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ON A NONLINEAR WAVE EQUATION WITH A NONLOCAL BOUNDARY CONDITION

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. Consider the initial-boundary value problem for the nonlinear wave equation

$$\begin{cases} u_{tt}-\mu(t)u_{xx}+K|u|^{p-2}u+\lambda|u_t|^{q-2}u_t=F(x,t),\ 0<x<1,\ 0<t<T,\\ \mu(t)u_x(0,t)=K_0u(0,t)+\int_0^t k\,(t-s)\,u\,(0,s)\,ds+g(t),\\ -\mu(t)u_x(1,t)=K_1u(1,t)+\lambda_1|u_t(1,t)|^{\alpha-2}u_t(1,t),\\ u(x,0)=\widetilde{u}_0(x),\ u_t(x,0)=\widetilde{u}_1(x), \end{cases}$$

where $p,\ q,\ \alpha\geq 2$; $K_0,\ K_1,\ K\geq 0$; $\lambda,\ \lambda_1>0$ are given constants and μ, F, g, k, $\widetilde{u}_0$, $\widetilde{u}_1$, are given functions. First, the existence and uniqueness of a weak solution are proved by using the Galerkin method. Next, with $\alpha=2$, we obtain an asymptotic expansion of the solution up to order N in two small parameters $\lambda,\ \lambda_1$ with error $\left(\sqrt{\lambda^2+\lambda_1^2}\right)^{N+\frac{1}{2}}$.

1. Introduction

In this paper, we consider the following initial-boundary value problem

$$u_{tt}-\mu(t)u_{xx}+f(u,u_t)=F(x,t),\ 0<x<1,\ 0<t<T, \tag{1.1}$$

$$\mu(t)u_x(0,t)=K_0u(0,t)+\int_0^t k\,(t-s)\,u\,(0,s)\,ds+g(t), \tag{1.2}$$

$$-\mu(t)u_x(1,t)=K_1u(1,t)+\lambda_1|u_t(1,t)|^{\alpha-2}u_t(1,t), \tag{1.3}$$

$$u(x,0)=\widetilde{u}_0(x),\ u_t(x,0)=\widetilde{u}_1(x), \tag{1.4}$$

where $f(u,u_t)=K|u|^{p-2}u+\lambda|u_t|^{q-2}u_t$, with K, K_0, K_1, λ, λ_1, p, q and α being given constants and $\widetilde{u}_0$, $\widetilde{u}_1$, g, k, μ, F being given functions satisfying conditions specified later.

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It is well known that many various problems in the form (1.1)–(1.4) have been investigated. For example, we refer to Cavalcanti et al. [4], Long, Dinh and Diem [9], Ngoc, Hang and Long [10], Qin [11, 12], Rivera [13], Santos [14], and the references therein. In these works, many interesting results about the unique existence, regularity, stability, asymptotic expansion or the decay of solutions are obtained.

Clearly, the boundary condition (1.2) is nonlocal and if we put

$$Y(t) = g(t) + K_0 u(0,t) + \int_0^t k(t-s)\, u(0,s)\, ds,$$

then (1.2) is written as follows

$$\mu(t) u_x(0,t) = Y(t), \tag{1.5}$$

and problem (1.1)–(1.4) can be reduced to problem (1.1), (1.3)-(1.5), in which an unknown function $u(x,t)$ and an unknown boundary value $Y(t)$ satisfy the following Cauchy problem for the ordinary differential equation

$$\begin{cases} Y''(t) + \gamma_1 Y'(t) + \gamma_2 Y(t) = \gamma_3 u_{tt}(0,t),\ 0 < t < T, \\ Y(0) = \widetilde{Y}_0,\ Y'(0) = \widetilde{Y}_1, \end{cases} \tag{1.6}$$

where γ_1, γ_2, γ_3, $\widetilde{Y}_0$, $\widetilde{Y}_1$ are certain constants such that $\gamma_1^2 - 4\gamma_2 < 0$.

In [1], An and Trieu studied a special case of problem (1.1), (1.4)-(1.6) associated with the following homogeneous boundary condition at $x = 1$:

$$u(1,t) = 0, \tag{1.7}$$

with $\mu(t) \equiv 1$, $F = \widetilde{u}_0 = \widetilde{u}_1 = \widetilde{Y}_0 = \gamma_1 = 0$, and $f(u, u_t) = Ku + \lambda u_t$, with γ_3, $K \geq 0$, $\lambda \geq 0$ being given constants. In the latter case, problem (1.1), (1.4)-(1.7) is a mathematical model describing the shock of a rigid body and a linear viscoelastic bar resting on a rigid base [1].

We note more that from (1.6), representing $Y(t)$ in terms of γ_1, γ_2, γ_3, $\widetilde{Y}_0$, $\widetilde{Y}_1$, $u_{tt}(0,t)$ and then integrating by parts, we shall obtain $Y(t)$ as below

$$Y(t) = \widehat{g}(t) + K_0 u(0,t) + \int_0^t \widehat{k}(t-s)\, u(0,s)\, ds,$$

where

$$\begin{aligned} \widehat{g}(t) = {} & \left(\widetilde{Y}_0 - \gamma_3 \widetilde{u}_0(0)\right) e^{-\gamma t} \cos \omega t \\ & + \left[-\widetilde{Y}_0 - \frac{1}{\gamma} \widetilde{Y}_1 + \frac{\gamma}{\omega} \gamma_3 \widetilde{u}_0(0) - \frac{1}{\omega} \gamma_3 \widetilde{u}_1(0) \right] e^{-\gamma t} \sin \omega t, \\ \widehat{k}(t) = {} & \gamma_3 \left[-2\gamma \cos \omega t + (\gamma^2 - \omega^2) \frac{\sin \omega t}{\omega} \right] e^{-\gamma t}, \end{aligned}$$

with $\gamma = \frac{1}{2}\gamma_1$, $\omega = \frac{1}{2}\sqrt{4\gamma_2 - \gamma_1^2}$, $K_0 = \gamma_3$. Therefore, problem (1.1), (1.3)-(1.5) leads to problem (1.1)-(1.4).

The paper consists of three sections. In Section 2, under the conditions $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$; $F,\ F' \in L^1(0,T;L^2)$; $g,\ k,\ \mu \in W^{2,1}(0,T)$, $\mu(t) \geq \mu_0 > 0$; $p,\ q,\ \alpha \geq 2$; $K,\ K_0,\ K_1 \geq 0$; $\lambda,\ \lambda_1 > 0$ and some other conditions, we prove a theorem of global existence and uniqueness of a weak solution u of problem (1.1)-(1.4). The proof is based on the Galerkin method associated to a priori estimates, the weak convergence and the compactness techniques. Finally, in Section 3, with $\alpha = 2$, we obtain an asymptotic expansion of the solution u up to order N in two small parameters $\lambda,\ \lambda_1$ with error $\left(\sqrt{\lambda^2+\lambda_1^2}\right)^{N+\frac{1}{2}}$. The results obtained here may be considered as the generalizations of those in An and Trieu [1] and in [2, 6, 7, 9, 10].

2. Existence and uniqueness of a weak solution

First, put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$ and denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$.

Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $||\cdot||$ stands for the norm in L^2 and we denote by $||\cdot||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that $||u||_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int\limits_0^T \|u(t)\|_X^p \, dt \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}\limits_{0<t<T} \|u(t)\|_X, & \text{if } p = \infty. \end{cases} \tag{2.1}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1 we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{1/2}. \tag{2.2}$$

Then we have the following lemma.

Lemma 2.1. [The imbedding] $H^1 \hookrightarrow C^0([0,1])$ *is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1} \ \text{ for all } v \in H^1. \tag{2.3}$$

The proof of Lemma 2.1 is straightforward, and we omit it.

Next, we make the following assumptions:

(H_1) $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$,

(H_2) $F,\ F' \in L^1(0,T;L^2)$,

(H_3) $\mu \in W^{2,1}(0,T)$, $\mu(t) \geq \mu_0 > 0$,

(H_4) $g,\ k \in W^{2,1}(0,T)$,

(H_5) $p,\ q,\ \alpha \geq 2;\ K,\ K_0,\ K_1 \geq 0;\ \lambda,\ \lambda_1 > 0$.

Finally, let us note more that the weak solution u of the initial and boundary value problem (1.1)-(1.4) will be obtained in Theorem 2.2 in the following manner:

Find $u \in \widetilde{W} = \{u \in L^\infty(0,T;H^2),\ u_t \in L^\infty(0,T;H^1),\ u_{tt} \in L^\infty(0,T;L^2)\}$, such that u satisfies the variational equation

$$\begin{aligned}&\left\langle u''(t), v\right\rangle + \mu(t)\left\langle u_x(t), v_x\right\rangle + Y(t)v(0) + \left[K_1 u(1,t) + \lambda_1 \Pi_\alpha(u'(1,t))\right] v(1)\\ &+ \left\langle K\Pi_p(u(t)) + \lambda \Pi_q(u'(t)), v\right\rangle = \left\langle F(t), v\right\rangle,\ \text{for all } v \in H^1,\ \text{a.e., } t \in (0,T),\end{aligned}$$

and the initial conditions

$$u(0) = \widetilde{u}_0,\ u_t(0) = \widetilde{u}_1,$$

where $\Pi_r(z) = |z|^{r-2}z$, $r \in \{p,\ q,\ \alpha\}$ and

$$Y(t) = g(t) + K_0 u(0,t) + \int_0^t k(t-s)\, u(0,s)\, ds.$$

Remark 2.1. If $u \in L^\infty(0,T;H^2)$ and $u_t \in L^\infty(0,T;H^1)$, then $u : [0,T] \to H^1$ is continuous ([5], Lemma 1.2, p.7), so it is clear that $u(0)$ is defined and $u(0)$ belongs to H^1. Similarly, with $u_t \in L^\infty(0,T;H^1)$ and $u_{tt} \in L^\infty(0,T;L^2)\}$, it implies that $u_t : [0,T] \to L^2$ is continuous, $u_t(0) \in L^2$ follows. In order to get the following result of existence, we need the assumptions (H_1)-(H_5), in which $\widetilde{u}_0 \in H^2,\ \widetilde{u}_1 \in H^1$. Then, $u(0) = \widetilde{u}_0 \in H^2$ and $u_t(0) = \widetilde{u}_1 \in H^1$.

