ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 2, 2011, pp. 375–396

SOLVABILITY OF A SYSTEM OF DUAL INTEGRAL EQUATIONS OF A MIXED BOUNDARY VALUE PROBLEM FOR THE BIHARMONIC EQUATION IN A STRIP

NGUYEN VAN NGOC AND NGUYEN THI NGAN

Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. The aim of the present work is to consider a mixed boundary value problem of the biharmonic equation in a strip. The problem may be interpreted as a deflection surface of a strip plate with the edges $y = 0, y = h$ having clamped conditions on intervals $|x| \leqslant a$ and hinged support conditions for $|x| > a$. Using the Fourier transform, the problem is reduced to studying a system of dual integral equations on the edges of the strip. The uniqueness and existence theorems of a solution of the system of dual integral equations are established in appropriate Sobolev spaces. A method for reducing the system of dual integral equations to an infinite system of linear algebraic equations is also proposed.

1. Introduction

Mixed boundary value problems for the biharmonic equation in a strip have been considered by many authors (see for example, [22,1,6]). V. B. Zelentsov [22] considered the problem of pending of Kirchhoff-Love plate in the shape of a strip under the impression of a thin liner rigid inclusion fastened at one of the edges of the plate when the other edge of the plate is rigidly clamped. The problem is reduced to the solution of convolution-type integral equations of the first kind in a finite segment with a regular kernel. In [1] some problems of the impression of one or two inclusions in the form of stiffner ribs into an infinite lying on an elastic foundation were considered. It follows from the properties of the kernel of the integral equations that their solutions have a non-integrable singularity. In [6] A. I. Fridman and S. D. Eidelman considered some boundary value problems for the biharmonic equation in a strip. New uniqueness theorems for nonnegative solutions of these problems are proved.

The aim of the present work is to consider a mixed boundary value problem of the biharmonic equation in a strip by the dual integral equation method.

Received February14, 2011; in revised form May 15, 2011.
2000 *Mathematics Subject Classification.* Primary: 45H05, 42A38, 46F05, 46F10, 47G30 .
Key words and phrases. Biharmonic equation, mixed boundary value problems, dual integral equations.
The first author is supported by NAFOSTED, grant code: 101.01.23.09.

The problem may be interpreted as a deflection surface of a strip plate with the edges $y = 0, y = h$ having clamped conditions on intervals $|x| \leqslant a$ and hinged support conditions for $|x| > a$. Using the Fourier transform, we reduce the problem to studying a system of dual integral equations on the edges of a strip. The uniqueness and existence theorems of a solution of the system of dual integral equations are established in appropriate Sobolev spaces. A method for reducing the dual integral equation to an infinite system of linear algebraic equations is also proposed.

Note that the theory of dual integral equations recently became very developed and is the subject of numerous investigations. Formal analytical methods for finding solutions to dual equations have been studied by many authors (see [11] and [18]), but much less attention has been paid to the solvability question of these equations. The dual integral equations of Titchmarsh's type were investigated in [21,5] by the distributional approach and in [3] on Lebesgue spaces. Recently, P. K. Banerji and Deshna Loonker [2] obtained the solution of dual integral equations involving Legendre functions in distributional spaces. Some results on the solvability and validity of solutions of dual integral equations involving Fourier transforms based on the Dirichlet problems for pseudo-differential operators have been considered in [12-16]. Note that dual integral equations for convolutions may be reduced to the dual equations involving Fourier transforms. Some classes of those equations on semi-axes of the real axis were considered in [7].

Our work is constructed as follows: In Section 2 we formulate the mixed boundary value problem for the biharmonic equation in a strip and reduce it to a system of dual integral equations. Section 3 is intended for the solvability of a system of dual integral equations involving Fourier transforms with decreasing symbols in appropriate Sobolev spaces. Finally in the last section, Section 4 we present a manner reducing the system of dual integral equations to a system of integral equations with logarithmic kernels, and reducing the latter to an infinite system of linear algebraic equations.

2. Formulation of the problem

We study the solutions $\Phi = \Phi(x, y)$ of a boundary-value problem for the biharmonic equation

$$\Delta^2\Phi(x,y) = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0 \tag{2.1}$$

in the strip

$$\Pi = \{(x, y) :\ -\infty < x < \infty,\ 0 < y < h\}.$$

Let $\mathbb{R}$ be the real axis, $(-a, a)$ denotes a certain bounded interval on $\mathbb{R}$. Consider the following mixed boundary value problem:

Find a solution $\Phi(x,y)$ of equation (2.1) in the strip Π that satisfies the boundary conditions

$$\Phi\Big|_{y=0} = r_1(x), \;\; x \in \mathbb{R}, \tag{2.2}$$

$$\Phi\Big|_{y=h} = r_2(x), \;\; x \in \mathbb{R}, \tag{2.3}$$

$$\begin{cases} \dfrac{\partial \Phi}{\partial y}\Big|_{y=0} = f_1(x), \;\; x \in (-a,a), \\ M[\Phi]\Big|_{y=0} = 0, \;\; x \in \mathbb{R}\setminus(-a,a), \end{cases} \tag{2.4}$$

$$\begin{cases} \dfrac{\partial \Phi}{\partial y}\Big|_{y=h} = f_2(x), \;\; x \in (-a,a), \\ M[\Phi]\Big|_{y=h} = 0, \;\; x \in \mathbb{R}\setminus(-a,a), \end{cases} \tag{2.5}$$

where

$$M[\Phi] = M[\Phi](x,y) = \frac{\partial^2 \Phi}{\partial y^2} + \nu \frac{\partial^2 \Phi}{\partial x^2}, \;\; 0 < \nu < 1. \tag{2.6}$$

This problem may be interpreted as a deflection surface of a strip plate with the edges $y = 0, y = h$ having clamped conditions on intervals $|x| \leqslant a$ and hinged support conditions for $|x| > a$. Physically, $M[\Phi]$ is the bending moment with respect to the axis Oy.

We shall solve the formulated problem by the method of Fourier transforms and reduce it to a system of dual equations involving inverse Fourier transforms. It is well-known that, for a suitable function $f(x), x \in \mathbb{R}$ (for example, $f \in L^1(\mathbb{R})$), direct and inverse Fourier transforms are defined by the formulas

$$\hat{f}(\xi) = F[f](\xi) = \int_{-\infty}^{\infty} f(x) e^{ix\xi} dx, \tag{2.7}$$

$$\check{f}(\xi) = F^{-1}[f](\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx. \tag{2.8}$$

The theory of the Fourier transforms of tempered generalized functions can be found, for example, in [4,19].

Taking the Fourier transform with respect to the variable x of the biharmonic equation (2.1), we obtain

$$\frac{d^4 \widehat{\Phi}(\xi,y)}{dy^4} - 2\xi^2 \frac{d^2 \widehat{\Phi}(\xi,y)}{dy^2} + \xi^4 \widehat{\Phi}(\xi,y) = 0, \tag{2.9}$$

where $\widehat{\Phi}(\xi,y) = F_x[\Phi(x,y)](\xi)$ is the Fourier transform with respect to x of the function $\Phi(x,y)$. The general solution of the differential equation (2.9) for $\xi \neq 0$ is taken in the form

(2.10)

$$\widehat{\Phi}(\xi,y) = A^*(\xi)\cosh(|\xi|y) + B^*(\xi) y \cosh(|\xi|y) + C^*(\xi)\sinh(|\xi|y) + D^*(\xi) y \sinh(|\xi|y),$$

where $A^*(\xi), B^*(\xi), C^*(\xi), D^*(\xi)$ are arbitrary functions of the variable ξ. The value $\widehat{\Phi}(0,y)$ is understood in the sense

$$\widehat{\Phi}(0,y) = \lim_{\xi\to 0} \widehat{\Phi}(\xi,y). \tag{2.11}$$

Taking the Fourier transforms of the conditions (2.2), (2.3) due to (2.10) we have

$$\widehat{\Phi}(\xi,0) = A^*(\xi) = \widehat{r}_1(\xi), \tag{2.12}$$

$$\widehat{\Phi}(\xi,h) = A^*(\xi)\cosh(|\xi|h) + B^*(\xi)h\cosh(|\xi|h) + C^*(\xi)\sinh(|\xi|h)$$
$$+D^*(\xi)h\sinh(|\xi|h) = \widehat{r}_2(\xi). \tag{2.13}$$

Denote

$$\widehat{u}_1(\xi) = \widehat{M}[\Phi](\xi,0) = \widehat{\Phi}_{yy}(\xi,0) - \nu\xi^2\widehat{\Phi}(\xi,0) = (1-\nu)\xi^2A^*(\xi) + 2|\xi|D^*(\xi), \tag{2.14}$$

$$\widehat{u}_2(\xi) = \widehat{M}[\Phi](\xi,h) = \widehat{\Phi}_{yy}(\xi,h) - \nu\xi^2\widehat{\Phi}(\xi,h) = (1-\nu)\xi^2\cosh(|\xi|h)A^*(\xi)$$
$$+\big[2|\xi|\sinh(|\xi|h) + (1-\nu)\xi^2 h\cosh(|\xi|h)\big]B^*(\xi) + (1-\nu)\xi^2\sinh(|\xi|h)C^*(\xi)$$
$$+\big[2|\xi|\cosh(|\xi|h) + (1-\nu)\xi^2 h\sinh(|\xi|h)\big]D^*(\xi). \tag{2.15}$$

