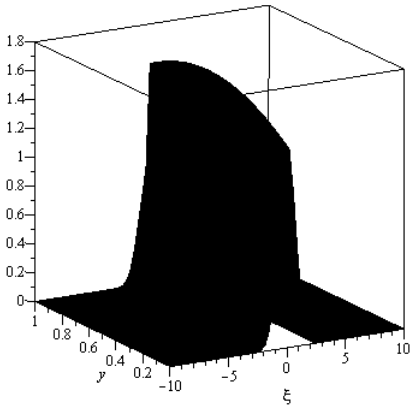

Figure 4: The Fourier transform of the regularized solution with $\epsilon_3 = 10^{-10}$

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