Theorem 2.2. *Let* $(H_1) - (H_5)$ *hold. For every* $T > 0$, *there exists a unique weak solution* u *of problem* (1.1)-(1.4), *such that*

$$\left\{\begin{array}{l} u \in L^\infty(0,T;H^2), u_t \in L^\infty(0,T;H^1), u_{tt} \in L^\infty(0,T;L^2),\\ |u_t|^{\frac{q}{2}-1}u_t \in H^1(Q_T), |u_t(1,\cdot)|^{\frac{\alpha}{2}-1}u_t(1,\cdot) \in H^1(0,T).\end{array}\right. \tag{2.4}$$

Remark 2.2. It follows from (2.4) that problem (1.1)-(1.4) has a unique weak solution u satisfying

$$\left\{\begin{array}{l} u \in H^2(Q_T) \cap C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right),\\ u_t \in L^\infty(0,T;H^1) \cap C^0\left(0,T;L^2\right),\\ u_{tt} \in L^\infty(0,T;L^2), |u_t|^{\frac{q}{2}-1}u_t \in H^1(Q_T),\ |u_t(1,\cdot)|^{\frac{\alpha}{2}-1}u_t(1,\cdot) \in H^1(0,T).\end{array}\right. \tag{2.5}$$

Proof of Theorem 2.2. The proof consists of Steps 1–4.
Step 1. (*The Galerkin approximation*). Let $\{w_j\}$ be a denumerable base of H^2. We find the approximate solution of problem (1.1)-(1.4) in the form

$$u_m(t) = \sum_{j=1}^m c_{mj}(t) w_j, \tag{2.6}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations

$$(2.7)\qquad \begin{cases} \langle u''_m(t), w_j\rangle + \mu(t)\langle u_{mx}(t), w_{jx}\rangle + Y_m(t)w_j(0) + K_1 u_m(1,t)w_j(1) \\ +\lambda_1\Pi_\alpha(u'_m(1,t))w_j(1) + \langle K\Pi_p(u_m(t)) + \lambda\Pi_q(u'_m(t)), w_j\rangle \\ = \langle F(t), w_j\rangle,\ 1 \le j \le m, \\ u_m(0) = u_{0m},\ u'_m(0) = u_{1m}, \end{cases}$$

in which

$$(2.8)\qquad Y_m(t) = g(t) + K_0 u_m(0,t) + \int_0^t k(t-s)u_m(0,s)ds,$$

and

$$(2.9)\qquad \begin{cases} u_{0m} &= \sum_{j=1}^m \alpha_{mj}w_j \to \widetilde{u}_0 \quad \text{strongly in } H^2, \\ u_{1m} &= \sum_{j=1}^m \beta_{mj}w_j \to \widetilde{u}_1 \quad \text{strongly in } H^1. \end{cases}$$

From the assumptions of Theorem 2.2, the system (2.7)–(2.8) has a solution u_m on an interval $[0, T_m] \subset [0, T]$. The following estimates allow one to take $T_m = T$ for all m, (see [3]).

Step 2. (*A priori estimates I*). Substituting (2.8) into (2.7), then multiplying the j^{th} equation of (2.7) by $c'_{mj}(t)$ and summing with respect to j, and afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$(2.10)\qquad \begin{aligned} S_m(t) = {} & S_m(0) + 2g(0)u_{0m}(0) - 2g(t)u_m(0,t) + \int_0^t \mu'(s)||u_{mx}(s)||^2 ds \\ & + 2\int_0^t g'(s)u_m(0,s)ds + 2\int_0^t \left\langle F(s), u'_m(s)\right\rangle ds \\ & - 2u_m(0,t)\int_0^t k(t-r)u_m(0,r)dr + 2k(0)\int_0^t u_m^2(0,s)ds \\ & + 2\int_0^t u_m(0,s)ds\int_0^s k'(s-r)u_m(0,r)dr \\ = {} & S_m(0) + 2g(0)u_{0m}(0) + \sum_{j=1}^7 I_j, \end{aligned}$$

where

$$S_m(t) = \left\|u_m'(t)\right\|^2 + \mu(t)||u_{mx}(t)||^2 + \frac{2K}{p}\left\|u_m(t)\right\|_{L^p}^p + 2\lambda\int_0^t \left\|u_m'(s)\right\|_{L^q}^q ds$$

$$+ 2\lambda_1 \int_0^t \left|u_m'(1,s)\right|^\alpha ds + K_0 u_m^2(0,t) + K_1 u_m^2(1,t). \tag{2.11}$$

By the assumptions (H_1), (H_3)-(H_5), and the imbedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$, we deduce from (2.9) that

$$S_m(0) + 2g(0)u_{0m}(0) + \left\|u_{0m}\right\|_{H^1}^2 \le C_0, \tag{2.12}$$

for all m, where C_0 is a constant depending only on p, K, K_0, K_1, $\mu(0)$, $g(0)$, $\widetilde{u}_0$, $\widetilde{u}_1$.

On the other hand, we deduce from (2.2), (2.11) and (2.12) that

$$\begin{cases} S_m(t) \ge \left\|u_m'(t)\right\|^2 + \mu_0\left\|u_{mx}(t)\right\|^2, \\ \left\|u_m(t)\right\|^2 = \left\|u_m(0) + \int_0^t u_m'(s)ds\right\|^2 \le 2C_0 + 2t\int_0^t S_m(s)ds, \\ \left\|u_m(t)\right\|_{H^1}^2 \le \frac{1}{\mu_0}S_m(t) + 2C_0 + 2t\int_0^t S_m(s)ds, \\ \int_0^t \left\|u_m(s)\right\|_{H^1}^2 ds \le 2TC_0 + \left(\frac{1}{\mu_0} + T^2\right)\int_0^t S_m(s)ds. \end{cases} \tag{2.13}$$

Using the assumptions (H_1)-(H_5) and (2.3), (2.13) and the following inequality

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \text{ for all } \varepsilon > 0,\ a,\ b \ge 0, \tag{2.14}$$

we shall estimate respectively the terms I_j, $j = 1, ..., 7$ on the right-hand side of (2.10) as follows

$$\begin{aligned} I_1 = \int_0^t \mu'(s)\left\|u_{mx}(s)\right\|^2 ds &\le \frac{1}{\mu_0}\left\|\mu'\right\|_{L^\infty(0,T)}\int_0^t S_m(s)ds \\ &\le C_T\int_0^t S_m(s)ds, \end{aligned} \tag{2.15}$$

where C_T always indicates a bound depending on T,

$$I_2 = -2g(t)u_m(0,t) \le \varepsilon \|u_m(t)\|_{H^1}^2 + \frac{2}{\varepsilon}\|g\|_{C^0([0,T])}^2$$
$$\le \frac{\varepsilon}{\mu_0}S_m(t) + \varepsilon C_T \int_0^t S_m(s)ds + (\varepsilon + \frac{1}{\varepsilon})C_T, \text{ for all } \varepsilon > 0, \tag{2.16}$$

$$I_3 = 2\int_0^t g'(s)u_m(0,s)ds \le 2\|g'\|_{L^\infty(0,T)}^2 + \int_0^t \|u_m(s)\|_{H^1}^2 ds$$
$$\le C_T\left[1 + \int_0^t S_m(s)ds\right], \tag{2.17}$$

$$I_4 = 2\int_0^t \left\langle F(s), u_m'(s)\right\rangle ds \le \int_0^t \|F(s)\| ds + \int_0^t \|F(s)\| \left\|u_m'(s)\right\|^2 ds$$
$$\le C_T + \int_0^t \|F(s)\| S_m(s)ds, \tag{2.18}$$

$$I_5 = -2u_m(0,t)\int_0^t k(t-r)u_m(0,r)dr$$
$$\le \varepsilon \|u_m(t)\|_{H^1}^2 + \frac{4}{\varepsilon}\|k\|_{L^2(0,T)}^2 \int_0^t \|u_m(r)\|_{H^1}^2 dr \tag{2.19}$$
$$\le \frac{\varepsilon}{\mu_0}S_m(t) + (\varepsilon + \frac{1}{\varepsilon})C_T\left[1 + \int_0^t S_m(s)ds\right], \text{for all } \varepsilon > 0,$$

$$I_6 = 2k(0)\int_0^t u_m^2(0,s)ds \le 4k(0)\int_0^t \|u_m(s)\|_{H^1}^2 ds$$
$$\le C_T\left[1 + \int_0^t S_m(s)ds\right], \tag{2.20}$$

$$I_7 = 2\int_0^t u_m(0,s)ds\int_0^s k'(s-r)u_m(0,r)dr$$
$$\le 4\sqrt{T}\left\|k'\right\|_{L^2(0,T)}\int_0^t \|u_m(s)\|_{H^1}^2 ds \le C_T\left[1 + \int_0^t S_m(s)ds\right]. \tag{2.21}$$

Combining (2.10), (2.12) and (2.15)-(2.21), we obtain after some rearrangements

$$S_m(t) \leq \frac{2\varepsilon}{\mu_0} S_m(t) + \frac{1}{2} k_{1T}(\varepsilon) + \frac{1}{2} \int_0^t q_{1T}(\varepsilon, s) S_m(s) ds, \tag{2.22}$$

for all $\varepsilon > 0$, where

$$\begin{cases} \frac{1}{2} k_{1T}(\varepsilon) & = C_0 + 2(\varepsilon + \frac{1}{\varepsilon} + 2) C_T, \\ \frac{1}{2} q_{1T}(\varepsilon, t) & = (2\varepsilon + \frac{1}{\varepsilon} + 4) C_T + \|F(t)\|, \quad q_{1T}(\varepsilon, \cdot) \in L^1(0,T). \end{cases} \tag{2.23}$$

Choosing $\varepsilon > 0$, with $\frac{2\varepsilon}{\mu_0} \leq \frac{1}{2}$, by Gronwall's lemma, we deduce from (2.22) that

$$S_m(t) \leq k_{1T}(\varepsilon) \exp\left(\int_0^t q_{1T}(\varepsilon, s) ds \right) \leq C_T, \ \forall m \in \mathbb{N}, \ \forall t \in [0,T], \ \forall T > 0. \tag{2.24}$$

(*A priori estimates II*). Now differentiating $(2.7)_1$ with respect to t, we have

$$\begin{aligned} &\left\langle u_m'''(t), w_j \right\rangle + \mu(t) \left\langle u_{mx}'(t), w_{jx} \right\rangle + \mu'(t) \left\langle u_{mx}(t), w_{jx} \right\rangle + Y_m'(t) w_j(0) \\ &+ K_1 u_m'(1,t) w_j(1) + \lambda_1 \Pi_\alpha'(u_m'(1,t)) u_m''(1,t) w_j(1) \\ &+ K \left\langle \Pi_p'(u_m(t)) u_m'(t), w_j \right\rangle + \lambda \left\langle \Pi_q'(u_m'(t)) u_m''(t), w_j \right\rangle = \left\langle F'(t), w_j \right\rangle, \end{aligned} \tag{2.25}$$

for all $1 \leq j \leq m$.

Multiplying the j^{th} equation of (2.25) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we have after some rearrangements

$$\begin{aligned} X_m(t) = X_m(0) &+ 2\mu'(0) \left\langle u_{0mx}, u_{1mx} \right\rangle \\ &+ 3 \int_0^t \mu'(s) ||u_{mx}'(s)||^2 ds - 2\mu'(t) \left\langle u_{mx}(t), u_{mx}'(t) \right\rangle \\ &+ 2 \int_0^t \mu''(s) \left\langle u_{mx}(s), u_{mx}'(s) \right\rangle ds - 2K \int_0^t \left\langle \Pi_p'(u_m(s)) u_m'(s), u_m''(s) \right\rangle ds \\ &+ 2 \int_0^t \left\langle F'(s), u_m''(s) \right\rangle ds - 2 \int_0^t g'(s) u_m''(0,s) ds \\ &- 2k(0) \int_0^t u_m(0,s) u_m''(0,s) ds - 2 \int_0^t u_m''(0,s) ds \int_0^s k'(s-r) u_m(0,r) dr \\ = X_m(0) &+ 2 \left\langle \mu'(0) u_{0mx}, u_{1mx} \right\rangle + \sum_{j=1}^{8} J_j, \end{aligned} \tag{2.26}$$

where

$$X_m(t) = \left\|u_m''(t)\right\|^2 + \mu(t)\left\|u_{mx}'(t)\right\|^2 + K_0\left|u_m'(0,t)\right|^2 + K_1\left|u_m'(1,t)\right|^2 + \frac{8\lambda_1(\alpha-1)}{\alpha^2}\int_0^t \left|\frac{\partial}{\partial s}\left(|u_m'(1,s)|^{\frac{\alpha}{2}-1}u_m'(1,s)\right)\right|^2 ds + \frac{8\lambda(q-1)}{q^2}\int_0^t \left\|\frac{\partial}{\partial s}\left(|u_m'(s)|^{\frac{q}{2}-1}u_m'(s)\right)\right\|^2 ds. \tag{2.27}$$

By the assumptions (H_1)-(H_3), (H_5), and the imbedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$, we deduce from (2.9) that

$$X_m(0) + 2\mu'(0)\left\langle u_{0mx}, u_{1mx}\right\rangle \le C_0, \tag{2.28}$$

for all m, where C_0 is a constant depending only on p, q, K, K_0, K_1 λ, μ, $F(0)$, $\widetilde{u}_0$, $\widetilde{u}_1$.

