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PRODUCT OF COMPOSITION AND MULTIPLICATION OPERATORS

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ABSTRACT. The paper discusses some properties for the operator $W_{(u,T)} = C_T M_u$ given by $f \mapsto u \circ T \cdot f \circ T$, on Orlicz spaces using the conditional expectation operator.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous convex function such that

(1) $\varphi(x) = 0$ if and only if $x = 0$,
(2) $\varphi(x) \to \infty$ as $x \to \infty$.

Such a function is known as an Orlicz function.

The Orlicz space $L_{\varphi,\Omega,\mathcal{A},\mu}$ consists of all those complex-valued measurable functions f on Ω such that

$$\int_\Omega \varphi(\alpha|f(\omega)|)d\mu \ < \ \infty$$

for some $\alpha \ > \ 0$.

The Orlicz space $L_{\varphi,\Omega,\mathcal{A},\mu}$ is a Banach space with respect to the Luxemburg norm

$$\|f\|_\varphi = \inf\left\{\epsilon \ > \ 0 : \int_\Omega \varphi\left(\frac{|f(\omega)|}{\epsilon}\right)d\mu \ \leq \ 1\right\}.$$

If there is no confusion about the measure space Ω or $\mathcal{A}$ or μ, then we simply denote the Orlicz space $L_{\varphi,\Omega,\mathcal{A},\mu}$ by L_φ. The Orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$,

$$\varphi(2x) \leq k\varphi(x) \text{ for all } x > 0.$$

Some well known facts are the following:
(i) If φ satisfies Δ_2-condition, then the class of simple functions is dense in L_φ.
(ii) If $\|f\|_\varphi \leq 1$ then $I_\varphi(f) = \int \varphi(|f|)d\mu \leq \|f\|_\varphi$. As a consequence $\|f_n - f\|_\varphi \to 0$ implies that $I_\varphi(f_n - f) \to 0$ for a sequence $\{f_n\}$ in L_φ. If φ satisfies Δ_2-condition then the converse of the above fact is also true.
(iii) Corresponding to the Orlicz function φ, we can associate another Orlicz

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function ψ (known as the complementary function to φ) such that the Banach dual L_φ^* of the Orlicz space L_φ is isometrically isomorphic to the Orlicz space L_ψ.

For more details on Orlicz spaces we refer [12].

An atom of a measure space $(\Omega, \mathcal{A}, \mu)$ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $E \in \mathcal{A}$, if $E \subset A$ then either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. A measure space having no atoms is called a non-atomic measure space. It is an interesting fact that every σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ can be decomposed as

$$\Omega = \Omega_1 \bigcup \Omega_2$$

where Ω_1 is non-atomic and $\Omega_2 = \bigcup_{n\in\mathbb{N}} A_n$ is a countable union of disjoint atoms of finite measure.

A measurable transformation $T : \Omega \to \Omega$ satisfying $\mu(T^{-1}(B)) = 0$ whenever $\mu(B) = 0$ for $B \in \mathcal{A}$ is said to be a non-singular measurable transformation. If T is non-singular, then the measure μT^{-1} given by

$$(\mu T^{-1})(B) = \mu(T^{-1}(B)) \text{ for } B \in \mathcal{A},$$

is absolutely continuous with respect to the measure μ. Hence by the Radon-Nikodym theorem, there exists a non-negative measurable function f_T such that

$$(\mu T^{-1})(B) = \int_B f_T d\mu,$$

for every $B \in \mathcal{A}$. The function f_T is called the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ. It is denoted by $f_T = d\mu T^{-1}/d\mu$.

Consider the σ-finite subalgebra $T^{-1}(\mathcal{A})$ of a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, then the conditional expectation with respect to $T^{-1}(\mathcal{A})$ is a transformation from $L^p(\Omega, \mathcal{A}, \mu)$ into $L^p(\Omega, T^{-1}(\mathcal{A}), \mu)$ and we denote this transformation by E.

For each $\mathcal{A}$-measurable function f, there exists a $\mathcal{A}$-measurable function g such that $E^{T^{-1}(\mathcal{A})}(f) = g \circ T$. We can assume that the support of g lies in the support of f_T, and then $E(f) = g \circ T$ for exactly one $\mathcal{A}$-measurable function. In particular, $g = E(f) \circ T^{-1}$ is a well defined measurable function. For a deeper study of the properties and applications of expectation operator, we refer [8] and [11].