Using the relations (2.12) –(2.15) we express the unknown functions $A^*(\xi), B^*(\xi), C^*(\xi), D^*(\xi)$ in terms of $\widehat{u}_1(\xi)$, $\widehat{u}_2(\xi)$, $\widehat{r}_1(\xi)$ and $\widehat{r}_2(\xi)$. For $\xi \neq 0$ after some transformations we obtain

$$A^*(\xi) = \widehat{r}_1(\xi), \tag{2.16}$$

$$D^*(\xi) = \frac{\widehat{u}_1}{2|\xi|} - \frac{(1-\nu)|\xi|\widehat{r}_1(\xi)}{2}, \tag{2.17}$$

$$B^*(\xi) = -\frac{\cosh(|\xi|h)}{2|\xi|\sinh(|\xi|h)}\widehat{u}_1(\xi) + \frac{\widehat{u}_2(\xi)}{2|\xi|\sinh(|\xi|h)}$$
$$+\frac{(1-\nu)|\xi|\cosh(|\xi|h)}{2\sinh(|\xi|h)}\widehat{r}_1(\xi) - \frac{(1-\nu)|\xi|}{2\sinh(|\xi|h)}\widehat{r}_2(\xi), \tag{2.18}$$

$$C^*(\xi) = \frac{h}{2|\xi|\sinh^2(|\xi|h)}\widehat{u}_1(\xi) - \frac{h\cosh(|\xi|h)}{2|\xi|\sinh^2(|\xi|h)}\widehat{u}_2(\xi)$$
$$-\frac{\sinh(2|\xi|h) + (1-\nu)|\xi|h}{2\sinh^2(|\xi|h)}\widehat{r}_1(\xi) + \frac{(1-\nu)|\xi|h\cosh(|\xi|h) + 2\sinh(|\xi|h)}{2\sinh^2(|\xi|h)}\widehat{r}_2(\xi). \tag{2.19}$$

Substituting (2.16)–(2.19) into (2.10) we obtain

$$\widehat{\Phi}(\xi,y) = \widehat{u}_1(\xi)\Big[\frac{h\sinh(|\xi|y) - y\sinh(|\xi|h)\cosh(|\xi|(h-y))}{2|\xi|\sinh^2(|\xi|h)}\Big]$$
$$+\widehat{u}_2(\xi)\Big[\frac{y\cosh(|\xi|y)\sinh(|\xi|h) - h\sinh(|\xi|y)\cosh(|\xi|h)}{2|\xi|\sinh^2(|\xi|h)}\Big]$$
$$+\widehat{r}_1(\xi)\Big[\frac{2\sinh(|\xi|h)\sinh(|\xi|(h-y))}{2\sinh^2(|\xi|h)}\Big]$$

$$+\widehat{r}_1(\xi)\Big[\frac{(1-\nu)|\xi|\big[y\sinh(|\xi|h)\cosh(|\xi|(h-y))-h\sinh(|\xi|y)\big]}{2\sinh^2(|\xi|h)}\Big]$$

$$+\widehat{r}_2(\xi)\Big[\frac{(1-\nu)|\xi|(h-y)\sinh(|\xi|y)\cosh(|\xi|h)}{2\sinh^2(|\xi|h)}\Big]$$

$$+\widehat{r}_2(\xi)\Big[\frac{2\sinh(|\xi|y)\sinh(|\xi|h)-(1-\nu)y|\xi|\sinh(|\xi|(h-y))}{2\sinh^2(|\xi|h)}\Big]. \tag{2.20}$$

From (2.20) it is easy to see that the terms in the big brackets have the asymptotical behavior

$$O(|\xi|e^{-|\xi|(h-y)}), \quad |\xi|\to\infty, \tag{2.21}$$

and besides, in the sense (2.11) we have

$$\widehat{\Phi}(0,y)=\alpha_1(y)\lim_{\xi\to 0}\widehat{u}_1(\xi)+\alpha_2(y)\lim_{\xi\to 0}\widehat{u}_2(\xi)+\beta_1(y)\lim_{\xi\to 0}\widehat{r}_1(\xi)+\beta_2(y)\lim_{\xi\to 0}\widehat{r}_2(\xi), \tag{2.22}$$

where

$$\alpha_1(y)=-\frac{y(h^2-y^2)+3y(h-y)^2}{12h}, \tag{2.23}$$

$$\alpha_2(y)=\frac{y(y^2-h^2)}{6h}, \tag{2.24}$$

$$\beta_1(y)=\frac{h-y}{h}, \tag{2.25}$$

$$\beta_2(y)=\frac{y}{h}. \tag{2.26}$$

Substituting (2.16)-(2.19) into the following relations

$$\frac{d\widehat{\Phi}(\xi,0)}{dy}=|\xi|C^*(\xi)+B^*(\xi),$$

$$\frac{d\widehat{\Phi}(\xi,h)}{dy}=[A^*(\xi)|\xi|+D^*(\xi)]\sinh(|\xi|h)+[C^*(\xi)|\xi|+B^*(\xi)]\cosh(|\xi|h)$$

$$+B^*(\xi)|\xi|y\sinh(|\xi|h)+D^*(\xi)|\xi|h\cosh(|\xi|h),$$

we get

$$\frac{d\widehat{\Phi}(\xi,0)}{dy}=-a_{11}(\xi)\widehat{u}_1(\xi)-a_{12}(\xi)\widehat{u}_2(\xi)-a_1(\xi)\widehat{r}_1(\xi)+a_2(\xi)\widehat{r}_2(\xi), \tag{2.27}$$

$$\frac{d\widehat{\Phi}(\xi,h)}{dy}=a_{21}(\xi)\widehat{u}_1(\xi)+a_{22}(\xi)\widehat{u}_2(\xi)-a_2(\xi)\widehat{r}_1(\xi)+a_1(\xi)\widehat{r}_2(\xi), \tag{2.28}$$

where

$$a_1(\xi) = \frac{|\xi|[(1+\nu)\sinh(|\xi|h)\cosh(|\xi|h) + (1-\nu)|\xi|h]}{2\sinh^2(|\xi|h)}, \tag{2.29}$$

$$a_2(\xi) = \frac{|\xi|[(1+\nu)\sinh(|\xi|h) + (1-\nu)|\xi|h\cosh(|\xi|h)]}{2\sinh^2(|\xi|h)}, \tag{2.30}$$

$$a_{11}(\xi) = a_{22}(\xi) = \frac{\sinh(|\xi|h)\cosh(|\xi|h) - |\xi|h}{2|\xi|\sinh^2(|\xi|h)}, \tag{2.31}$$

$$a_{21}(\xi) = a_{12}(\xi) = \frac{|\xi|h\cosh(|\xi|h) - \sinh(|\xi|h)}{2|\xi|\sinh^2(|\xi|h)}. \tag{2.32}$$

In order to determine the unknown functions $\widehat{u}_1(\xi)$ and $\widehat{u}_2(\xi)$, we use the mixed conditions (2.4) and (2.5). Satisfying these conditions, from (2.14), (2.15), (2.27) and (2.28), we have the system of dual integral equations with respect to $\widehat{u}_1(\xi), \widehat{u}_2(\xi)$:

$$\begin{cases} F^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x) = \widetilde{\mathbf{f}}(x), & x \in (-a,a), \\ F^{-1}[\widehat{\mathbf{u}}(\xi)](x) = \mathbf{0}, & x \in \mathbb{R}\setminus(-a,a), \end{cases} \tag{2.33}$$

where

$$u_1(x) = M[\Phi](x,0), \quad u_2(x) = M[\Phi](x,h), \quad \mathbf{u}(x) = (u_1(x), u_2(x))^T, \tag{2.34}$$

$$\widehat{\mathbf{u}}(\xi) = F[\mathbf{u}(x)](\xi), \quad \widetilde{\mathbf{f}}(x) = (\widetilde{f}_1(x), \widetilde{f}_2(x))^T, \tag{2.35}$$

$$\widetilde{f}_1(x) = -f_1(x) - F^{-1}[a_1(\xi)\widehat{r}_1(\xi)](x) + F^{-1}[a_2(\xi)\widehat{r}_2(\xi)](x), \tag{2.36}$$

$$\widetilde{f}_2(x) = f_2(x) + F^{-1}[a_2(\xi)\widehat{r}_1(\xi)](x) - F^{-1}[a_1(\xi)\widehat{r}_2(\xi)](x), \tag{2.37}$$

$$\mathbf{A}(\xi) = \begin{pmatrix} a_{11}(\xi) & a_{12}(\xi) \\ a_{21}(\xi) & a_{22}(\xi) \end{pmatrix}. \tag{2.38}$$

3. Solvability of systems of dual equations

3.1. **Functional spaces.** Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ be the Schwartz spaces of basic and generalized functions, respectively [4,19,20]. Denote by F and F^{-1} the Fourier transform and inverse Fourier transform defined on $\mathcal{S}'$. It is known that these operators are automorphisms on $\mathcal{S}'$. For a suitable ordinary function $f(x)$ (for example, $f \in L^1(\mathbb{R})$), the direct and inverse Fourier transforms defined by formulas (2.7) and (2.8), respectively [19]. The symbol $< f, \varphi >$ denotes the value of the generalized function $f \in \mathcal{S}'$ on the basic function $\varphi \in \mathcal{S}$, besides, $(f,\varphi) :=< f, \overline{\varphi} >$.

Definition 3.1. Let $H^s := H^s(\mathbb{R})(s \in \mathbb{R})$ be the Sobolev-Slobodeskii space defined as a closure of the set $C_o^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support with respect to the norm [4, 19, 20]

$$||u||_s := \Big[\int_{-\infty}^{\infty} (1+\xi^2)^s |\hat{u}(\xi)|^2 d\xi\Big]^{1/2} < \infty, \quad \hat{u} = F[u]. \tag{3.1}$$

The space H^s is a Hilbert space with the following scalar product

$$(u,v)_s := \int_{-\infty}^{\infty} (1+\xi^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi. \tag{3.2}$$

Let Ω be a certain interval in $\mathbb{R}$. The subspace of $H^s(\mathbb{R})$ consisting of functions $u(x)$ with support in $\overline{\Omega}$ is denoted by $H^s_o(\Omega)$ [20], while the space of functions $v(x) = pu(x)$, where $u \in H^s(\mathbb{R})$ and p is the restriction operator to Ω is denoted by $H^s(\Omega)$. The norm in $H^s(\Omega)$ is defined by

$$||v||_{H_s(\Omega)} = \inf_l ||lv||_s,$$

where the infimum is taken over all possible extensions $lv \in H^s(\mathbb{R})$.