On the other hand, (2.2), (2.3), (2.27) and (2.28) yield

$$\begin{cases} X_m(t) \ge \left\|u_m''(t)\right\|^2 + \mu_0\|u_{mx}'(t)\|^2, \\ \left\|u_m'(t)\right\|^2 \le 2\left\|u_{1m}\right\|^2 + 2t\int_0^t X_m(s)ds \le 2C_0 + 2t\int_0^t X_m(s)ds, \\ \left\|u_m'(t)\right\|_{H^1}^2 \le \frac{1}{\mu_0}X_m(t) + 2C_0 + 2t\int_0^t X_m(s)ds, \\ \int_0^t \left\|u_m'(s)\right\|_{H^1}^2 ds \le C_T\left[1 + \int_0^t X_m(s)ds\right], \end{cases} \tag{2.29}$$

where C_T always indicates a bound depending on T.

We again use the inequalities (2.3), (2.14), (2.29) and by (H_1)-(H_5), we shall estimate the terms J_j, $1 \le j \le 8$ on the right-hand side of (2.26) as follows

$$J_1 = 3\int_0^t \mu'(s)||u_{mx}'(s)||^2 ds \le \frac{3}{\mu_0}\left\|\mu'\right\|_{L^\infty(0,T)}\int_0^t X_m(s)ds \le C_T\int_0^t X_m(s)ds, \tag{2.30}$$

where C_T always indicates a bound depending on T,

$$J_2 = -2\mu'(t)\left\langle u_{mx}(t), u_{mx}'(t)\right\rangle \le 2\left\|\mu'\right\|_{L^\infty(0,T)}\left\|u_{mx}(t)\right\|\left\|u_{mx}'(t)\right\| \le 2\left\|\mu'\right\|_{L^\infty(0,T)} C_T\sqrt{\frac{1}{\mu_0}X_m(t)} \le \varepsilon X_m(t) + \frac{1}{\varepsilon}C_T, \tag{2.31}$$

for all $\varepsilon > 0$,

$$J_3 = 2\int_0^t \mu''(s)\left\langle u_{mx}(s), u'_{mx}(s)\right\rangle ds \le 2\int_0^t \left|\mu''(s)\right| \|u_{mx}(s)\| \left\|u'_{mx}(s)\right\| ds \tag{2.32}$$

$$\le 2C_T \int_0^t \left|\mu''(s)\right| \sqrt{\frac{1}{\mu_0} X_m(s)} ds \le C_T + \int_0^t \left|\mu''(s)\right| X_m(s) ds,$$

$$J_4 = -2K \int_0^t \left\langle \Pi'_p(u_m(s))u'_m(s), u''_m(s)\right\rangle ds \tag{2.33}$$

$$\le 2K(p-1)\left(\sqrt{2}\,\|u_m\|_{L^\infty(0,T;H^1)}\right)^{p-2} \int_0^t \sqrt{S_m(s)}\sqrt{X_m(s)} ds$$

$$\le C_T + \int_0^t X_m(s) ds,$$

$$J_5 = 2\int_0^t \left\langle F'(s), u''_m(s)\right\rangle ds \le \int_0^T \left\|F'(s)\right\| ds + \int_0^t \left\|F'(s)\right\| \left\|u''_m(s)\right\|^2 ds \tag{2.34}$$

$$\le C_T + \int_0^t \left\|F'(s)\right\| X_m(s) ds,$$

$$J_6 = -2\int_0^t g'(s) u''_m(0,s) ds$$

$$= 2g'(0)u_{1m}(0) - 2g'(t)u'_m(0,t) + 2\int_0^t g''(s)u'_m(0,s) ds$$

$$\le 2\left|g'(0)u_{1m}(0)\right| + \frac{2}{\varepsilon}\left\|g'\right\|^2_{C^0([0,T])} + \varepsilon\left\|u'_m(t)\right\|^2_{H^1} \tag{2.35}$$

$$+ 2\int_0^T \left|g''(s)\right| ds + \int_0^t \left|g''(s)\right| \left\|u'_m(s)\right\|^2_{H^1} ds$$

$$\le C_0 + (\varepsilon + \frac{1}{\varepsilon})C_T + \frac{\varepsilon}{\mu_0} X_m(t) + \int_0^t \left(C_T + \varepsilon C_T + \frac{1}{\mu_0}\left|g''(s)\right|\right) X_m(s) ds,$$

for all $\varepsilon > 0$,

$$
\begin{aligned}
J_7 &= -2k(0)\int_0^t u_m(0,s)u_m''(0,s)ds \\
&= 2k(0)u_{0m}(0)u_{1m}(0) - 2k(0)u_m(0,t)u_m'(0,t) + 2k(0)\int_0^t \left|u_m'(0,s)\right|^2 ds \\
&\leq C_0 + \frac{1}{\varepsilon}C_T + \varepsilon\left\|u_m'(t)\right\|_{H^1}^2 + 4\left|k(0)\right|\int_0^t \left\|u_m'(s)\right\|_{H^1}^2 ds \\
&\leq C_0 + (1+\varepsilon+\frac{1}{\varepsilon})C_T + \frac{\varepsilon}{\mu_0}X_m(t) + C_T\int_0^t X_m(s)ds \ \forall\,\varepsilon > 0, \qquad (2.36)
\end{aligned}
$$

(2.37)

$$
\begin{aligned}
J_8 =& -2\int_0^t u_m''(0,s)ds\int_0^s k'(s-r)u_m(0,r)dr \\
=& -2u_m'(0,t)\int_0^t k'(t-r)u_m(0,r)dr + 2k'(0)\int_0^t \left|u_m'(0,s)\right|^2 ds \\
&+ 2\int_0^t u_m'(0,s)ds\int_0^s k''(s-r)u_m(0,r)dr \\
\leq&\ 4\left\|u_m'(t)\right\|_{H^1}\int_0^t \left|k'(t-r)\right|\left\|u_m(r)\right\|_{H^1} dr + 4\left|k'(0)\right|\int_0^t \left\|u_m'(s)\right\|_{H^1}\left\|u_m(s)\right\|_{H^1} ds \\
&+ 4\int_0^t \left\|u_m'(s)\right\|_{H^1} ds\int_0^s \left|k''(s-r)\right|\left\|u_m(r)\right\|_{H^1} dr \\
\leq&\ 4C_T\left\|k'\right\|_{L^1(0,T)}\left\|u_m'(t)\right\|_{H^1} + 4C_T\left(\left|k'(0)\right| + \left\|k''\right\|_{L^1(0,T)}\right)\int_0^t \left\|u_m'(s)\right\|_{H^1} ds \\
\leq&\ \frac{1}{\varepsilon}C_T + \varepsilon\left\|u_m'(t)\right\|_{H^1}^2 + C_T + \int_0^t \left\|u_m'(s)\right\|_{H^1}^2 ds \\
\leq&\ (1+\varepsilon+\frac{1}{\varepsilon})C_T + \frac{\varepsilon}{\mu_0}X_m(t) + C_T\int_0^t X_m(s)ds \ \ \forall\ \varepsilon > 0.
\end{aligned}
$$

Combining (2.26), (2.28), (2.30) – (2.37), we obtain

$$X_m(t) \leq \varepsilon\left(1+\frac{3}{\mu_0}\right) X_m(t)+\frac{1}{2}k_{2T}(\varepsilon)+\frac{1}{2}\int_0^t q_{2T}(\varepsilon,s)X_m(s)ds, \tag{2.38}$$

for all $\varepsilon > 0$, where

$$\begin{cases} \frac{1}{2}k_{2T}(\varepsilon)=3C_0+5(1+\varepsilon+\frac{1}{\varepsilon})C_T, \\ \frac{1}{2}q_{2T}(\varepsilon,t)=1+(2+\varepsilon)\,C_T+\frac{1}{\mu_0}\left|g''(t)\right|+\left|\mu''(t)\right|+\left\|F'(t)\right\|, \\ \qquad\qquad q_{2T}(\varepsilon,\cdot)\in L^1\left(0,T\right). \end{cases} \tag{2.39}$$

Choosing $\varepsilon > 0$, with $\varepsilon\left(1+\frac{3}{\mu_0}\right) \leq \frac{1}{2}$, by Gronwall's lemma, it follows from (2.38) that
(2.40)

$$X_m(t) \leq k_{2T}(\varepsilon)\exp\left(\int_0^t q_{2T}(\varepsilon,s)ds\right) \leq C_T, \quad \forall m\in\mathbb{N},\ \forall t\in[0,T], \quad \forall T>0.$$

On the other hand, by (H_1), (H_3)-(H_5), combining (2.2), (2.3), (2.9), (2.11), (2.24), (2.27) and (2.40), we conclude that

$$\begin{cases} \left\|u_m(t)\right\|_{H^1}^2 \leq \frac{1}{\mu_0}S_m(t)+2\left\|u_{0m}\right\|^2+2t\int_0^t S_m(s)ds \leq C_T, \\ \left\|u_m'(t)\right\|_{H^1}^2 \leq \frac{1}{\mu_0}X_m(t)+2\left\|u_{1m}\right\|^2+2t\int_0^t X_m(s)ds \leq C_T, \end{cases} \tag{2.41}$$

$$\begin{cases} \left\|\Pi_p(u_m)\right\|_{H^1(Q_T)} \leq C_T, \\ \left\|\Pi_q(u_m')\right\|_{H^1(Q_T)} \leq C_T, \\ \left\|\Pi_\alpha(u_m'(1,\cdot))\right\|_{L^{\alpha'}(0,T)}^{\alpha'} = \int_0^T \left|u_m'(1,s)\right|^\alpha ds \leq C_T, \\ \left\||u_m(0,\cdot)\right\|_{W^{1,\infty}(0,T)} \leq C_T, \\ \left\||u_m(1,\cdot)\right\|_{W^{1,\infty}(0,T)} \leq C_T, \\ \left\|\left|u_m'(1,\cdot)\right|^{\frac{\alpha}{2}-1}u_m'(1,\cdot)\right\|_{H^1(0,T)} \leq C_T. \end{cases} \tag{2.42}$$

Step 3. (*Limiting process*). From (2.11), (2.24), (2.27), (2.40)-(2.42), we deduce the existence of a subsequence of $\{u_m\}$, denoted by the same symbol such that

$$(2.43)\qquad \begin{cases} u_m \to u & \text{in } L^\infty(0,T;H^1) & \text{weakly*}, \\ u'_m \to u' & \text{in } L^\infty(0,T;H^1) & \text{weakly*}, \\ u''_m \to u'' & \text{in } L^\infty(0,T;L^2) & \text{weakly*}, \\ \Pi_p(u_m) \to \widetilde{\Pi}_p & \text{in } H^1(Q_T) & \text{weakly}, \\ \Pi_q(u'_m) \to \widetilde{\Pi}_q & \text{in } H^1(Q_T) & \text{weakly}, \\ |u'_m(1,\cdot)|^{\frac{\alpha}{2}-1}u'_m(1,\cdot) \to \rho_1^* & \text{in } H^1(0,T) & \text{weakly}, \\ u_m(0,\cdot) \to u(0,\cdot) & \text{in } W^{1,\infty}(0,T) & \text{weakly*}, \\ u_m(1,\cdot) \to u(1,\cdot) & \text{in } W^{1,\infty}(0,T) & \text{weakly*}, \\ u'_m(1,\cdot) \to u'(1,\cdot) & \text{in } L^\infty(0,T) & \text{weakly*}. \end{cases}$$