Let $T : \Omega \to \Omega$ be a non-singular transformation and u be a complex-valued measurable function defined on Ω. The bounded linear transformation $M_u : f \mapsto u \cdot f$ on a Banach function space is called a *multiplication operator* induced by u and the bounded linear transformation $C_T : f \mapsto f \circ T$ on a Banach function space is called a *composition operator* induced by T. These operators are discussed on Orlicz spaces by Komal and Gupta in [9]. Weighted composition operators induced by u and T, given by $f \mapsto u \cdot f \circ T$, are studied on Orlicz spaces in [6], Lorentz spaces in [2] and on Orlicz-Lorentz spaces in [1]. In [8], Jabbarzadeh studied some properties for this class of operators on L^p spaces and Orlicz spaces using conditional expectation operators. A weighted composition operator is the product of a multiplication operator with a composition operator. It is possible to find u and T inducing the weighted composition operator, whereas the composition operator C_T alone may not be defined (see [8], Remark 2.6). Now, the present paper extend the study for the operators $W_{(u,T)}$ given by

$$W_{(u,T)}(f)(\omega) = u(T(\omega))f(T(\omega)),$$

where u is a complex-valued measurable function and T is a non-singular measurable transformation, on the Orlicz space L_φ.

We denote the class of all bounded operators on a Banach space X by $\mathfrak{B}(X)$ and the kernel and range of an operator P on X by $\mathrm{Ker}(P)$ and $R(P)$, respectively. If $W_{(u,T)}$ is bounded with range in L_φ, then we denote it by writing $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$. Although $W_{(u,T)}f = C_T M_u f$, but one can still find u and T inducing a bounded operator $W_{(u,T)}$ and not inducing the composition operator C_T. If $u \equiv 1$, then $W_{(u,T)} \equiv C_T$ and if T is the identity mapping, then $W_{(u,T)} \equiv M_u$. If we take $\Omega = [0,1], u \equiv 0, \varphi(\omega) = \omega^p, 1 \leq p < \infty$ and $T(\omega) = \omega^3$, then $W_{(u,T)} = 0 \in \mathfrak{B}(L_\varphi)$. However, T does not induce the composition operator C_T on L_φ.

The adjoint of $W_{(u,T)}$ is obtained using the conditional expectation operator. Corollary 2.8 of the paper provides a characterization for the compactness of $W_{(u,T)}$. However, the same result is obtained for the weighted composition operator on Lorentz spaces in [2].

2. Adjoint of $W_{(u,T)}$

Now onwards, we assume that φ satisfies Δ_2-condition, u is a complex-valued measurable mapping and T is a non-singular measurable transformation. The non-singularity of T guarantees that $W_{(u,T)}$ is well defined as a mapping from L_φ into $L(\mu)$, the linear space of all complex-valued measurable functions. The following result can be easily proved along the lines of arguments given in [8, Theorem 3.1].

Theorem 2.1. *If the mapping $W_{(u,T)}$ from L_φ into $L(\mu)$, the linear space of all complex-valued measurable functions, is such that $W_{(u,T)}(L_\varphi) \subseteq L_\varphi$ then $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$.*

Proof. Let a sequence $\{f_n\}$ and elements f and g in L_φ be such that

$$\|f_n - f\|_\varphi \to 0 \text{ and } \|W_{(u,T)}f_n - g\|_\varphi \to 0$$

as $n \to \infty$. Then along the lines of arguments in [7, Theorem 3.1], we find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ satisfying $\varphi(|f_{n_k} - f|) \to 0$ a.e. on Ω and hence $\varphi(|u \cdot f_{n_k} - u \cdot f|) \to 0$ a.e. Since T is non-singular, this means that $\varphi(|u \circ T \cdot f_{n_k} \circ T - u \circ T \cdot f \circ T|) \to 0$ a.e. on Ω. Moreover, we can find a subsequence $\{f_{n_{k'}}\}$ of $\{f_{n_k}\}$ such that $\varphi(|u \circ T \cdot f_{n_{k'}} \circ T - g|) \to 0$ a.e. on Ω. Also, we have $\varphi(|u \circ T \cdot f_{n_{k'}} \circ T - u \circ T \cdot f \circ T|) \to 0$ a.e. on Ω. Therefore, since φ satisfies Δ_2-condition,