Let X be a linear topological space. We denote the direct product of two elements X by X^2. Topology in X^2 is given by the usual topology of the direct product. We shall use bold letters for denoting vector-values and matrices. Denote by $\mathbf{u}$ a vector of the form (u_1, u_2), and $\mathcal{S}^2 = \mathcal{S} \times \mathcal{S}, \quad (S')^2 = \mathcal{S}' \times \mathcal{S}'$. For the vectors $\mathbf{u} \in (\mathcal{S}')^2, \ \boldsymbol{\varphi} \in \mathcal{S}^2$ we set

$$< \mathbf{u}, \boldsymbol{\varphi} >= \sum_{j=1}^{2} < u_j, \varphi_j > .$$

The Fourier transform and inverse Fourier transform of a vector $\mathbf{u} \in (S')^2$ are the vectors $\hat{\mathbf{u}} = F^{\pm 1}[\mathbf{u}] = (F^{\pm 1}[u_1], F^{\pm 1}[u_2])^T$, defined by the equalities [19]:

$$< F[\mathbf{u}], \boldsymbol{\varphi} >=< \mathbf{u}, F[\boldsymbol{\varphi}] >, \ < F^{-1}[\mathbf{u}], \boldsymbol{\varphi} >= \frac{1}{2\pi} < \mathbf{u}, F[\boldsymbol{\varphi}](-x) >, \ \boldsymbol{\varphi} \in \mathcal{S}^2. \tag{3.3}$$

Let $H^{s_j}, \ H^{s_j}_o(\Omega), \ H^{s_j}(\Omega)$ be the Sobolev spaces, where $j = 1, 2$; Ω is a certain set of intervals in $\mathbb{R}$. We put $\vec{s} = (s_1, s_2)^T$ and

$$\mathbb{H}^{\vec{s}} = H^{s_1} \times H^{s_2}, \ \mathbb{H}^{\vec{s}}_o(\Omega) = H^{s_1}_o(\Omega) \times H^{s_2}_o(\Omega), \ \mathbb{H}^{\vec{s}}(\Omega) = H^{s_1}(\Omega) \times H^{s_2}(\Omega).$$

The scalar product and the norm in $\mathbb{H}^{\vec{s}}$ and $\mathbb{H}^{\vec{s}}_o(\Omega)$ are given by the formulas

$$(\mathbf{u}, \mathbf{v})_{\vec{s}} = \sum_{j=1}^{2} (u_j, v_j)_{s_j}, \ ||\mathbf{u}||_{\vec{s}} = \Big(\sum_{j=1}^{2} ||u_j||^2_{s_j} \Big)^{1/2},$$

where $||u_j||_{s_j}$ and $(u_j, v_j)_{s_j}$ are given by formulas (3.1) and (3.2), respectively. The norm in $\mathbb{H}^{\vec{s}}(\Omega)$ is defined by the equality

$$||u||_{\mathbb{H}^{\vec{s}}(\Omega)} := \Big(\sum_{j=1}^{2} \inf_{l_j} ||l_j u_j||^2_{s_j} \Big)^{1/2},$$

where l_j are extension operators of the $u_j \in H^{s_j}(\Omega)$ from Ω to $\mathbb{R}$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}, \mathbf{u} = (u_1, u_2)^T \in \mathbb{H}^{\vec{s}}(\Omega), \mathbf{f} \in \mathbb{H}^{-\vec{s}}(\Omega)$ and $\mathbf{lf} = (l_1 f_1, l_2 f_2)^T$ be an extension of $\mathbf{f}$ from Ω to $\mathbb{R}$ belonging to $\mathbb{H}^{-\vec{s}}(\mathbb{R})$. Then the* *integral*

$$[\mathbf{f}, \mathbf{u}] := (\mathbf{lf}, \mathbf{u})_o := \sum_{j=1}^{2} \int_{-\infty}^{\infty} \widehat{l_j f_j}(\xi)\overline{\widehat{u_j(\xi)}}d\xi \tag{3.4}$$

does not depend on the choice of the extension $\mathbf{lf}$*. Therefore, this formula defines a linear continuous functional on* $\mathbb{H}_o^{\vec{s}}(\Omega)$*. Conversely, for every linear continuous functional* $\Phi(\mathbf{u})$ *on* $\mathbb{H}_o^{\vec{s}}(\Omega)$ *there exists an element* $\mathbf{f} \in \mathbb{H}^{-\vec{s}}(\Omega)$ *such that* $\Phi(\mathbf{u}) = [\mathbf{u}, \mathbf{f}]$ *and* $||\Phi|| = ||\mathbf{f}||_{\mathbb{H}^{-\vec{s}}(\Omega)}$.

Proof. Let $\mathbf{l'f}$ be another extension of the element $\mathbf{f}$. Then we have $\mathbf{lf} - \mathbf{l'f} = \mathbf{0}$ on Ω, i.e.

$$(\mathbf{lf} - \mathbf{l'f}, \mathbf{w})_o = \mathbf{0} \quad \forall \mathbf{w} \in (C_o^{\infty}(\Omega))^2. \tag{3.5}$$

Since $(C_o^{\infty}(\Omega))^2$ is dense in $\mathbb{H}_o^{\vec{s}}(\Omega)$, then from (3.5) it follows that

$$(\mathbf{lf} - \mathbf{l'f}, \mathbf{u})_o = 0 \quad \forall \mathbf{u} \in \mathbb{H}_o^{\vec{s}}(\Omega),$$

that is $(\mathbf{l'f}, \mathbf{u})_o = (\mathbf{lf}, \mathbf{u})_o$. Thus the integral in (3.4) does not depend on the choice of the extension $\mathbf{lf}$. From (3.4) we obtain

$$|(\mathbf{lf}, \mathbf{u})_o| \leqslant ||\mathbf{u}||_{\vec{s}}.||\mathbf{lf}||_{-\vec{s}}.$$

Since $(\mathbf{lf}, \mathbf{u})$ does not depend on the choice of $\mathbf{lf}$, we have

$$|[\mathbf{f}, \mathbf{u}]| = |(\mathbf{lf}, \mathbf{u})_o| \leqslant ||\mathbf{u}||_{\vec{s}} \inf_{\mathbf{l}} ||\mathbf{lf}||_{-\vec{s}} = ||\mathbf{u}||_{\vec{s}}.||\mathbf{f}||_{\mathbb{H}^{-\vec{s}}(\Omega)}. \tag{3.6}$$

Thus, every element $\mathbf{f} \in \mathbb{H}^{-\vec{s}}(\Omega)$ gives a continuous functional on $\mathbb{H}_o^{\vec{s}}(\Omega)$ by the formula (3.4).

The second part of Lemma 3.1 can be pvoved by using the Riesz theorem. The proof is complete. □

3.2. **Pseudo-differential operators.** Consider pseudo-differential operators of the form

$$(A\mathbf{u})(x) := F^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x),$$

where $\mathbf{A}(\xi) = ||a_{ij}(\xi)||_{2\times 2}$ is a square matrix of order two, $\mathbf{u} = (u_1, u_2)^T$ is a vector, transposed to the line vector (u_1, u_2), and $\widehat{\mathbf{u}}(\xi) := F[\mathbf{u}] = (F[u_1], F[u_2])^T$. We introduce the following classes.

Definition 3.2. Let $\alpha \in \mathbb{R}$. We say that a function $a(\xi)$ belongs to the class $\sigma^{\alpha}(\mathbb{R})$, if $|a(\xi)| \leqslant C_1(1 + |\xi|)^{\alpha}$, $\forall \xi \in \mathbb{R}$, and belongs to the class $\sigma_+^{\alpha}(\mathbb{R})$, if $C_2(1 + |\xi|)^{\alpha} \leqslant a(\xi) \leqslant C_1(1 + |\xi|)^{\alpha}, \forall \xi \in \mathbb{R}$, where C_1 and C_2 are certain positive constants.

Lemma 3.2. [13]. *Let* $a(\xi) > 0$ *be such that* $(1+|\xi|)^{-\alpha}a(\xi)$ *is a bounded continuous function on* $\mathbb{R}$*. Suppose moreover that there are positive limits of the function* $(1 + |\xi|)^{-\alpha}a(\xi)$ *when* $\xi \to \pm\infty$*. Then* $a(\xi) \in \sigma_+^{\alpha}(\mathbb{R})$.

Lemma 3.3. *Let* $a(\xi) \in \sigma^{\alpha}(\mathbb{R})$, $u(x) \in H^s(\mathbb{R})$, $a(\xi)\widehat{u}(\xi) \in \mathcal{S}'(\mathbb{R})$, *then* $F^{-1}[a(\xi)\widehat{u}(\xi)](x) \in H^{s-\alpha}(\mathbb{R})$.

Proof. Indeed, put

$$v(x) = F^{-1}[a(\xi)\widehat{u}(\xi)](x), \ \ \widehat{v}(\xi) = a(\xi)\widehat{u}(\xi).$$

We have $\widehat{v}(\xi) = a(\xi)\widehat{u}(\xi)$ and

$$(3.7) \quad (1+|\xi|)^{s-\alpha}|\widehat{v}(\xi)| = (1+|\xi|)^{-\alpha}|a(\xi)| \cdot (1+|\xi|)^{s}|\widehat{u}(\xi)| \leqslant C(1+|\xi|)^{s}|\widehat{u}(\xi)|.$$

From (3.7) it follows immediately that $||v||^2_{s-\alpha} \leqslant C||u||^2_s < \infty$.