By the Aubin-Lions lemma ([5, p.57]) and the imbeddings $H^1(Q_T) \hookrightarrow L^2(Q_T)$, $H^1(0,T) \hookrightarrow C^0([0,T])$, $W^{1,\alpha}(0,T) \hookrightarrow C^0([0,T])$, we can deduce from $(2.43)_{1-8}$ the existence of a subsequence still denoted by $\{u_m\}$, such that

$$(2.44)\qquad \begin{cases} u_m \to u & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ u'_m \to u' & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \Pi_p(u_m) \to \widetilde{\Pi}_p & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \Pi_q(u'_m) \to \widetilde{\Pi}_q & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ u_m(0,\cdot) \to u(0,\cdot) & \text{strongly in } C^0([0,T]), \\ u_m(1,\cdot) \to u(1,\cdot) & \text{strongly in } C^0([0,T]), \\ |u'_m(1,\cdot)|^{\frac{\alpha}{2}-1}u'_m(1,\cdot) \to \rho_1^* & \text{strongly in } C^0([0,T]). \end{cases}$$

By means of the continuity of Π_p, we have

$$(2.45)\qquad \Pi_p(u_m) \to \Pi_p(u) \text{ a.e. } (x,t) \text{ in } Q_T.$$

By $\|\Pi_p(u_m)\|^{p'}_{L^{p'}(Q_T)} = \|u_m\|^p_{L^p(Q_T)} \le T\left(\sqrt{2}\,\|u_m\|_{L^\infty(0,T;H^1)}\right)^p \le C_T$, we deduce from (2.45) and Lions' lemma ([5, Lemma 1.3, p.12]) that

$$(2.46)\qquad \Pi_p(u_m) \to \Pi_p(u) \text{ in } L^{p'}(Q_T) \text{ weakly.}$$

Hence it follows from $(2.44)_3$ and (2.46) that

$$(2.47)\qquad \widetilde{\Pi}_p = \Pi_p(u) \text{ a.e. } (x,t) \text{ in } Q_T.$$

Similarly

$$(2.48)\qquad \widetilde{\Pi}_q = \Pi_q(u') \text{ a.e. } (x,t) \text{ in } Q_T.$$

On the other hand, it follows from (2.8) and $(2.44)_5$ that

(2.49)

$$Y_m(t) \to g(t) + K_0 u(0,t) + \int_0^t k(t-s)u(0,s)ds \equiv Y(t) \text{ strongly in } C^0([0,T]).$$

Lemma 2.3.

$$(2.50)\qquad \Pi_\alpha(u'_m(1,\cdot)) \to \Pi_\alpha(u'(1,\cdot)) \textit{ strongly in } C^0([0,T]).$$

Proof. Put $\rho_m = |u'_m(1,\cdot)|^{\frac{\alpha}{2}-1}u'_m(1,\cdot)$, it follows from $(2.44)_7$, that

$$\rho_m \to \rho_1^* \text{ strongly in } C^0([0,T]). \tag{2.51}$$

Hence

$$u'_m(1,\cdot) = |\rho_m|^{\frac{2}{\alpha}-1}\rho_m \to |\rho_1^*|^{\frac{2}{\alpha}-1}\rho_1^* \text{ strongly in } C^0([0,T]). \tag{2.52}$$

From $(2.43)_9$, we obtain

$$u'(1,\cdot) = |\rho_1^*|^{\frac{2}{\alpha}-1}\rho_1^*. \tag{2.53}$$

Hence it follows from (2.52), (2.53) that

$$\Pi_\alpha(u'_m(1,\cdot)) \to \Pi_\alpha(u'(1,\cdot)) \text{ strongly in } C^0([0,T]). \tag{2.54}$$

The proof of Lemma 2.3 is complete. □

We now continue the proof of Theorem 2.2.

Passing to the limit in (2.7)-(2.9) by $(2.43)_{1,3}$, $(2.44)_{3,4}$, (2.47)-(2.50), we obtain that u satisfies the system

(2.55)
$$\begin{cases} \langle u''(t), v\rangle + \mu(t)\langle u_x(t), v_x\rangle + Y(t)v(0) + K_1u(1,t)v(1) + \lambda_1\Pi_\alpha(u'(1,t))v(1) \\ \qquad + \langle K\Pi_p(u(t)) + \lambda\Pi_q(u'(t)), v\rangle = \langle F(t), v\rangle, \text{ for all } v \in H^1, \\ Y(t) = g(t) + K_0u(0,t) + \int_0^t k(t-s)u(0,s)ds. \end{cases}$$

Obviously, $u(0) = \widetilde{u}_0$ and $u'(0) = \widetilde{u}_1$. Indeed, $\forall w \in L^2$, we have

$$T\langle u(0), w\rangle = \int_0^T \langle u(t), w\rangle\, dt - \int_0^T \langle u'(t), (T-t)w\rangle\, dt,$$

$$T\langle u_m(0), w\rangle = \int_0^T \langle u_m(t), w\rangle\, dt - \int_0^T \langle u'_m(t), (T-t)w\rangle\, dt.$$

By $(2.9)_1$, $(2.43)_{1,2}$, we get

$$T\langle \widetilde{u}_0, w\rangle = \int_0^T \langle u(t), w\rangle\, dt - \int_0^T \langle u'(t), (T-t)w\rangle\, dt.$$

Consequently,

$$\langle \widetilde{u}_0, w\rangle = \langle u(0), w\rangle \quad \forall w \in L^2.$$

Hence $u(0) = \widetilde{u}_0$.

Similarly, it follows from

$$T\langle u'(0), w\rangle = \int_0^T \langle u'(t), w\rangle\, dt - \int_0^T \langle u''(t), (T-t)w\rangle\, dt, \quad \forall w \in L^2,$$

$$T\left\langle u_m'(0),w\right\rangle=\int_0^T\left\langle u_m'(t),w\right\rangle dt-\int_0^T\left\langle u_m''(t),(T-t)w\right\rangle dt,\quad \forall w\in L^2,$$

and $(2.9)_2$, $(2.43)_{2,3}$ that

$$T\left\langle \widetilde{u}_1,w\right\rangle=\int_0^T\left\langle u'(t),w\right\rangle dt-\int_0^T\left\langle u''(t),(T-t)w\right\rangle dt=T\left\langle u'(0),w\right\rangle,\quad \forall w\in L^2.$$

So $u'(0)=\widetilde{u}_1$.

On the other hand, we have from $(2.43)_{1-5}$, $(2.55)_1$ and the assumptions (H_2), (H_3), and (H_5) that

$$u_{xx}=\frac{1}{\mu(t)}\left(u''+K\Pi_p(u(t))+\lambda\Pi_q(u'(t))-F\ \right)\in L^\infty\left(0,T;L^2\right). \tag{2.56}$$

So $u\in L^\infty\left(0,T;H^2\right)$. The proof of the existence is complete.

Step 4. (*Uniqueness of the solution*). Let u_1, u_2 be two weak solutions of problem (1.1)-(1.4) such that

$$\begin{cases}u_i\in L^\infty(0,T;H^2),\ u_i'\in L^\infty(0,T;H^1),\ u_i''\in L^\infty(0,T;L^2),\\ |u_i'|^{\frac{q}{2}-1}u_i'\in H^1\left(Q_T\right),\ |u_i'(1,\cdot)|^{\frac{\alpha}{2}-1}u_i'(1,\cdot)\in H^1\left(0,T\right).\end{cases} \tag{2.57}$$

Then $u=u_1-u_2$ satisfies the variational problem

(2.58)

$$\begin{cases}\langle u''(t),v\rangle+\mu(t)\left\langle u_x(t),v_x\right\rangle+K_0u(0,t)v(0)+K_1u(1,t)v(1)\\ \qquad +\left(\int_0^t k(t-s)u(0,s)ds\right)v(0)+\lambda_1\left[\Pi_\alpha(u_1'(1,t))-\Pi_\alpha(u_2'(1,t))\right]v(1)\\ \qquad +K\left\langle \Pi_p(u_1(t))-\Pi_p(u_2(t)),v\right\rangle+\lambda\left\langle \Pi_q(u_1'(t))-\Pi_q(u_2'(t)),v\right\rangle=0,\\ \text{ for all } v\in H^1,\\ u(0)=u'(0)=0.\end{cases}$$

Taking $v=u'$ in $(2.58)_1$ and integrating with respect to t, we get

$$\begin{aligned}\sigma(t)=&\int_0^t\mu'(s)\left\|u_x(s)\right\|^2ds-2K\int_0^t\left\langle \Pi_p(u_1(s))-\Pi_p(u_2(s)),u'(s)\right\rangle ds\\ &+2k(0)\int_0^t u^2(0,s)ds-2u(0,t)\int_0^t k(t-r)u(0,r)dr\\ &+2\int_0^t u(0,s)ds\int_0^s k'(s-r)u(0,r)dr\\ \equiv&\sum_{i=1}^5\sigma_i,\end{aligned} \tag{2.59}$$

where

(2.60)

$$\sigma(t) = \left\|u'(t)\right\|^2 + \mu(t)\left\|u_x(t)\right\|^2 + K_0 u^2(0,t) + K_1 u^2(1,t)$$
$$+ 2\lambda_1 \int_0^t \left[\Pi_\alpha(u_1'(1,s)) - \Pi_\alpha(u_2'(1,s))\right] u'(1,s)ds$$
$$+ 2\lambda \int_0^t \left\langle \Pi_q(u_1'(s)) - \Pi_q(u_2'(s)), u'(s)\right\rangle ds \geq \left\|u'(t)\right\|^2 + \mu_0 \left\|u_x(t)\right\|^2.$$

We again use the inequalities (2.3) and (2.14), and the following inequalities

$$\begin{cases} \|u(t)\|^2 = \left\|\int_0^t u'(s)ds\right\|^2 \leq t\int_0^t \|u'(s)\|^2 ds \leq t\int_0^t \sigma(s)ds, \\ \|u(t)\|_{H^1}^2 \leq \frac{1}{\mu_0}\sigma(t) + t\int_0^t \sigma(s)ds, \\ \int_0^t \|u(s)\|_{H^1}^2 ds \leq \left(\frac{1}{\mu_0} + \frac{1}{2}T^2\right)\int_0^t \sigma(s)ds, \end{cases} \tag{2.61}$$

to respectively estimate the terms on the right-hand side of (2.59) as follows

$$\sigma_1 = \int_0^t \mu'(s)\left\|u_x(s)\right\|^2 ds \leq \frac{1}{\mu_0}\left\|\mu'\right\|_{L^\infty(0,T)} \int_0^t \sigma(s)ds \equiv \widehat{\sigma}_1 \int_0^t \sigma(s)ds, \tag{2.62}$$

$$\sigma_2 = -2K\int_0^t \left\langle \Pi_p(u_1(s)) - \Pi_p(u_2(s)), u'(s)\right\rangle ds \tag{2.63}$$
$$\leq 2K(p-1)R^{p-2}\int_0^t \|u(s)\|\left\|u'(s)\right\| ds \leq \widehat{\sigma}_2 \int_0^t \sigma(s)ds,$$

where $R = \sqrt{2}\max_{i=1,2}\|u_i\|_{L^\infty(0,T;H^1)}$,

$$\sigma_3 = 2k(0)\int_0^t u^2(0,s)ds \leq \widehat{\sigma}_3 \int_0^t \sigma(s)ds, \tag{2.64}$$

$$\sigma_4 = -2u(0,t)\int_0^t k(t-r)u(0,r)dr \tag{2.65}$$
$$\leq \varepsilon\left\|u(t)\right\|_{H^1}^2 + \frac{4}{\varepsilon}\left\|k\right\|_{L^2(0,T)}^2 \int_0^t \|u(r)\|_{H^1}^2 dr$$

$$\leq \frac{\varepsilon}{\mu_0}\sigma(t) + \widehat{\sigma}_4(\varepsilon)\int_0^t \sigma(s)ds,$$

$$\sigma_5 = 2\int_0^t u(0,s)ds\int_0^s k'(s-r)u(0,r)dr \tag{2.66}$$
$$\leq 4\int_0^t \|u(s)\|_{H^1} ds\int_0^s |k'(s-r)|\,\|u(r)\|_{H^1} dr$$
$$\leq 4\int_0^t \|u(s)\|_{H^1}^2 ds + T\left\|k'\right\|_{L^2(0,T)}^2\int_0^t \|u(r)\|_{H^1}^2 dr \leq \widehat{\sigma}_5\int_0^t \sigma(s)ds.$$