□

Definition 3.3. Let $\mathbf{A}(\xi) = ||a_{ij}(\xi)||_{2\times 2}, \xi \in \mathbb{R}$, be a square matrix of second order, where $a_{ij}(\xi)$ are continuous functions on $\mathbb{R}, \alpha_j \in \mathbb{R}, (j = 1, 2),\ \vec{\alpha} = (\alpha_1, \alpha_2)^T$. Denote by $\mathbf{\Sigma}^{\vec{\alpha}}(\mathbb{R})$ the class of square matrices $\mathbf{A}(\xi) = ||a_{ij}(\xi)||_{2\times 2}$, such that

$$a_{ii}(\xi) \in \sigma^{\alpha_i}(\mathbb{R}), \ \ a_{ij}(\xi) \in \sigma^{\alpha_{ij}}(\mathbb{R}), \ \ \alpha_{ij} \leqslant \frac{1}{2}(\alpha_i + \alpha_j).$$

We shall say that the matrix $\mathbf{A}(\xi)$ belongs to the class $\mathbf{\Sigma}^{\vec{\alpha}}_{+}(\mathbb{R})$, if $\mathbf{A}(\xi) \in \mathbf{\Sigma}^{\vec{\alpha}}(\mathbb{R})$ and it is Hermitian, i.e. $\overline{(\mathbf{A}(\xi))^T} = \mathbf{A}(\xi)$, and satisfies the condition:

$$\overline{\mathbf{w}^T}\mathbf{A}\mathbf{w} \geq C_1 \sum_{j=1}^{2} (1+|\xi|)^{\alpha_j}|w_j|^2 \ \forall \mathbf{w} = (w_1, w_2)^T \in \mathbb{C}^2,$$

where C_1 is a positive constant. Finally, we say that the matrix $\mathbf{A}(\xi) \in \mathbf{\Sigma}^{\vec{\alpha}}(\mathbb{R})$ belongs to the class $\mathbf{\Sigma}^{\vec{\alpha}}_{o}(\mathbb{R})$, if it is positive-definite for almost all $\xi \in \mathbb{R}$.

Lemma 3.4. *Let the matrix* $\mathbf{A}(\xi)$ *belong to the class* $\mathbf{\Sigma}^{\vec{\alpha}}_{+}(\mathbb{R})$. *Then the scalar product and the norm in* $\mathbb{H}^{\vec{\alpha}/2}(\mathbb{R})$ *can be defined by the formulas*

$$(3.8) \qquad (\mathbf{u}, \mathbf{v})_{\mathbf{A}, \vec{\alpha}/2} = \int_{-\infty}^{\infty} \overline{F[\mathbf{v}^T](\xi)}\mathbf{A}(\xi)F[\mathbf{u}](\xi)d\xi,$$

$$(3.9) \qquad ||\mathbf{u}||_{\mathbf{A}, \vec{\alpha}/2} = \Big(\int_{-\infty}^{\infty} \overline{F[\mathbf{u}^T](\xi)}\mathbf{A}(\xi)F[\mathbf{u}](\xi)d\xi\Big)^{1/2},$$

respectively.

Proof. Using the Cauchy-Schwarz inequality one can show that

$$(3.10) \qquad \overline{\mathbf{w}(\xi)}^T\mathbf{A}\mathbf{w}(\xi) \leqslant C_2 \sum_{j=1}^{2} (1+|\xi|)^{\alpha_j}|w_j(\xi)|^2,$$

where C_2 is a positive constant. Replacing in (3.3) and (3.10) $w_j(\xi)$ by $\widehat{u}_j(\xi) = F[u_j](\xi)$ and $\mathbf{w}(\xi)$ by $F[\mathbf{u}](\xi)$, and after that, integrating on $(-\infty, \infty)$, we have

$$C_1 \sum_{j=1}^{2} \int_{-\infty}^{\infty} (1+|\xi|)^{\alpha_j}|F[u_j](\xi)|^2 dt \leqslant \int_{-\infty}^{\infty} \overline{F[\mathbf{u}^T](\xi)}\mathbf{A}(\xi)F[\mathbf{u}](\xi)d\xi$$

$$(3.11) \qquad \leqslant C_2 \sum_{j=1}^{2} \int_{-\infty}^{\infty} (1+|\xi|)^{\alpha_j}|F[u_j](\xi)|^2 d\xi.$$

From (3.11), due to (3.1) and (3.3), we get (3.9). It is clear that, the integral (3.8) defines a scalar product in $\mathbb{H}^{\vec{\alpha}/2}(\mathbb{R})$. □

Theorem 3.5. *Let* $\mathbf{A}(\xi) \in \mathbf{\Sigma}^{\alpha}(\mathbb{R}), \mathbf{u} \in \mathbb{H}^{\vec{s}}(\mathbb{R})$, *where*

$$\vec{s} = \vec{\alpha}/2 \pm \vec{\epsilon}, \;\; \vec{\epsilon} = (\epsilon, \epsilon)^T, \;\; \epsilon \geq 0. \tag{3.12}$$

Then the pseudo-differential operator $A\mathbf{u}$ *defined by the formula* $F^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x)$, $x \in \mathbb{R}$, *is bounded from* $\mathbb{H}^{\vec{s}}(\mathbb{R})$ *into* $\mathbb{H}^{\vec{s}-\vec{\alpha}}(\mathbb{R})$.

Proof. Let $\mathbf{v}(x) := (A\mathbf{u})(x) = F^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x)$. Hence we have

$$\widehat{\mathbf{v}}(\xi) = \mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi). \tag{3.13}$$

Consider an m-component of the vector (3.13). We have

$$\widehat{v}_m(\xi) = \sum_{j=1}^{2} a_{mj}(\xi)\widehat{u}_j(\xi), \;\; m = 1, 2. \tag{3.14}$$

Multiplying both parts of (3.14) by $(1+|\xi|)^{s_m-\alpha_m}$, we have

$$(1+|\xi|)^{s_m-\alpha_m}\widehat{v}_m(\xi) = \sum_{j=1}^{2}[a_{mj}(\xi)(1+|\xi|)^{s_m-\alpha_m-s_j}][(1+|\xi|)^{s_j}\widehat{u}_j(\xi)]. \tag{3.15}$$

Applying the Cauchy-Schwarz inequality to equality (3.15), we get

$$(1+|\xi|)^{2(s_m-\alpha_m)} \leqslant \sum_{j=1}^{2}|a_{mj}(\xi)|^2(1+|\xi|)^{2(s_m-\alpha_m-s_j)}\sum_{j=1}^{2}(1+|\xi|)^{2s_j}|\widehat{u}_j(\xi)|^2. \tag{3.16}$$

Since

$$|a_{mj}(\xi)| \leqslant C(1+|\xi|)^{\alpha/2+\alpha_j/2} \;\; \forall\xi \in \mathbb{R},$$

due to (3.12) we get

$$\sum_{j=1}^{2}|a_{mj}(\xi)|^2(1+|\xi|)^{2(s_m-\alpha_m-s_j)} \leqslant C \;\; \forall\xi \in \mathbb{R}. \tag{3.17}$$

From (3.12) and (3.17) we have

$$||v_m||^2_{s_m-\alpha_m} \leqslant C\sum_{j=1}^{2}||u_j||^2_{s_j},$$

that is, $\mathbf{v} = (A\mathbf{u})(x) \in \mathbf{H}^{\vec{s}-\vec{\alpha}}(\mathbb{R})$. The proof is complete. □

Theorem 3.6. *Let* Ω *be a bounded subset of intervals in* $\mathbb{R}$. *Then the imbedding* $\mathbb{H}^{\vec{s}}(\Omega)$ *into* $\mathbb{H}^{\vec{s}-\vec{\varepsilon}}(\Omega)$ *is compact, where* $\vec{\varepsilon} = (\varepsilon, \varepsilon)^T > \mathbf{0} \Leftrightarrow \;\varepsilon > 0$.

Proof. The proof is based on the fact that the imbedding $H^{s_j}(\Omega)$ into $H^{s_j-\varepsilon}(\Omega)$, $\varepsilon > 0$ is completely continuous if Ω is bounded in $\mathbb{R}$ (see [20]). □

3.3. **Solvability of the system of dual equations** (2.33)**.** The system (2.33) can be rewritten in the form

$$\begin{cases} pF^{-1}[\mathbf{A}(\xi)\hat{\mathbf{u}}(\xi)](x) = \widetilde{\mathbf{f}}(x), & x \in (-a,a), \\ p'\mathbf{u} := p'F^{-1}[\widehat{\mathbf{u}}(\xi)](x) = \mathbf{0}, & x \in \mathbb{R} \setminus (-a,a), \end{cases} \tag{3.18}$$

where the operator F^{-1} is understood in the generalized sense (3.3).

The following propositions hold.

Lemma 3.7. *The matrix* $\mathbf{A}(\xi)$ *defined by formulas* (2.38)*,* (2.31)*,* (2.32) *is positve-definite for all* $\xi \neq 0$.

Due to Lemma 3.7, we have $\mathbf{A}(\xi) \in \mathbf{\Sigma}_o^{-\vec{\alpha}}$, $\vec{\alpha} = (1,1)^T$.

Lemma 3.8. *Let* $a_1(\xi)$, $a_2(\xi)$, $a_{11}(\xi)$ *and* $a_{12}(\xi)$ *be determined by formulas* (2.29)*,* (2.30)*,* (2.31) *and* (2.32)*. Then*

(i) $a_{11}(-\xi) = a_{11}(\xi) > 0, \quad a_{12}(-\xi) = a_{12}(\xi) > 0 \quad \forall \xi \neq 0,$
$a_1(-\xi) = a_1(\xi) > 0, \quad a_2(-\xi) = a_2(\xi) > 0 \quad \forall \xi \neq 0.$

(ii) $a_{11}(0) = \lim_{\xi\to 0} a_{11}(\xi) = \dfrac{h}{3}, \quad a_{12}(0) = \lim_{\xi\to 0} a_{12}(\xi) = \dfrac{h}{6},$
$a_1(0) = \lim_{\xi\to 0} C_1(\xi) = \dfrac{1}{h}, \quad a_2(0) = \lim_{\xi\to 0} a_2(\xi) = \dfrac{1}{h}.$

(iii) $\lim_{h\to\infty} a_{11}(\xi) = \dfrac{1}{2|\xi|}, \qquad \lim_{h\to\infty} a_{12}(\xi) = 0,$
$\lim_{h\to\infty} a_1(\xi) = \dfrac{(1+\nu)|\xi|}{2}, \qquad \lim_{h\to\infty} a_2(\xi) = 0.$

By virtue of Lemmas 3.2 and 3.8, from (2.29)-(2.32) we get

$$a_{11}(\xi) = a_{22}(\xi) \in \sigma_+^{-1} \cap C(\mathbb{R}), \quad a_1(\xi) \in \sigma_+^1 \cap C(\mathbb{R}), \tag{3.19}$$

$$a_{12}(\xi) = a_{21}(\xi) \text{ and } a_2(\xi) \in \sigma^{-\beta} \cap C(\mathbb{R}) \ \forall \beta > 1. \tag{3.20}$$

We make the following assumptions for the traces of $\Phi(x,y)$ on the edges $y = 0$ and $y = h$ of the strip Π:

$$r_1(x) := \Phi(x,0) \text{ and } r_2(x) := \Phi(x,h) \in H^{\frac{3}{2}}(\mathbb{R}), \tag{3.21}$$

$$u_1(x) := M[\Phi](x,0) \text{ and } u_2(x) := M[\Phi](x,h) \in H^{-\frac{1}{2}}(\mathbb{R}). \tag{3.22}$$

By virtue of relations (3.19), (3.41) and Lemma 3.3 we have the following.

Theorem 3.9. *Let conditions* (3.21) *and* (3.22) *be fulfilled. Then*

$$F^{-1}[a_{11}(\xi)\widehat{u}_1(\xi)](x) \text{ and } F^{-1}[a_{22}(\xi)\widehat{u}_2(\xi)](x) \in H^{\frac{1}{2}}(\mathbb{R}),$$
$$F^{-1}[a_{21}(\xi)\widehat{u}_1(\xi)](x) \text{ and } F^{-1}[a_{12}(\xi)\widehat{u}_2(\xi)](x) \in H^{\beta}(\mathbb{R}) \ \forall \beta > 1,$$
$$F^{-1}[a_1(\xi)\widehat{r}_1(\xi)](x) \text{ and } F^{-1}[a_1(\xi)\widehat{r}_2(\xi)](x) \in H^{\frac{1}{2}}(\mathbb{R})$$
$$F^{-1}[a_2(\xi)\widehat{r}_1(\xi)](x) \text{ and } F^{-1}[a_2(\xi)\widehat{r}_2(\xi)](x) \in H^{\beta}(\mathbb{R}) \ \forall \beta > 1.$$

Due to Theorem 3.9 we suppose the following assumptions

$$r_1(x) \text{ and } r_2(x) \in H^{\frac{3}{2}}(\mathbb{R}), \tag{3.23}$$

$$f_1(x) \text{ and } f_2(x) \in H^{\frac{1}{2}}(-a,a). \tag{3.24}$$

Hence we have the conditions

$$\widetilde{\mathbf{f}}(x) \in \mathbb{H}^{\vec{\alpha}/2}(-a,a), \;\; \vec{\alpha} = (1,1)^T. \tag{3.25}$$

Theorem 3.10. (Uniqueness). *Let conditions* (3.25) *be fulfilled. Then the system of dual equations* (3.18) *has at most one solution* $\mathbf{u} \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$.

Proof. Let $\mathbf{u} \in \mathbb{H}^{\vec{\alpha}/2}(-a,a)$ be a solution of the homogeneous system of system (3.18). Using formulas (3.4)-(3.9) and Theorem 3.1 we can show that

$$[A\mathbf{u},\mathbf{u}] = \int_{-\infty}^{\infty} \overline{\widehat{\mathbf{u}}^{\mathbf{T}}(\xi)}\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)d\xi = 0,$$

from which $\mathbf{u} \equiv 0$. □

Denote

$$(A\mathbf{u})(x) = pF^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x) \tag{3.26}$$

and rewrite (3.18) in the form

$$(A\mathbf{u})(x) = \mathbf{f}(x), \quad x \in (-a,a). \tag{3.27}$$

Our purpose now is to establish an existence result for the solution of the system (3.27) in the space $\mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$, $\vec{\alpha} = (1,1)^T$.

We introduce the matrices

$$\mathbf{A}_+(\xi) = \begin{pmatrix} \dfrac{\tanh(|\xi|h)}{2|\xi|} & 0 \\ & \\ 0 & \dfrac{\tanh(|\xi|h)}{2|\xi|} \end{pmatrix}, \;\; \mathbf{B}(\xi) = \mathbf{A}(\xi) - \mathbf{A}_+(\xi). \tag{3.28}$$

Lemma 3.11. *We have* $\mathbf{A}_+(\xi) \in \mathbf{\Sigma}_+^{-\vec{\alpha}}$, $\vec{\alpha} = (1,1)^T$.

Proof. Let

$$\widehat{u}_1 = a_1 + ib_1, \quad \widehat{u}_2 = a_2 + ib_2, \quad a_1, b_1, a_2, b_2 \in \mathbb{R}.$$

We have

$$|\hat{u}_1|^2 = a_1^2 + b_1^2, \;\; |\hat{u}_2|^2 = a_2^2 + b_2^2.$$

It is not difficult to show that

$$\overline{\hat{\mathbf{u}}}^T\mathbf{A}_+\hat{\mathbf{u}} = \frac{\tanh(|\xi|h)}{2|\xi|}[2(a_1^2 + b_1^2 + a_2^2 + b_2^2)] \geq 0.$$

Thus,

$$\overline{\hat{\mathbf{u}}}^T\mathbf{A}_+\hat{\mathbf{u}} = \frac{\tanh(|\xi|h)}{|\xi|}(|\widehat{u}_1|^2 + |\widehat{u}_2|^2). \tag{3.29}$$

Using Lemma 3.2, we can show that $\dfrac{\tanh(|\xi|h)}{|\xi|} \in \sigma_+^{-1}(\mathbb{R})$, that means there exists a positive constant C such that

$$\frac{\tanh(|\xi|h)}{|\xi|} \geq C\frac{1}{(1+|\xi|)} \ \forall \xi \in \mathbb{R}. \tag{3.30}$$

From (3.29) and (3.30) it follows that $\mathbf{A}_+(\xi) \in \mathbf{\Sigma}_+^{-\vec{\alpha}}$, $\vec{\alpha} = (1,1)^T$. The theorem is proved. □

It is not difficult to show that

$$\mathbf{B}(\xi) \in \mathbf{\Sigma}^{-\vec{\beta}},\ \vec{\beta} = (\beta,\beta)^T \ \forall \beta > 1.$$

We have

Lemma 3.12. *The scalar product and the norm in* $\mathbb{H}_o^{-\vec{\alpha}/2}(\mathbb{R})$ $(\vec{\alpha} = (1,1)^T)$ *are equivalent to the followings:*

$$(\mathbf{u},\mathbf{v})_{\mathbf{A}_+,-\vec{\alpha}/2} = \int_{-\infty}^{\infty} \overline{\widehat{\mathbf{v}}(\xi)}^T \mathbf{A}_+(\xi)\widehat{\mathbf{u}}(\xi)d\xi, \tag{3.31}$$

$$||\mathbf{u}||_{\mathbf{A}_+,-\vec{\alpha}/2} = \Big(\int_{-\infty}^{\infty} \overline{\widehat{\mathbf{u}}(\xi)}^T \mathbf{A}_+(\xi)\widehat{\mathbf{u}}(\xi)d\xi\Big)^{1/2}, \tag{3.32}$$

where the matrix $\mathbf{A}_+(\xi)$ *is defined by formula* (3.28).

Theorem 3.13. (Existence). *Let assumptions* (3.23) *and* (3.24) *hold. Then the system of dual equations* (3.18) *has a unique solution* $\mathbf{u} = F^{-1}[\widehat{\mathbf{u}}] \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$.