Combining (2.59) and (2.62)-(2.66), we obtain

$$\sigma(t) \leq \frac{\varepsilon}{\mu_0}\sigma(t) + (\widehat{\sigma}_1 + \widehat{\sigma}_2 + \widehat{\sigma}_3 + \widehat{\sigma}_4(\varepsilon) + \widehat{\sigma}_5)\int_0^t \sigma(s)ds. \tag{2.67}$$

Choosing $\varepsilon > 0$, with $\frac{\varepsilon}{\mu_0} \leq \frac{1}{2}$, using Gronwall's lemma, we obtain that $\sigma(t) \equiv 0$, i.e., $u_1 \equiv u_2$.

Theorem 2.2 is completely proved. □

3. An asymptotic expansion of the weak solution with respect to two small parameters

In this part, we assume that $\alpha = 2$, and

(H_6) $K,\ K_0,\ K_1 \geq 0$, $p > N+2$, $q > N+1$, $N \geq 1$.

and ($\widetilde{u}_0$, $\widetilde{u}_1$, F, μ, g, k, K, K_0, K_1) satisfy the assumptions (H_1)-(H_4), (H_6). Let $\lambda,\ \lambda_1 > 0$. By Theorem 2.2, problem (1.1)-(1.4) has a unique weak solution u depending on $\overrightarrow{\lambda} = (\lambda,\ \lambda_1)$:

$$u = u_{\overrightarrow{\lambda}} = u(\lambda,\ \lambda_1).$$

We consider the following perturbed problem

$$\left(P_{\overrightarrow{\lambda}}\right)\begin{cases} u_{tt} - \mu(t)u_{xx} + K\Pi_p(u) + \lambda\Pi_q(u_t) = F(x,t),\ 0 < x < 1,\ 0 < t < T,\\ \mu(t)u_x(0,t) = K_0u(0,t) + \int_0^t k\left(t-s\right)u\left(0,s\right)ds + g(t),\\ -\mu(t)u_x(1,t) = K_1u(1,t) + \lambda_1u_t(1,t),\\ u(x,0) = \widetilde{u}_0(x),\ \ u_t(x,0) = \widetilde{u}_1(x), \end{cases}$$

where $\lambda,\ \lambda_1$ are small parameters such that $\lambda,\ \lambda_1 > 0$, $\sqrt{\lambda^2 + \lambda_1^2} < 1$.

First, we note that if the small parameters $\lambda,\ \lambda_1>0$ satisfy $\sqrt{\lambda^2+\lambda_1^2}<1$, then a priori estimates of the sequence $\{u_m\}$ in the proof of Theorem 2.2 for problem $\left(P_{\overrightarrow{\lambda}}\right)$ satisfy

$$(3.1)\quad \begin{cases} \left\|u_m'(t)\right\|^2+\mu_0\|u_{mx}(t)\|^2+K\left\|u_m(t)\right\|_{L^p}^p+K_0u_m^2(0,t) \\ \quad +K_1u_m^2(1,t)+\lambda\int\limits_0^t\left\|u_m'(s)\right\|_{L^q}^q ds+\lambda_1\int\limits_0^t\left|u_m'(1,s)\right|^2 ds\le C_T, \\ \left\|u_m''(t)\right\|^2+\mu_0\|u_{mx}'(t)\|^2+K_0\left|u_m'(0,t)\right|^2+K_1\left|u_m'(1,t)\right|^2 \\ \quad +\lambda_1\int\limits_0^t\left|u_m''(1,s)\right|^2 ds+\lambda\int\limits_0^t\left\|\frac{\partial}{\partial s}\left(|u_m'(s)|^{\frac{q}{2}-1}u_m'(s)\right)\right\|^2 ds\le C_T, \end{cases}$$

where C_T is a constant depending only on T, $\widetilde{u}_0$, $\widetilde{u}_1$, F, μ, g, k (independent of $\lambda,\ \lambda_1$). Hence the limit $u=u_{\overrightarrow{\lambda}}=u(\lambda,\ \lambda_1)$ of the sequence $\{u_m\}$ as $m\to+\infty$ in suitable function spaces is a unique weak solution of problem $\left(P_{\overrightarrow{\lambda}}\right)$ satisfying

$$(3.2)\quad \begin{cases} \left\|u_{\overrightarrow{\lambda}}'(t)\right\|^2+\mu_0\|u_{\overrightarrow{\lambda}x}(t)\|^2+K\left\|u_{\overrightarrow{\lambda}}(t)\right\|_{L^p}^p+K_0u_{\overrightarrow{\lambda}}^2(0,t) \\ \quad +K_1u_{\overrightarrow{\lambda}}^2(1,t)+\lambda\int\limits_0^t\left\|u_{\overrightarrow{\lambda}}'(s)\right\|_{L^q}^q ds+\lambda_1\int\limits_0^t\left|u_{\overrightarrow{\lambda}}'(1,s)\right|^2 ds\le C_T, \\ \left\|u_{\overrightarrow{\lambda}}''(t)\right\|^2+\mu_0\|u_{\overrightarrow{\lambda}x}'(t)\|^2+K_0\left|u_{\overrightarrow{\lambda}}'(0,t)\right|^2+K_1\left|u_{\overrightarrow{\lambda}}'(1,t)\right|^2 \\ \quad +\lambda_1\int\limits_0^t\left|u_{\overrightarrow{\lambda}}''(1,s)\right|^2 ds+\lambda\int\limits_0^t\left\|\frac{\partial}{\partial s}\left(|u_{\overrightarrow{\lambda}}'(s)|^{\frac{q}{2}-1}u_{\overrightarrow{\lambda}}'(s)\right)\right\|^2 ds\le C_T. \end{cases}$$

It follows from (3.2) that

$$(3.3)\quad \begin{cases} \left\|\Pi_p(u_{\overrightarrow{\lambda}})\right\|_{H^1(Q_T)}\le C_T,\ \left\|\Pi_q(u_{\overrightarrow{\lambda}}')\right\|_{H^1(Q_T)}\le C_T, \\ \left\|\sqrt{\lambda_1}u_{\overrightarrow{\lambda}}'(1,\cdot)\right\|_{L^2(0,T)}^2=\lambda_1\int\limits_0^T\left|u_{\overrightarrow{\lambda}}'(1,s)\right|^2 ds\le C_T, \\ \left\|\sqrt{\lambda_1}u_{\overrightarrow{\lambda}}''(1,\cdot)\right\|_{L^2(0,T)}^2=\lambda_1\int\limits_0^T\left|u_{\overrightarrow{\lambda}}''(1,s)\right|^2 ds\le C_T, \\ \left\|\sqrt{\lambda_1}u_{\overrightarrow{\lambda}}'(1,\cdot)\right\|_{H^1(0,T)}^2\le C_T. \end{cases}$$

Let $\{\overrightarrow{\lambda}_j\}$, $\overrightarrow{\lambda}_j=(\lambda_j,\ \lambda_{1j})$, be a sequence such that $\lambda_j,\ \lambda_{1j}>0$, $\overrightarrow{\lambda}_j\to 0$ as $j\to\infty$. We put $u_j=u_{\overrightarrow{\lambda}_j}$ and deduce from (3.2), (3.3) that there exists a subsequence of the sequence $\{u_j\}$ denoted again by $\{u_j\}$, such that

$$(3.4)\qquad \begin{cases} u_j \to u_0 & \text{in } L^\infty(0,T;H^1) & \text{weakly*}, \\ u'_j \to u'_0 & \text{in } L^\infty(0,T;H^1) & \text{weakly*}, \\ u''_j \to u''_0 & \text{in } L^\infty(0,T;L^2) & \text{weakly*}, \\ \Pi_p(u_j) \to \overline{\Pi}_p & \text{in } H^1(Q_T) & \text{weakly}, \\ \Pi_q(u'_j) \to \overline{\Pi}_q & \text{in } H^1(Q_T) & \text{weakly}, \\ \sqrt{\lambda_{1j}}u'_j(1,\cdot) \to \chi_1 & \text{in } H^1(0,T) & \text{weakly}, \\ u_j(0,\cdot) \to u_0(0,\cdot) & \text{in } W^{1,\infty}(0,T) & \text{weakly*}, \\ u_j(1,\cdot) \to u_0(1,\cdot) & \text{in } W^{1,\infty}(0,T) & \text{weakly*}, \\ u'_j(1,\cdot) \to u'_0(1,\cdot) & \text{in } L^\infty(0,T) & \text{weakly*}. \end{cases}$$

By the Aubin-Lions lemma ([5, p. 57]) and the imbeddings $H^1(Q_T) \hookrightarrow L^2(Q_T)$, $H^1(0,T) \hookrightarrow C^0([0,T])$, we can deduce from $(3.4)_{1-8}$ the existence of a subsequence denoted again by $\{u_j\}$ such that

$$(3.5)\qquad \begin{cases} u_j \to u_0 & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ u'_j \to u'_0 & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \Pi_p(u_j) \to \overline{\Pi}_p & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ \Pi_q(u'_j) \to \overline{\Pi}_q & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ u_j(0,\cdot) \to u_0(0,\cdot) & \text{strongly in } C^0([0,T]), \\ u_j(1,\cdot) \to u_0(1,\cdot) & \text{strongly in } C^0([0,T]), \\ \sqrt{\lambda_{1j}}u'_j(1,\cdot) \to \chi_1 & \text{strongly in } C^0([0,T]). \end{cases}$$

Similarly, by $(3.3)_1$ and $(3.5)_{1-4}$, it is easy to prove that

$$(3.6)\qquad \overline{\Pi}_p = \Pi_p(u_0),\ \overline{\Pi}_q = \Pi_q(u'_0),$$

and

$$(3.7)\qquad \begin{aligned} Y_j(t) &= g(t) + K_0 u_j(0,t) + \int_0^t k(t-s)u_j(0,s)ds \\ &\to g(t) + K_0 u_0(0,t) + \int_0^t k(t-s)u_0(0,s)ds \\ &\equiv Y_0(t) \ \text{ strongly in } C^0([0,T]). \end{aligned}$$

Now, we shall prove that $\chi_1 = 0$.

It follows from $(3.5)_7$ that

$$(3.8)\qquad \sqrt{\lambda_{1j}}u'_j(1,\cdot) \to \chi_1 \text{ strongly in} C^0([0,T]).$$

On the other hand, it follows from $(3.4)_9$ that

$$(3.9)\qquad \sqrt{\lambda_{1j}}u'_j(1,\cdot) \to 0 \text{ in } L^\infty(0,T) \text{ weakly*}.$$

Then, (3.8) and (3.9) imply

$$(3.10)\qquad \chi_1 = 0.$$

Hence we obtain from (3.8), (3.10) that

$$\sqrt{\lambda_{1j}}u_j'(1,\cdot) \to 0 \text{ strongly in } C^0([0,T]). \tag{3.11}$$

Similarly,

$$\lambda_j \Pi_q(u_j') \to 0 \text{ strongly in } L^2(Q_T). \tag{3.12}$$

By passing to the limit, as in the proof of Theorem 2.2, we conclude that u_0 is a unique weak solution of problem (P_0) corresponding to $\overrightarrow{\lambda} = 0$ satisfying

$$\begin{cases} u_0 \in H^2(Q_T) \cap C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\ u_0' \in L^\infty(0,T;H^1),\ u_0'' \in L^\infty(0,T;L^2). \end{cases} \tag{3.13}$$

Note more that, u_0 is a unique weak solution of problem (P_0), i.e.,

$$(P_0)\begin{cases} u_0'' - \mu(t)u_{0xx} + K\Pi_p(u_0) = F(x,t), \quad 0 < x < 1, 0 < t < T, \\ \mu(t)u_{0x}(0,t) = K_0u_0(0,t) + \int_0^t k(t-s)\,u_0(0,s)\,ds + g(t), \\ -\mu(t)u_{0x}(1,t) = K_1u_0(1,t), \\ u_0(x,0) = \widetilde{u}_0(x), u_0'(x,0) = \widetilde{u}_1(x), \\ u_0 \in H^2(Q_T) \cap C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\ u_0' \in L^\infty(0,T;H^1), u_0'' \in L^\infty(0,T;L^2). \end{cases}$$

We shall study the asymptotic expansion of the solution of problem $(P_{\overrightarrow{\lambda}})$ with respect to two small parameters λ and λ_1.

We use the following notations. For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $\overrightarrow{\lambda} = (\lambda,\ \lambda_1) \in \mathbb{R}^2$, we put

$$\begin{cases} |\alpha| = \alpha_1 + \alpha_2,\ \alpha! = \alpha_1!\alpha_2!, \\ \left\|\overrightarrow{\lambda}\right\| = \sqrt{\lambda^2 + \lambda_1^2},\ \overrightarrow{\lambda}^\alpha = \lambda^{\alpha_1}\lambda_1^{\alpha_2}, \\ \alpha,\ \beta \in \mathbb{Z}_+^2,\ \alpha \le \beta \Longleftrightarrow \alpha_i \le \beta_i \quad \forall i = 1,2. \end{cases}$$

First, we shall need the following lemma.

Lemma 3.1. *Let* $m,\ N \in \mathbb{N}$ *and* $u_\alpha \in \mathbb{R},\ \alpha \in \mathbb{Z}_+^2,\ 1 \le |\alpha| \le N$. *Then*

$$\left(\sum_{1\le|\alpha|\le N} u_\alpha \overrightarrow{\lambda}^\alpha\right)^m = \sum_{m\le|\alpha|\le mN} T_N^{(m)}[u]_\alpha \overrightarrow{\lambda}^\alpha, \tag{3.14}$$

where the coefficients $T_N^{(m)}[u]_\alpha$, $m \le |\alpha| \le mN$ *depend on* $u = (u_\alpha)$, $\alpha \in \mathbb{Z}_+^2$, $1 \le |\alpha| \le N$ *defined by the recurrent formulas*

(3.15)

$$\begin{cases} T_N^{(1)}[u]_\alpha = u_\alpha, \ 1 \le |\alpha| \le N, \\ T_N^{(m)}[u]_\alpha = \sum\limits_{\beta \in A_\alpha^{(m)}(N)} u_{\alpha-\beta} T_N^{(m-1)}[u]_\beta, \ m \le |\alpha| \le mN, \ m \ge 2, \\ A_\alpha^{(m)}(N) = \{\beta \in \mathbb{Z}_+^2 : \beta \le \alpha, \ 1 \le |\alpha - \beta| \le N, \ m-1 \le |\beta| \le (m-1)N\}. \end{cases}$$

The proof of Lemma 3.1 can be found in [8].

Let us consider the sequence of the weak solutions u_γ, $\gamma \in \mathbb{Z}_+^2$, $1 \le |\gamma| \le N$, defined by the following problems:

$$(\widetilde{P}_\gamma) \begin{cases} u_\gamma'' - \mu(t) u_{\gamma xx} = F_\gamma(x,t), \ 0 < x < 1, 0 < t < T, \\ \mu(t) u_{\gamma x}(0,t) = K_0 u_\gamma(0,t) + \int\limits_0^t k(t-s)\, u_\gamma(0,s)\, ds, \\ -\mu(t) u_{\gamma x}(1,t) = K_1 u_\gamma(1,t) + G_{1\gamma}(t), \\ u_\gamma(x,0) = u_\gamma'(x,0) = 0, \\ u_\gamma \in H^2(Q_T) \cap C^0(0,T;H^1) \cap C^1(0,T;L^2) \cap L^\infty(0,T;H^2), \\ u_\gamma' \in L^\infty(0,T;H^1), u_\gamma'' \in L^\infty(0,T;L^2), \end{cases}$$

where F_γ, $G_{1\gamma}$, $\gamma \in \mathbb{Z}_+^2$, $1 \le |\gamma| \le N$, are defined by the recurrent formulas

$$(3.16) \qquad \begin{aligned} F_\gamma = F_\gamma(x,t) &= \begin{cases} -K\Pi_p'(u_0)\, u_{1,0} - \Pi_q(u_0'), & \gamma_1 = 1, \gamma_2 = 0, \\ -K\Pi_p'(u_0)\, u_{0,1}, & \gamma_1 = 0, \gamma_2 = 1, \\ -K\Phi_\gamma[\Pi_p, u] - \Phi_\gamma^{[1]}[\Pi_q, u'], & 2 \le |\gamma| \le N, \end{cases} \\ G_{1\gamma} = G_{1\gamma}(t) &= \dot{u}_\gamma^{[2]}(1,t), \ 1 \le |\gamma| \le N, \end{aligned}$$

with $\Phi_\gamma^{[1]}[\Pi_p, u]$, $\dot{u}_\gamma^{[2]}(1,t)$, $1 \le |\gamma| \le N$, defined by the formulas

$$(3.17) \qquad \begin{aligned} \Phi_\gamma^{[1]}[\Pi_q, u'] &= \begin{cases} \Phi_{\gamma_1 - 1, \gamma_2}[\Pi_q, u'], & \gamma_1 \ge 1, \\ 0, & \gamma_1 = 0, \end{cases} \\ \dot{u}_\gamma^{[2]}(1,t) &= \begin{cases} u'_{\gamma_1, \gamma_2 - 1}(1,t), & \gamma_2 \ge 1, \\ 0, & \gamma_2 = 0, \end{cases} \end{aligned}$$

and

$$(3.18) \qquad \begin{cases} \Phi_\gamma[\Pi_p, u] &= \sum\limits_{m=1}^{|\gamma|} \frac{1}{m!} \Pi_p^{(m)}(u_0) T_N^{(m)}[u]_\gamma, \\ \Phi_\gamma[\Pi_q, u'] &= \sum\limits_{m=1}^{|\gamma|} \frac{1}{m!} \Pi_q^{(m)}(u_0') T_N^{(m)}[u']_\gamma. \end{cases}$$

Here $u = (u_\gamma)$, $u' = (u_\gamma')$, $|\gamma| \le N$.

Then, we have the following theorem.

Theorem 3.2. *Let* (H_1)-(H_4), (H_6) *hold. Then, for every* $\overrightarrow{\lambda}$, *with* $\left\|\overrightarrow{\lambda}\right\| < 1$, *problem* $\left(P_{\overrightarrow{\lambda}}\right)$ *has a unique weak solution* $u = u_{\overrightarrow{\lambda}}$ *satisfying the asymptotic estimation up to order* N *as follows*

$$\begin{aligned}(3.19)\qquad &\left\|u'_{\overrightarrow{\lambda}} - \sum_{|\gamma|\leq N} u'_\gamma \overrightarrow{\lambda}^\gamma\right\|_{L^\infty(0,T;L^2)} + \left\|u_{\overrightarrow{\lambda}} - \sum_{|\gamma|\leq N} u_\gamma \overrightarrow{\lambda}^\gamma\right\|_{L^\infty(0,T;H^1)} \\ &+\sqrt{\lambda_1}\left\|u'_{\overrightarrow{\lambda}}(1,\cdot) - \sum_{|\gamma|\leq N} u'_\gamma(1,\cdot)\overrightarrow{\lambda}^\gamma\right\|_{L^2(0,T)} \leq C_T\left\|\overrightarrow{\lambda}\right\|^{N+\frac{1}{2}},\end{aligned}$$

where C_T *is a constant depending only on* T, *the functions* u_γ, $|\gamma| \leq N$, *are the weak solutions of problems* (P_0), $(\widetilde{P}_\gamma)$, $1 \leq |\gamma| \leq N$, *respectively.*

Proof. Let $u = u_{\overrightarrow{\lambda}}$ be the unique weak solution of problem $\left(P_{\overrightarrow{\lambda}}\right)$. Then v, with

$$(3.20)\qquad v = u - \sum_{|\gamma|\leq N} u_\gamma \overrightarrow{\lambda}^\gamma = u - h,$$

satisfies

$$(3.21)\qquad \begin{cases} v'' - \mu(t)v_{xx} = -K\left[\Pi_p(v+h) - \Pi_p(h)\right] - \lambda\left[\Pi_q\left(v'+h'\right) - \Pi_q\left(h'\right)\right] \\ \qquad\qquad + E_N(\overrightarrow{\lambda}),\ 0<x<1,\ 0<t<T, \\ \mu(t)v_x(0,t) = K_0v(0,t) + \int_0^t k(t-s)\,v(0,s)\,ds, \\ -\mu(t)v_x(1,t) = K_1v(1,t) + \lambda_1v'(1,t) + E_{1N}(\overrightarrow{\lambda}), \\ v(x,0) = v'(x,0) = 0, \\ v \in H^2(Q_T)\cap C^0\left(0,T;H^1\right)\cap C^1\left(0,T;L^2\right)\cap L^\infty\left(0,T;H^2\right), \\ v' \in L^\infty(0,T;H^1),\quad v'' \in L^\infty(0,T;L^2), \end{cases}$$

where

(3.22)

$$\begin{cases} E_N(\overrightarrow{\lambda}) \equiv E_N(\overrightarrow{\lambda};x,t) = -K\left[\Pi_p(h) - \Pi_p(u_0)\right] - \lambda\Pi_q(h') - \sum_{1\leq|\gamma|\leq N} F_\gamma\overrightarrow{\lambda}^\gamma, \\ E_{1N}(\overrightarrow{\lambda}) \equiv E_{1N}(\overrightarrow{\lambda},t) = \sum_{|\gamma|\leq N}\lambda_1u'_\gamma(1,t)\ \overrightarrow{\lambda}^\gamma - \sum_{1\leq|\gamma|\leq N} G_{1\gamma}\overrightarrow{\lambda}^\gamma. \end{cases}$$

Then, we have the following lemma.