Proof. Since the assumptions (3.23) and (3.24) hold, due to (2.36), (2.37) and Theorem 3.9, we have $\widetilde{\mathbf{f}}(x) \in \mathbb{H}^{\vec{\alpha}/2}(-a,a)$, $\vec{\alpha} = (1,1)^T$ (see (3.25)). We represent the operator $\mathbf{A}$ defined by formula (3.26) in the form $\mathbf{A} = \mathbf{A}_+ + \mathbf{B}$, where

$$A_+\mathbf{u} = pF^{-1}[\mathbf{A}_+\hat{\mathbf{u}}], \quad B\mathbf{u} = pF^{-1}[\mathbf{B}\hat{\mathbf{u}}], \quad \mathbf{u} = F[\mathbf{u}]. \tag{3.33}$$

First we consider the system of equations

$$A_+\mathbf{u}(x) = \mathbf{g}(x), \quad \mathbf{u}(x) \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a), \tag{3.34}$$

where $\mathbf{g}(x) \in \mathbb{H}^{\vec{\alpha}/2}(-a,a)$ is a given vector-function. From (3.4), (3.8) and (3.31) we have

$$[\mathbf{A}_+\mathbf{u},\mathbf{v}] = \int_{-\infty}^{\infty} \overline{F[\mathbf{v}^T](\xi)}\mathbf{A}_+(\xi)F[\mathbf{u}](\xi)d\xi = (\mathbf{u},\mathbf{v})_{\mathbf{A}_+,-\vec{\alpha}/2}$$

for arbitrary vector-functions $\mathbf{u}$ and $\mathbf{v}$ belonging to $\mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$. Therefore, if $\mathbf{u} \in \mathbb{H}_o^{\vec{\alpha}/2}(-a,a)$ satisfies (3.34), then

$$(\mathbf{u},\mathbf{v})_{\mathbf{A}_+,-\tilde{\alpha}/\mathbf{2}} = [\mathbf{g},\mathbf{v}], \ \forall \mathbf{v} \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a). \tag{3.35}$$

Since $[\mathbf{g},\mathbf{v}]$ is a linear continuous functional on the Hilbert space $\mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$, by virtue of the Riesz theorem there exists a unique element $\mathbf{u}_o \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$ such that

$$[\mathbf{g},\mathbf{v}] = (\mathbf{u_o},\mathbf{v})_{\mathbf{A}_+,-\vec{\alpha}/2}, \quad \mathbf{v} \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a). \tag{3.36}$$

From (3.35) and (3.36) we get $\mathbf{u} = \mathbf{u}_0$. Moreover, the estimation

$$||\mathbf{u_o}||_{\mathbf{A}_+,-\vec{\alpha}/2} = ||A^{-1}\mathbf{g}||_{\mathbf{A}_+,-\vec{\alpha}/2} \leqslant C||\mathbf{g}||_{\mathbb{H}^{\vec{\alpha}/2}(-a,a)} \tag{3.37}$$

holds, where C is a positive constant. Next, we represent system (3.18) in the form

$$A_+\mathbf{u} + B\mathbf{u} = \tilde{\mathbf{f}}.$$

Hence we have

$$\mathbf{u} + A_+^{-1}B\mathbf{u} = A_+^{-1}\tilde{\mathbf{f}}. \tag{3.38}$$

In virtue of Theorem 3.6, the operator $B\mathbf{u}$ defined by (3.33) is completely continuous from $\mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$ into $\mathbb{H}^{\vec{\alpha}/2}(-a,a)$, due to (3.37) the operator A_+^{-1} is bounded. Thus, the operator $A_+^{-1}B$ is completely continuous. It follows that the system of equations (3.38) is Fredholm. Due to the uniqueness of its solution (Theorem 3.10) it follows that this system has a unique solution $\mathbf{u} \in \mathbb{H}_o^{-\vec{\alpha}/2}(-a,a)$. □

3.4. **Regularity.** We have the following result.

Theorem 3.14. *Assume that* (3.23) *and* (3.24) *hold. There exists a unique solution* Φ *of problem* (2.1)–(2.5) *which belongs to the Sobolev spaces* $H^1(\Pi)$, $H^m(\Pi_\varepsilon)(\forall m \geq 2, \forall \varepsilon > 0)$, *where*

$$\Pi_\varepsilon := \{(x,y) : -\infty < x < \infty,\ \ \varepsilon < y < h - \varepsilon\}.$$

Proof. Write $\Phi(x,y) := F^{-1}[\widehat{\Phi}(\xi,y)](x)$, where $\widehat{\Phi}(\xi,y)$ is given by (2.20). First we pvove that the function $\Phi(x,y)$ satisfies equation (2.1) in the strip Π. Indeed, in (2.20) the functions $\widehat{u}_1(\xi), \widehat{u}_2(\xi), \widehat{r}_1(\xi)$, and $\widehat{r}_2(\xi)$ are tempered, then due to (2.21) and (2.9) for $|x| < \infty, 0 < y < h$ we get

$$\Delta^2\Phi(x,y) = F^{-1}\Big[\frac{d^4\widehat{\Phi}(\xi,y)}{dy^4} - 2\xi^2\frac{d^2\widehat{\Phi}(\xi,y)}{dy^2} + \xi^4\widehat{\Phi}(\xi,y)\Big](x) = 0.$$

It is not difficult to verify that the boundary conditions (2.2) and (2.3) are fulfilled, and that

$$M[\Phi](x,0) = u_1(x),\ \ M[\Phi](x,h) = u_2(x),\ \ x \in \mathbb{R},$$

where $M[\Phi]$ is defined by (2.6). The functions $u_1(x), u_2(x)$ are determined by boundary conditions (2.4) and (2.5), that are equivalent to the system of dual integral equations (2.33). According to Theorem 3.13, if conditions (3.23) and (3.24) are fulfilled, then the system of dual equations (2.33) has a unique solution $\mathbf{u} = (u_1,u_2) \in H_o^{-1/2}(-a,a) \times H_o^{-1/2}(-a,a)$. Thus, if conditions (3.23) and (3.24) hold, then there exists a unique solution $\Phi(x,y) = F^{-1}[\widehat{\Phi}(\xi,y)](x)$ of problem (2.1)-(2.5), where $\widehat{\Phi}(\xi,y)$ is given by (2.20).

We now prove $\Phi \in H^1(\Pi)$. First we pvove $\Phi_x = \dfrac{\partial\Phi}{\partial x} \in L^2(\Pi)$. Using Parseval's equality, we have

$$\int_{-\infty}^{\infty} |\Phi_x(x,y)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi^2|\widehat{\Phi}(\xi,y)|^2 d\xi,\ \ 0 < y < h. \tag{3.39}$$

Substituting $\widehat{\Phi}(\xi, y)$ from (2.20) into the right hand side of (3.39) and applying the Cauchy-Schwarz inequality we get

$$\int_{-\infty}^{\infty} |\Phi_x(x,y)|^2 dx$$
$$\leqslant \frac{2}{\pi}\int_{-\infty}^{\infty} \xi^2|\widehat{u}_1(\xi)|^2\Big|\frac{h\sinh(|\xi|y) - y\sinh(|\xi|h)\cosh(|\xi|(h-y))}{2|\xi|\sinh^2(|\xi|h)}\Big|^2 d\xi$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty} \xi^2|\widehat{u}_2(\xi)|^2\Big|\frac{y\cosh(|\xi|y)\sinh(|\xi|h) - h\sinh(|\xi|y)\cosh(|\xi|h)}{2|\xi|\sinh^2(|\xi|h)}\Big|^2 d\xi$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty} \xi^2|\widehat{r}_1(\xi)|^2\Big|\frac{2\sinh(|\xi|h)\sinh(|\xi|(h-y))}{2\sinh^2(|\xi|h)}$$
$$+\frac{(1-\nu)|\xi|\big[y\sinh(|\xi|h)\cosh(|\xi|(h-y)) - h\sinh(|\xi|y)\big]}{2\sinh^2(|\xi|h)}\Big|^2 d\xi$$
$$+\frac{2}{\pi}\int_{-\infty}^{\infty} \xi^2|\widehat{r}_2(\xi)|^2\Big|\frac{(1-\nu)|\xi|(h-y)\sinh(|\xi|y)\cosh(|\xi|h)}{2\sinh^2(|\xi|h)}$$
$$+\frac{2\sinh(|\xi|y)\sinh(|\xi|h) - (1-\nu)y|\xi|\sinh(|\xi|(h-y))}{2\sinh^2(|\xi|h)}\Big|^2 d\xi. \tag{3.40}$$

From (3.40) it follows that in order to prove $||\Phi_x||_{L^2(\Pi)} < +\infty$, it suffices to show that

$$J_1 := \int_{|\xi|>1} |\widehat{u}_1(\xi)|^2 \frac{d\xi}{\sinh^2(|\xi|h)}\int_0^h \cosh^2(|\xi|(h-y))dy < +\infty,$$
$$J_2 := \int_{|\xi|>1} |\widehat{u}_2(\xi)|^2 \frac{d\xi}{\sinh^2(|\xi|h)}\int_0^h \cosh^2(|\xi|y)dy < +\infty,$$
$$J_3 := \int_{|\xi|>1} |\widehat{r}_1(\xi)|^2 \frac{\xi^4 d\xi}{\sinh^2(|\xi|h)}\int_0^h \cosh^2(|\xi|(h-y))dy < +\infty,$$
$$J_4 := \int_{|\xi|>1} |\widehat{r}_2(\xi)|^2 \frac{\xi^4 d\xi}{\sinh^2(|\xi|h)}\int_0^h \cosh^2(|\xi|y)dy < +\infty.$$

Using the identity

$$\int_0^h \cosh^2(|\xi|(h-y))dy = \frac{\sinh(2|\xi|h) + 2|\xi|h}{4|\xi|}$$

we have

$$J_1 = \int_{|\xi|>1} \frac{|\widehat{u}_1(\xi)|^2 d\xi}{1+|\xi|}\Big[\frac{(1+|\xi|)[\sinh(2|\xi|h) + 2|\xi|h]}{4|\xi|\sin^2(|\xi|h)}\Big] \leqslant C\int_{|\xi|>1} \frac{|\widehat{u}_1(\xi)|^2 d\xi}{1+|\xi|}, \tag{3.41}$$

where C is a certain positive constant. As $u_1(x) \in H_o^{-1/2}(-a,a) \subset H^{-1/2}(\mathbb{R})$, the integral in the right-hand side of (3.41) is finite. It implies $J_1 < +\infty$. By the same reason $J_2 < +\infty$. Next we have

$$J_3 = \int_{|\xi|>1} (1+|\xi|)^3|\widehat{r}_1(\xi)|^2 d\xi\Big[\frac{\xi^4[\sinh(2|\xi|h) + 2|\xi|h]}{(1+|\xi|)^3 4|\xi|\sin^2(|\xi|h)}\Big]$$

$$\leqslant C\int_{|\xi|>1}(1+|\xi|)^3|\widehat{r}_1(\xi)|^2 d\xi < +\infty$$

because $r_1(x) \in H^{3/2}(\mathbb{R})$. By the same reason we have $J_4 < +\infty$. From the above we get $\Phi_x \in L^2(\Pi)$. Similarly, we get Φ and $\Phi_y \in L^2(\Pi)$. We thus have $\Phi \in H^1(\Pi)$. In the same manner, we can see that $\Phi \in H^m(\Pi_\varepsilon)(\forall m \geq 2, \forall \varepsilon > 0)$, where

$$\Pi_\varepsilon := \{(x,y) : -\infty < x < \infty,\ \ \varepsilon < y < h-\varepsilon\}.$$

The proof is complete. □

4. Reduction to an infinite system of linear algebraic equations

In this section we propose a method for reducing the system of dual integral equations (2.33) to an infinite system of linear algebraic equations of second kind.