Lemma 3.3. *Let* (H_1)-(H_4), (H_6) *hold. Then*

$$(3.23)\qquad \begin{aligned} &\text{(i)}\quad \left\|E_N(\overrightarrow{\lambda})\right\|_{L^\infty(0,T;L^2)} \leq \widetilde{C}_N\left\|\overrightarrow{\lambda}\right\|^{N+1}, \\ &\text{(ii)}\quad 2\left|\int_0^t E_{1N}(\overrightarrow{\lambda},s)v'(1,s)\,ds\right| \leq \frac{1}{2}\lambda_1\int_0^t\left|v'(1,s)\right|^2ds + \widetilde{C}_{1N}\left\|\overrightarrow{\lambda}\right\|^{2N+1}, \end{aligned}$$

for all $\overrightarrow{\lambda}$, $\left\|\overrightarrow{\lambda}\right\| < 1$, $\widetilde{C}_N$, $\widetilde{C}_{1N}$, *are the constants depending only on* N, T, p, q *and the constants* $\|u_\gamma\|_{L^\infty(0,T;H^1)}$, $\left\|u'_\gamma\right\|_{L^\infty(0,T;H^1)}$, $|\gamma| \le N$.

Proof. (i) *Estimate* $\left\|E_N(\overrightarrow{\lambda})\right\|_{L^\infty(0,T;L^2)}$. In case $N = 1$, the proof of Lemma 3.3 is easy, hence we omit the details, therefore we only prove for $N \ge 2$. Put

$$h = \sum_{|\gamma| \le N} u_\gamma \overrightarrow{\lambda}^\gamma = u_0 + h_1 \text{ and } h_1 = \sum_{1 \le |\gamma| \le N} u_\gamma \overrightarrow{\lambda}^\gamma. \tag{3.24}$$

By using Taylor's expansion of the function $\Pi_p(h) = \Pi_p(u_0 + h_1)$ around the point u_0 up to order N, we obtain

$$\Pi_p(h) = \Pi_p(u_0) + \sum_{m=1}^{N} \frac{1}{k!} \Pi_p^{(m)}(u_0) h_1^m + \frac{1}{(N+1)!} \Pi_p^{(N+1)}(u_0 + \theta_1 h_1) h_1^{N+1}, \tag{3.25}$$

where $0 < \theta_1 < 1$. By Lemma 3.1, we obtain from $(3.18)_1$, (3.24), after some rearrangements in order to of $\overrightarrow{\lambda}$, that

$$\begin{aligned}
\Pi_p(h) - \Pi_p(u_0) &= \sum_{1 \le |\gamma| \le N} \left(\sum_{m=1}^{|\gamma|} \frac{1}{m!} \Pi_p^{(m)}(u_0) T_N^{(m)}[u]_\gamma \right) \overrightarrow{\lambda}^\gamma \\
&\quad + \sum_{m=1}^{N} \frac{1}{m!} \Pi_p^{(m)}(u_0) \left(\sum_{N+1 \le |\gamma| \le m(N+1)} T_N^{(m)}[u]_\gamma \overrightarrow{\lambda}^\gamma \right) \\
&\quad + \frac{1}{(N+1)!} \Pi_p^{(N+1)}(u_0 + \theta_1 h_1) h_1^{N+1} \\
&= \sum_{1 \le |\gamma| \le N} \Phi_\gamma[\Pi_p, u] \overrightarrow{\lambda}^\gamma + \left\|\overrightarrow{\lambda}\right\|^{N+1} \overline{R}_N(\Pi_p, u, \overrightarrow{\lambda}), \qquad (3.26)
\end{aligned}$$

where

$$\begin{aligned}
\left\|\overrightarrow{\lambda}\right\|^{N+1} \overline{R}_N(\Pi_p, u, \overrightarrow{\lambda}) &= \sum_{m=1}^{N} \frac{1}{m!} \Pi_p^{(m)}(u_0) \left(\sum_{N+1 \le |\gamma| \le m(N+1)} T_N^{(m)}[u]_\gamma \overrightarrow{\lambda}^\gamma \right) \\
&\quad + \frac{1}{(N+1)!} \Pi_p^{(N+1)}(u_0 + \theta_1 h_1) h_1^{N+1}. \qquad (3.27)
\end{aligned}$$

Similarly, using Taylor's expansion of the function $\Pi_q(h') = \Pi_q(u'_0 + h'_1)$ up to order $N-1$, we obtain

$$\begin{aligned}
\Pi_q(h') &= \Pi_q(u'_0 + h'_1) \\
&= \Pi_q(u'_0) + \sum_{1 \le |\gamma| \le N-1} \Phi_\gamma[\Pi_q, u'] \overrightarrow{\lambda}^\gamma + \left\|\overrightarrow{\lambda}\right\|^{N} \overline{R}_{N-1}(\Pi_q, u', \overrightarrow{\lambda}), \qquad (3.28)
\end{aligned}$$

where

$$\left\|\overrightarrow{\lambda}\right\|^N \overline{R}_{N-1}(\Pi_q, u', \overrightarrow{\lambda}) = \sum_{m=1}^{N-1} \frac{1}{m!}\Pi_q^{(m)}(u_0') \left(\sum_{N\leq|\gamma|\leq mN} T_N^{(m)}[u']_\gamma \overrightarrow{\lambda}^\gamma \right) + \frac{1}{N!}\Pi_q^{(N)}(u_0' + \theta_2 h_1') \left(h_1'\right)^N , \tag{3.29}$$

and $0 < \theta_2 < 1$. Combining (3.16), $(3.22)_1$, (3.26), and (3.28), we obtain

$$E_N(\overrightarrow{\lambda}) \equiv E_N(\overrightarrow{\lambda}; x, t) = -\left\|\overrightarrow{\lambda}\right\|^{N+1} \left[K\overline{R}_N(\Pi_p, u, \overrightarrow{\lambda}) + \frac{\lambda}{\left\|\overrightarrow{\lambda}\right\|}\overline{R}_{N-1}(\Pi_q, u', \overrightarrow{\lambda}) \right]. \tag{3.30}$$

We shall respectively estimate the following terms on the right-hand side of (3.30).

Estimate $K \left\|\overrightarrow{\lambda}\right\|^{N+1} \overline{R}_N(\Pi_p, u, \overrightarrow{\lambda})$.

First, we note that if we put $D_T = \sqrt{2} \sum_{|\gamma|\leq N} \left(\|u_\gamma\|_{L^\infty(0,T;H^1)} + \left\|u_\gamma'\right\|_{L^\infty(0,T;H^1)} \right)$, then by the boundedness of the functions u_γ, u_γ', $|\gamma| \leq N$, in the function space $L^\infty(0, T; H^1)$, we obtain from (3.24) that

$$\begin{cases} |h_1| & \leq \sqrt{2} \sum_{1\leq|\gamma|\leq N} \|u_\gamma\|_{H^1} \left|\overrightarrow{\lambda}^\gamma\right| \leq \sqrt{2} \sum_{1\leq|\gamma|\leq N} \|u_\gamma\|_{L^\infty(0,T;H^1)} \left\|\overrightarrow{\lambda}\right\|^{|\gamma|} \\ & \leq \sqrt{2}\left\|\overrightarrow{\lambda}\right\| \sum_{1\leq|\gamma|\leq N} \|u_\gamma\|_{L^\infty(0,T;H^1)} \leq D_T \left\|\overrightarrow{\lambda}\right\| \leq D_T, \\ & |u_0 + \theta_1 h_1| \leq |u_0| + |h_1| \leq \sqrt{2} \sum_{|\gamma|\leq N} \|u_\gamma\|_{L^\infty(0,T;H^1)} \left\|\overrightarrow{\lambda}\right\|^{|\gamma|} \leq D_T. \end{cases} \tag{3.31}$$

Hence

$$\frac{1}{(N+1)!} \left|\Pi_p^{(N+1)}(u_0 + \theta_1 h_1) h_1^{N+1}\right| \leq \left(D_T \left\|\overrightarrow{\lambda}\right\| \right)^{N+1} \frac{1}{(N+1)!} \sup_{|z|\leq D_T} \left|\Pi_p^{(N+1)}(z)\right| \equiv D_{1T}(p) \left\|\overrightarrow{\lambda}\right\|^{N+1}, \tag{3.32}$$

and

$$\left\|\overrightarrow{\lambda}\right\|^{-N-1} \left\| \sum_{m=1}^{N} \frac{1}{m!}\Pi_p^{(m)}(u_0) \left(\sum_{N+1\leq|\gamma|\leq m(N+1)} T_N^{(m)}[u]_\gamma \overrightarrow{\lambda}^\gamma \right) \right\| \leq \left(\max_{1\leq m\leq N} \sup_{|z|\leq D_T} \left|\Pi_p^{(m)}(z)\right| \right) \sum_{m=1}^{N} \frac{1}{m!} \sum_{N+1\leq|\gamma|\leq m(N+1)} \left\|T_N^{(m)}[u]_\gamma\right\|_{L^\infty(0,T;L^2)} \left\|\overrightarrow{\lambda}\right\|^{|\gamma|-N-1} \tag{3.33}$$

$$\leq \left(\max_{1\leq m\leq N} \sup_{|z|\leq D_T} \left|\Pi_p^{(m)}(z)\right|\right) \sum_{m=1}^{N} \frac{1}{m!} \sum_{N+1\leq|\gamma|\leq m(N+1)} \left\|T_N^{(m)}[u]_\gamma\right\|_{L^\infty(0,T;L^2)} \equiv D_{2T}(p).$$

This implies

(3.34)

$$\begin{aligned} K\left\|\overrightarrow{\lambda}\right\|^{N+1}\left|\overline{R}_N(\Pi_p,u,\overrightarrow{\lambda})\right| &\leq K\left|\sum_{m=1}^{N}\frac{1}{m!}\Pi_p^{(m)}(u_0)\left(\sum_{N+1\leq|\gamma|\leq m(N+1)} T_N^{(m)}[u]_\gamma \overrightarrow{\lambda}^\gamma\right)\right| \\ &\quad + K\left|\frac{1}{(N+1)!}\Pi_p^{(N+1)}(u_0+\theta_1 h_1)h_1^{N+1}\right| \\ &\leq (D_{2T}(p)+D_{1T}(p))\left\|\overrightarrow{\lambda}\right\|^{N+1} \equiv D_{*T}(p)\left\|\overrightarrow{\lambda}\right\|^{N+1}. \end{aligned}$$

Estimate $\left\|\overrightarrow{\lambda}\right\|^{N}\lambda\overline{R}_{N-1}(\Pi_q,u',\overrightarrow{\lambda})$.

We also obtain from (3.29) in a similar manner corresponding to the above part that

$$\begin{aligned} \lambda\left\|\overrightarrow{\lambda}\right\|^{N}\left|\overline{R}_{N-1}(\Pi_q,u',\overrightarrow{\lambda})\right| &= \lambda\left|\sum_{m=1}^{N-1}\frac{1}{m!}\Pi_q^{(m)}(u_0')\left(\sum_{N\leq|\gamma|\leq mN} T_N^{(m)}[u']_\gamma \overrightarrow{\lambda}^\gamma\right)\right| \\ &\quad + \lambda\left|\frac{1}{N!}\Pi_q^{(N)}(u_0'+\theta_2 h_1')\left(h_1'\right)^N\right| \\ &\leq (D_{2T}(q)+D_{1T}(q))\left\|\overrightarrow{\lambda}\right\|^{N+1} \equiv D_{*T}(q)\left\|\overrightarrow{\lambda}\right\|^{N+1}, \end{aligned} \tag{3.35}$$

where

(3.36)

$$\begin{cases} D_{1T}(q) = \frac{D_T^N}{N!}\sup\limits_{|z|\leq D_T}\left|\Pi_q^{(N)}(z)\right|, \\ D_{2T}(q) = \left(\max\limits_{1\leq m\leq N-1}\sup\limits_{|z|\leq D_T}\left|\Pi_q^{(m)}(z)\right|\right)\sum\limits_{m=1}^{N-1}\frac{1}{m!}\sum\limits_{N\leq|\gamma|\leq mN}\left\|T_N^{(m)}[u']_\gamma\right\|_{L^\infty(0,T;L^2)}. \end{cases}$$

Therefore, it follows from (3.30), (3.34) and (3.35) that

$$\left|E_N(\overrightarrow{\lambda})\right| \leq [D_{*T}(p)+D_{*T}(q)]\left\|\overrightarrow{\lambda}\right\|^{N+1} \equiv \widetilde{C}_N\left\|\overrightarrow{\lambda}\right\|^{N+1}, \tag{3.37}$$

where $\widetilde{C}_N = D_{*T}(p)+D_{*T}(q)$ and the proof of (i) is complete.