4.1. **Some preliminary considerations.**

Definition 4.1. Let $\rho(x) = \sqrt{a^2 - x^2}$ $(-a < x < a)$. We denote by $L^2_{\rho^{\pm1}}(-a,a)$ the Hilbert spaces of functions with respect to the scalar products and the norms

$$(u,v)_{L^2_{\rho^{\pm1}}} = \int_{-a}^{a}\rho^{\pm1}(x)u(x)\overline{v(x)}dx, \quad ||u||_{L^2_{\rho^{\pm1}}} = \sqrt{(u,u)_{L^2_{\rho^{\pm1}}}} < +\infty.$$

We will need the following result [14].

Lemma 4.1. *Let $\varphi \in L^2_\rho(-a,a)$. Denote by φ_o the zero-extension of the function φ on $\mathbb{R}$. Then $\varphi_o \in H_o^{-1/2}(-a,a)$.*

We shall need some relations for Chebyshev polynomials. Let $T_k(x)$ and $U_k(x)$ be the Chebyshev polynomials of first and second kind, respectively. We have the following relations [17]:

$$T_n(\cos\theta) = \cos n\theta, \tag{4.1}$$

$$\int_{-a}^{a}\frac{T_k[\eta(x)]T_j[\eta(x)]}{\rho(x)}dx = \alpha_k\delta_{kj}, \tag{4.2}$$

$$\int_{-a}^{a}\ln\Big|\frac{1}{x-y}\Big|\frac{T_k[\eta(y)]}{\rho(y)}dy = \sigma_k T_k[\eta(x)],\ \ (k=0,1,2,...), \tag{4.3}$$

where δ_{kj} is the Kronecker symbol and

$$\alpha_k = \begin{cases}\pi, & k=0,\\ \dfrac{\pi}{2}, & k=1,2,...\end{cases},\quad \sigma_k = \begin{cases}\pi(\ln 2 - \ln a), & k=0,\\ \dfrac{\pi}{k}, & k=1,2,...,\end{cases},\quad \eta(x) = \frac{x}{a}. \tag{4.4}$$

Consider the following system of linear algebraic equations [9]:

$$x_i = \sum_{k=1}^{\infty} c_{i,k}x_k + b_i \quad (i=1,2,...), \tag{4.5}$$

where the numbers x_i are to be determined.

Definition 4.2. [9] The infinite system (4.5) is called regular if

$$\sum_{k=1}^{\infty} |c_{i,k}| < 1 \quad (i = 1, 2, ...) \tag{4.6}$$

and completely regular if

$$\sum_{k=1}^{\infty} |c_{i,k}| \leqslant 1 - \theta < 1 \quad (i = 1, 2, ...). \tag{4.7}$$

If the inequalities (4.6) (respectively, (4.7)) hold only for $i = N + 1, N + 2, ...$, then system (4.5) is called quasi-regular (respectively, quasi-completely regular).

The theory and applications of regular infinite systems can be found in [9].

4.2. **Reduction to a system of integral equations with logarithmic kernel.** Now we turn to system (2.33) and rewrite it in the form

$$\begin{cases} \sum_{n=1}^{2} F^{-1}\Big[\dfrac{a^*_{mn}(\xi)}{|\xi|}\widehat{u}_n(\xi)\Big](x) = \widetilde{f}_m(x), \ \ x \in (-a, a), \\ u_m(x) = F^{-1}[\widehat{u}_m](x) = 0, \ \ x \in \mathbb{R} \setminus (-a, a), \ \ m = 1, 2, \end{cases} \tag{4.8}$$

where

$$a^*_{11}(\xi) = a^*_{22}(\xi) = \frac{\sinh(|\xi|h)\cosh(|\xi|h) - |\xi|h}{2\sinh^2(|\xi|h)}, \tag{4.9}$$

$$a^*_{21}(\xi) = a^*_{12}(\xi) = \frac{|\xi|h\cosh(|\xi|h) - \sinh(|\xi|h)}{2\sinh^2(|\xi|h)}. \tag{4.10}$$

Theorem 4.2. *The system of dual integral equations* (4.8) *with respect to* $(\widehat{u}_1(\xi), \widehat{u}_2(\xi))$ *is equivalent to the following system of integral equations on* $(-a, a)$:

(4.11)

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-a}^{a} \ln\Big|\frac{1}{x-t}\Big| u_1(t)dt + \int_{-a}^{a} u_1(t)k_{11}(x-t)dt + \int_{-a}^{a} u_2(t)k_{12}(x-t)dt = \widetilde{f}_1(x), \\ \dfrac{1}{2\pi}\displaystyle\int_{-a}^{a} \ln\Big|\frac{1}{x-t}\Big| u_2(t)dt + \int_{-a}^{a} u_1(t)k_{21}(x-t)dt + \int_{-a}^{a} u_2(t)k_{22}(x-t)dt = \widetilde{f}_2(x), \end{cases}$$

where

(4.12)

$$\widehat{u}_m(\xi) = F[u_m](\xi) = \int_{-a}^{a} e^{it\xi}u_m(t)dt, \ \ \mathrm{supp} u_m \subset [-a, a], \ \ m = 1, 2,$$

(4.13)

$$k_{11}(x) = k_{22}(x) = \frac{1}{2\pi}\ln\Big|x\coth\frac{\pi x}{4h}\Big| + \frac{1}{2\pi}\int_0^{\infty} \frac{2a^*_{11}(\xi) - \tanh(\xi h)}{\xi}\cos\xi x d\xi,$$

(4.14)

$$k_{12}(x) = k_{21}(x) = \frac{1}{\pi}\int_0^{\infty} \frac{a^*_{12}(\xi)}{\xi}\cos(\xi x)d\xi.$$

Proof. From the second equation in (4.8) we obtain (4.12). Substituting (4.8) into the first equation in (4.8), using the convolution theorem for the Fourier transforms we get

$$\sum_{n=1}^{2}\int_{-a}^{a} u_n(t)K_{mn}(x-t)dt = \widetilde{f}_m(x),\ \ x\in(-a,a), \tag{4.15}$$

where

$$K_{mn}(x) = F^{-1}\Big[\frac{a^*_{mn}(\xi)}{|\xi|}\Big](x) = \frac{1}{\pi}\int_0^\infty \frac{a^*_{mn}(\xi)}{\xi}\cos(\xi x) = k_{mn}(x)\ (m\neq n). \tag{4.16}$$

Since

$$a^*_{mm}(x) = O\Big(\frac{\tanh(|\xi|h)}{2}\Big)\ \ (\xi\to 0,\ \ \xi\to+\infty)$$

we transform $K_{mm}(x)$ as follows

$$K_{mm}(x) = \frac{1}{2\pi}\int_0^\infty \frac{\tanh(\xi h)}{\xi}\cos(\xi x)d\xi + \frac{1}{2\pi}\int_0^\infty \frac{2a^*_{mm}(\xi)-\tanh(\xi h)}{\xi}\cos(\xi x)d\xi.$$

Using the formula 4.116 (2), p.530 [8]:

$$\int_0^\infty \frac{\tanh(\beta\xi)}{\xi}\cos(\alpha\xi)d\xi = \ln\Big|\coth\big(\frac{\alpha\pi}{4\beta}\big)\Big|$$

we get

$$K_{mm}(x) = \frac{1}{2\pi}\ln\Big|\coth\big(\frac{\pi x}{4h}\big)\Big| + \frac{1}{2\pi}\int_0^\infty \frac{2a^*_{mm}(\xi)-\tanh\big(\xi h\big)}{\xi}\cos(\xi x)d\xi$$

(4.17)

$$= \frac{1}{2\pi}\ln\Big|\frac{1}{x}\Big| + \frac{1}{2\pi}\ln\Big|x\coth\big(\frac{\pi x}{4h}\big)\Big| + \frac{1}{2\pi}\int_0^\infty \frac{2a^*_{mm}(\xi)-\tanh(\xi h)}{\xi}\cos(\xi x)d\xi.$$