(ii) *Estimate* $2\int_0^t E_{1N}(\overrightarrow{\lambda},s)v'(1,s)\,ds$.

We note that

$$\begin{aligned} E_{1N}(\overrightarrow{\lambda}) \equiv E_{1N}(\overrightarrow{\lambda},t) &= \sum_{|\gamma|\leq N}\lambda_1 u_\gamma'(1,t)\overrightarrow{\lambda}^\gamma - \sum_{1\leq|\gamma|\leq N} G_{1\gamma}\overrightarrow{\lambda}^\gamma \\ &= \lambda_1\sum_{|\gamma|=N} u_\gamma'(1,t)\overrightarrow{\lambda}^\gamma. \end{aligned} \tag{3.38}$$

Using Cauchy's inequality, we obtain

$$\begin{aligned}
&\left| 2\int_0^t E_{1N}(\overrightarrow{\lambda}, s)v'(1,s)\,ds \right| \\
&\leq 2\lambda_1 \int_0^t \sum_{|\gamma|=N} \left\| \overrightarrow{\lambda} \right\|^{|\gamma|} \left| u'_\gamma(1,s) \right| \left| v'(1,s) \right| ds \\
&\leq 2\sqrt{2}\lambda_1 \left\| \overrightarrow{\lambda} \right\|^N \sum_{|\gamma|=N} \|u'_\gamma\|_{L^\infty(0,T;H^1)} \int_0^t \left| v'(1,s) \right| ds \qquad (3.39)\\
&\leq \frac{1}{2}\lambda_1 \int_0^t \left| v'(1,s) \right|^2 ds + 4\lambda_1 \left\| \overrightarrow{\lambda} \right\|^{2N} T \left(\sum_{|\gamma|=N} \|u'_\gamma\|_{L^\infty(0,T;H^1)} \right)^2 \\
&\leq \frac{1}{2}\lambda_1 \int_0^t \left| v'(1,s) \right|^2 ds + \widetilde{C}_{1N} \left\| \overrightarrow{\lambda} \right\|^{2N+1},
\end{aligned}$$

where

$$\widetilde{C}_{1N} = 4T \left(\sum_{|\gamma|=N} \|u'_\gamma\|_{L^\infty(0,T;H^1)} \right)^2. \qquad (3.40)$$

The proof of (ii) is complete.

The proof of Lemma 3.3 is complete. □

We now continue the proof of Theorem 3.2.

Next, by multiplying the two sides of $(3.21)_1$ with v', after integration with respect to t, we find without difficulty from Lemma 3.3 that

$$\begin{aligned}
Z(t) &\leq T\widetilde{C}_N^2 \left\| \overrightarrow{\lambda} \right\|^{2N+2} + \left(2 + \frac{1}{\mu_0} \|\mu'\|_{L^\infty(0,T)} \right) \int_0^t Z(s)ds \\
&\quad - 2\int_0^t E_{1N}(\overrightarrow{\lambda}, s)v'(1,s)\,ds - 2v(0,t)\int_0^t k(t-r)v(0,r)dr + 2k(0)\int_0^t v^2(0,s)ds \\
&\quad + 2\int_0^t v(0,s)ds \int_0^s k'(s-r)v(0,r)dr + K^2 \int_0^t \|\Pi_p(v+h) - \Pi_p(h)\|^2 ds \qquad (3.41)\\
&\equiv T\widetilde{C}_N^2 \left\| \overrightarrow{\lambda} \right\|^{2N+2} + \left(2 + \frac{1}{\mu_0} \|\mu'\|_{L^\infty(0,T)} \right) \int_0^t Z(s)ds + \sum_{i=1}^5 Z_i,
\end{aligned}$$

where

$$Z(t) = \left\|v'(t)\right\|^2 + \mu(t)\left\|v_x(t)\right\|^2 + K_0 v^2(0,t) + K_1 v^2(1,t)$$

$$+ 2\lambda \int_0^t \left\langle \Pi_q\left(v' + h'\right) - \Pi_q\left(h'\right), v'\right\rangle ds + 2\lambda_1 \int_0^t \left|v'(1,s)\right|^2 ds. \tag{3.42}$$

Using the following inequality

$$\forall \alpha \geq 2, \exists C_\alpha > 0 : \left(\Pi_\alpha(x) - \Pi_\alpha(y)\right)(x-y) \geq C_\alpha \left|x-y\right|^\alpha, \ \forall x, y \in \mathbb{R}, \tag{3.43}$$

we deduce from (3.42) that

$$Z(t) \geq \left\|v'(t)\right\|^2 + \mu_0 \left\|v_x(t)\right\|^2 + 2\lambda C_q \int_0^t \left\|v'(s)\right\|_{L^q}^q ds + 2\lambda_1 \int_0^t \left|v'(1,s)\right|^2 ds. \tag{3.44}$$

Using (3.44) and the inequalities

$$\begin{cases} \left\|v(t)\right\|^2 = \left(\int_0^t \left\|v'(s)\right\| ds\right)^2 \leq t\int_0^t \left\|v'(s)\right\|^2 ds \leq t \int_0^t Z(s)ds, \\ \left\|v(t)\right\|_{H^1}^2 \leq \dfrac{1}{\mu_0} Z(t) + t\int_0^t Z(s)ds, \\ \int_0^t \left\|v(s)\right\|_{H^1}^2 ds \leq \left(\dfrac{1}{\mu_0} + \dfrac{1}{2}T^2\right) \int_0^t Z(s)ds, \\ \left|v(x,t)\right| \leq \left\|v(t)\right\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\left\|v(t)\right\|_{H^1}, \quad \forall x \in [0,1], \end{cases} \tag{3.45}$$

we shall respectively estimate the terms Z_i, $i = 1, ..., 5$ on the right-hand side of (3.41) as follows

$$Z_1 = -2\int_0^t E_{1N}(\overrightarrow{\lambda}, s) v'(1,s)\, ds \leq \frac{1}{2}\lambda_1 \int_0^t \left|v'(1,s)\right|^2 ds + \widetilde{C}_{1N} \left\|\overrightarrow{\lambda}\right\|^{2N+1}$$

$$\leq \frac{1}{4} Z(t) + \widetilde{C}_{1N} \left\|\overrightarrow{\lambda}\right\|^{2N+1}, \tag{3.46}$$

$$Z_2 = -2v(0,t)\int_0^t k(t-r) v(0,r) dr \leq \beta \left\|v(t)\right\|_{H^1}^2 + \frac{4}{\beta}\left\|k\right\|_{L^2(0,T)}^2 \int_0^t \left\|v(r)\right\|_{H^1}^2 dr$$

$$\leq \frac{\beta}{\mu_0} Z(t) + \left[\beta T + \frac{4}{\beta}\left\|k\right\|_{L^2(0,T)}^2 \left(\frac{1}{\mu_0} + \frac{1}{2}T^2\right)\right] \int_0^t Z(s)ds$$

(3.47)
$$\equiv \frac{\beta}{\mu_0} Z(t) + Z_{2T}(\beta) \int_0^t Z(s)ds, \forall \beta > 0,$$

(3.48)
$$Z_3 = 2k(0) \int_0^t v^2(0,s)ds \leq 4|k(0)| \left(\frac{1}{\mu_0} + \frac{1}{2}T^2 \right) \int_0^t Z(s)ds \equiv Z_{3T} \int_0^t Z(s)ds,$$

$$Z_4 = 2 \int_0^t v(0,s)ds \int_0^s k'(s-r)v(0,r)dr$$

$$\leq 4\sqrt{T} \left\| k' \right\|_{L^2(0,T)} \int_0^t \|v(s)\|_{H^1}^2 ds \leq Z_{4T} \int_0^t Z(s)ds. \tag{3.49}$$

With Z_5, first, by using the same arguments as in the above part, we can show that the weak solution $u_{\overrightarrow{\lambda}}$ of problem $\left(P_{\overrightarrow{\lambda}}\right)$ satisfies

$$\left\| u_{\overrightarrow{\lambda}} \right\|_{L^\infty(0,T;H^1)} \leq C_T, \ \left\| \overrightarrow{\lambda} \right\| < 1, \tag{3.50}$$

where C_T is a constant independent of $\overrightarrow{\lambda}$. On the other hand, we have

$$\begin{cases} \|h\|_{L^\infty(0,T;H^1)} \leq \sum_{|\gamma| \leq N} \|u_\gamma\|_{L^\infty(0,T;H^1)} \equiv C_{1T}, \\ \|v+h\|_{L^\infty(0,T;H^1)} = \left\| u_{\overrightarrow{\lambda}} \right\|_{L^\infty(0,T;H^1)} \leq C_T. \end{cases} \tag{3.51}$$

Next, with $R_T = \max\{C_{1T}, C_T\}$, it follows from (3.51) that

$$Z_5 = K^2 \int_0^t \left\| \Pi_p(v+h) - \Pi_p(h) \right\|^2 ds \leq K^2 (p-1)^2 R_T^{2p-4} \int_0^t \|v(s)\|^2 ds$$

$$\leq \frac{1}{2} T^2 (p-1)^2 R_T^{2p-4} \int_0^t Z(s)ds \equiv Z_{5T} \int_0^t Z(s)ds. \tag{3.52}$$

Combining (3.41), (3.46)-(3.49) and (3.52), we then obtain

$$Z(t) \leq \frac{1}{2} \widehat{Z}_{1T} \left\| \overrightarrow{\lambda} \right\|^{2N+1} + \left(\frac{1}{4} + \frac{\beta}{\mu_0} \right) Z(t) + \frac{1}{2} \widehat{Z}_{2T}(\beta) \int_0^t Z(s)ds, \tag{3.53}$$

for all $\beta > 0$, where

$$\begin{cases} \frac{1}{2} \widehat{Z}_{1T} = T \widetilde{C}_N^2 + \widetilde{C}_{1N}, \\ \frac{1}{2} \widehat{Z}_{2T}(\beta) = 2 + \frac{1}{\mu_0} \|\mu'\|_{L^\infty(0,T)} + Z_{2T}(\beta) + Z_{3T} + Z_{4T} + Z_{5T}. \end{cases} \tag{3.54}$$

Choosing $\beta > 0$, with $\frac{1}{4} + \frac{\beta}{\mu_0} \leq \frac{1}{2}$, it follows from (3.53), that

$$Z(t) \leq \widehat{Z}_{1T} \left\| \overrightarrow{\lambda} \right\|^{2N+1} + \widehat{Z}_{2T}(\beta) \int_0^t Z(s)ds. \tag{3.55}$$

Using Gronwall's lemma, it follows that

$$Z(t) \leq \widehat{Z}_{1T} \left\| \overrightarrow{\lambda} \right\|^{2N+1} \exp(T\widehat{Z}_{2T}(\beta)) \equiv \widehat{Z}_{3T}(\beta) \left\| \overrightarrow{\lambda} \right\|^{2N+1}. \tag{3.56}$$

This implies (3.19). Theorem 3.2 is proved completely. □

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