Substituting (4.16) and (4.17) in (4.15) we obtain (4.11). □

It is not difficult to show that for $\xi\to+\infty$ the following asymptotical behaviors

$$2a^*_{mm}(\xi)-\tanh(\xi h) = O(\xi e^{-2\xi h}),\ \ a^*_{mn}(\xi) = O(\xi e^{-\xi h})(m\neq n)$$

hold. Therefore from (4.13), (4.14) and the above relations it follows that

$$k_{mm}(x)\ \text{ and }\ k_{mn}(x)\in C^\infty[-a,a]\ (m,n=1,2;m\neq n).$$

4.3. **Reduction to an infinite system of linear algebraic equations.** We will seek the solution of system (4.11) in the class $L^2_\rho(-a,a)$, which is represented in the form

$$u_m(x) = \frac{v_m(x)}{\rho(x)},\ \ m=1,2, \tag{4.18}$$

where $v_m(x)\in L^2_{\rho^{-1}}(-a,a)$. Then, due to Lemma 4.1 the functions $u_m(x)$ will belong to the space $H_o^{-1/2}(-a,a)$. We make the following assumptions

$$\widetilde{f}_m(x)\in H^{1/2}(-a,a)\cap L^2_{\rho^{-1}}(-a,a),\ \ m=1,2. \tag{4.19}$$

Substituting (4.18) into (4.11) we have the following system of integral equations

(4.20)
$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-a}^{a}\ln\Big|\frac{1}{x-t}\Big|\frac{v_1(t)dt}{\rho(t)}+\int_{-a}^{a}\frac{v_1(t)}{\rho(t)}k_{11}(x-t)dt+\int_{-a}^{a}\frac{v_2(t)}{\rho(t)}k_{12}(x-t)dt=\widetilde{f}_1(x),\\ \dfrac{1}{2\pi}\displaystyle\int_{-a}^{a}\ln\Big|\frac{1}{x-t}\Big|\frac{v_2(t)dt}{\rho(t)}+\int_{-a}^{a}\frac{v_1(t)}{\rho(t)}k_{21}(x-t)dt+\int_{-a}^{a}\frac{v_2(t)}{\rho(t)}k_{22}(x-t)dt=\widetilde{f}_2(x).\end{cases}$$

Further, we expand the functions $v_1(t)$ and $v_2(t)$ into the series

$$v_m(t)=\sum_{j=0}^{\infty}A_j^{(m)}T_j[\eta(t)],\quad (m=1,2), \tag{4.21}$$

where $A_j^{(m)}$ are unknown constants that are to be determined so that $\{A_j^{(m)}\}_{j=0}^{\infty}\in l_2 (m=1,2)$. Substituting (4.21) into (4.20), changing the order of integrations and summations and using formula (4.3), we have the following system

$$\begin{cases}\dfrac{1}{2\pi}\alpha_n\sigma_n A_n^{(1)}+\displaystyle\sum_{j=0}^{\infty}(A_j^{(1)}C_{nj}^{(11)}+A_j^{(2)}C_{nj}^{(12)})=F_n^{(1)},\\ \dfrac{1}{2\pi}\alpha_n\sigma_n A_n^{(2)}+\displaystyle\sum_{j=0}^{\infty}(A_j^{(1)}C_{nj}^{(21)}+A_j^{(2)}C_{nj}^{(22)})=F_n^{(2)},\\ \quad n=0,1,2,...,\end{cases} \tag{4.22}$$

where

$$C_{nj}^{(11)}=C_{nj}^{(22)}=\int_{-a}^{a}\frac{T_n[\eta(x)]}{\rho(x)}dx\int_{-a}^{a}\frac{T_j[\eta(t)]}{\rho(t)}k_{11}(x-t)dt, \tag{4.23}$$

$$C_{nj}^{(12)}=C_{nj}^{(21)}=\int_{-a}^{a}\frac{T_n[\eta(x)]}{\rho(x)}dx\int_{-a}^{a}\frac{T_j[\eta(t)]}{\rho(t)}k_{12}(x-t)dt, \tag{4.24}$$

$$F_n^{(1)}=\int_{-a}^{a}\frac{T_n[\eta(x)]}{\rho(x)}\widetilde{f}_1(x)dx,\ \ F_n^{(2)}=\int_{-a}^{a}\frac{T_n[\eta(x)]}{\rho(x)}\widetilde{f}_2(x)dx. \tag{4.25}$$

From the above reasoning and relation (4.3) we obtain readily.

Theorem 4.3. *The system of integral equations* (4.20) *with respect to* $v_1(t), v_2(t)\in L^2_{\rho^{-1}}(-a,a)$ *is equivalent to the system of linear algebraic equations* (4.22) *with respect to* $\{A_j^{(m)}\}_{j=0}^{\infty}\in l_2$ $(m=1,2)$.

We introduce the notations

$$X_{2n+1}=A_n^{(1)},\qquad X_{2n+2}=A_n^{(2)},\qquad (n=0,1,2,...), \tag{4.26}$$

$$E_{2n+1}=\frac{2\pi}{\alpha_n\sigma_n}F_n^{(1)},\qquad E_{2n+2}=\frac{2\pi}{\alpha_n\sigma_n}F_n^{(2)},\qquad (n=0,1,2,...), \tag{4.27}$$

$$C_{2n+1,2j+1}=\frac{2\pi}{\alpha_n\sigma_n}C_{nj}^{(11)},\quad C_{2n+1,2j+2}=\frac{2\pi}{\alpha_n\sigma_n}C_{nj}^{(12)},\ (n,j=0,1,2,...), \tag{4.28}$$

$$C_{2n+2,2j+1}=\frac{2\pi}{\alpha_n\sigma_n}C_{nj}^{(21)},\quad C_{2n+2,2j+2}=\frac{2\pi}{\alpha_n\sigma_n}C_{nj}^{(22)},\ (n,j=0,1,2,...). \tag{4.29}$$

Then system (4.22) can be written in the form

$$X_m + \sum_{k=1}^{\infty} C_{m,k} X_k = E_m \ \ (m = 1, 2, ...). \tag{4.30}$$

Lemma 4.4. *It holds*

$$|C_{n,j}| \leqslant \frac{L}{nj^2} \ \ (n \geq 2, \ \ j \geq 2), \tag{4.31}$$

where L is a certain positive constant.

Proof. Taking changes of variables

$$x = a\cos\theta, \quad t = a\cos\varphi,$$

due to (4.28) and (4.29), we have

$$C_{n,j} = \frac{2\pi}{\alpha_n \sigma_n} \int_0^{\pi} \cos(n\theta) d\theta \int_0^{\pi} \cos(j\varphi) k_{mk}[a(\cos\theta - \cos\varphi)] d\varphi. \tag{4.32}$$

Denote the internal integral in (4.32) by $K_{mk,j}(\cos\theta)$ and integrate it twice by parts, we obtain

$$K_{mk,j}(\cos\theta) = \frac{a^2}{j(j-1)} \int_0^{\pi} \sin(j-1)\varphi \sin\varphi k''_{mk}[a(\cos\theta - \cos\varphi)] d\varphi$$

$$-\frac{a^2}{j(j+1)} \int_0^{\pi} \sin(j+1)\varphi \sin\varphi k''_{mk}[a(\cos\theta - \cos\varphi)] d\varphi. \tag{4.33}$$

Because $k_{mk}(x)$ are infinitely differentiable bounded functions on $[-a, a]$, then from (4.33) it follows that

$$|K_{mk,j}(\cos\theta)| \leqslant \frac{L}{j^2} \ \ (j \geq 2). \tag{4.34}$$

Consider now the integral

$$L_{nj,mk} := \int_0^{\pi} \cos(n\theta) K_{mk,j}(\cos\theta) d\theta.$$

Similarly as above, we get

$$|L_{nj,mk}| \leqslant \frac{L}{n^2} \ \ (n \geq 2). \tag{4.35}$$

From (4.32)– (4.35), due to (4.4), we obtain (4.31). □

The following lemma may be proved in the same way as Lemma 4.4.

Lemma 4.5. *If the derivatives $\widetilde{f}_m^{(k)}(x), m = 1, 2$ are continuous functions on $[-a, a]$, then the following inequalities hold:*

$$|E_n| \leqslant \frac{L}{n^{k-1}} \ \ (n = 1, 2, ...; k = 1, 2, ...). \tag{4.36}$$

Theorem 4.6. *Let* $\widetilde{f}_1(x)$ *and* $\widetilde{f}_2(x)$ *satisfy conditions* (4.19) *and be such that the set* $\{E_n\}_{n=1}^{\infty}$, *defined by* (4.27) *belongs to* l_2. *Then the infinite system of linear algebraic equations* (4.30) *possesses a unique solution* $\{X_n\}_{n=1}^{\infty} \in l_2$. *This infinite system is quasi-completely regular.*

Proof. Denote by H the infinite coefficient matrix in the left-hand side of (4.30). According to (4.31) the double series formed of the squares of components of the H is convergent, so the infinite matrix H defines a completely continuous operator mapping the Hilbert space l_2 into itself. Therefore, the infinite system (4.30) is Fredholm in l_2. The uniqueness of a solution of this system follows from the uniqueness of that of the system of dual equations (2.33). Hence it follows that the infinite system (4.30) has a unique solution in l_2. For a sufficiently large number $n = N$, we have

$$\sum_{j=1}^{\infty} |C_{nj}| \leqslant \frac{L}{n} \sum_{j=1}^{\infty} \frac{1}{j^2} \leqslant 1 - \theta < 1 \ \ (n = N+1, N+2, \ldots).$$

Therefore the infinite system (4.30) is quasi-completely regular [9]. □

For the approximate calculations of the coefficients of the finite system (4.30) one can use the quadrature formulas, for example [10].

Acknowledgement

The authors would like to thank the referee for valuable comments and suggestions to the original manuscript.

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Hanoi Institute of Mathemayics
18 Hoang Quoc Viet, 10307 Hanoi, Vietnam
E-mail address: nvngoc@math.ac.vn

Department of Mathematics, Thainguyen Education University
20 Luong Ngoc Quyen, Thainguyen City, Vietnam
E-mail address: ngandhsphtn@yahoo.com.vn