$$\|W_{(u,T)}f_{n_{k'}} - g\|_\varphi \to 0 \text{ and } \|W_{(u,T)}f_{n_{k'}} - W_{(u,T)}f\|_\varphi \to 0$$

as $n_{k'} \to \infty$. This yields that $W_{(u,T)}f = g$ and hence by the closed graph theorem, $W_{(u,T)}$ is bounded on L_φ. $\square$

Theorem 2.2. *If u and f_T belong to $L^\infty(\mu)$ then $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$.*

Proof. Let $f \in L_\varphi$. Along the lines of computations made in [8, Theorem 3.1], we find that

$$\|W_{(u,T)}f\|_\varphi \leq k\|u\|_\infty \|f\|_\varphi,$$

where $k = \max(1, \|f_T\|_\infty)$. Therefore, $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$. □

Theorem 2.2 can also be obtained by using the boundedness of the multiplication operator M_u (see [9]) and the composition operator C_T (see [4]) to the fact that $W_{(u,T)} = C_T M_u$. Now, we consider the multiplication operator $M_u : L_\varphi \mapsto L(\mu)$ given by

$$M_u f = u \cdot f$$

for $f \in L_\varphi$. Then

$$\begin{aligned} \|W_{(u,T)}f\|_\varphi &= \inf\left\{\epsilon > 0 : \int_\Omega \varphi\left(\frac{|u \circ T \cdot f \circ T|}{\epsilon}\right) d\mu \leq 1\right\} \\ &= \inf\left\{\epsilon > 0 : \int_\Omega \varphi\left(\frac{|u \cdot f|}{\epsilon}\right) d\mu T^{-1} \leq 1\right\} \\ &= \|M_u f\|_{\varphi, \mu T^{-1}}, \end{aligned}$$

where M_u is the multiplication operator from the Orlicz space $L_{\varphi,\Omega,\mathcal{A},\mu}$ into the Orlicz space $L_{\varphi,\Omega,\mathcal{A},\mu T^{-1}}$.

Now, using the above observation and the result for multiplication operators in [9], we extend the results for the weighted composition operator on the Orlicz space L_φ as follows:

Theorem 2.3. *The linear transformation $W_{(u,T)} : L_\varphi \mapsto L(\mu)$, is a bounded operator on L_φ if and only if $u \in L^\infty(\mu T^{-1})$.*

Proof. Suppose $W_{(u,T)} : L_\varphi \mapsto L(\mu)$, is a bounded operator on L_φ. Assume on the contrary that, u is not essentially bounded with respect to the measure μT^{-1}, equivalently $u \circ T$ is not essentially bounded with respect to the measure μ. Then it is easy to verify (similarly as in [9, Theorem 2.1]) that, for each natural number n, $\chi_{E_n} \in L_\varphi$ and

$$\|W_{(u,T)}\chi_{E_n}\|_\varphi = \|M_u \chi_{E_n}\|_{\varphi,\mu T^{-1}} = \|M_{u\circ T}\chi_{E_n}\|_\varphi \geq n\, \|\chi_{E_n}\|_\varphi,$$

where $E_n = \{x \in \Omega : |u(T(x))| > n\}$. This contradicts the boundedness of $W_{(u,T)}$. Hence $u \in L^\infty(\mu T^{-1})$.

The converse follows easily as, if $u \in L^\infty(\mu T^{-1})$ i.e. $u \circ T \in L^\infty$ then by using [9, Theorem 2.1], $f \mapsto u \circ T \cdot f$ is a bounded operator on L_φ. Also $f \mapsto f \circ T$ is bounded on L_φ since $f_T \in L^\infty$. Hence $W_{(u,T)}$ is bounded on L_φ. □

Theorem 2.4. *Let $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$. Then*

1. *$W_{(u,T)}$ is a compact operator if and only if for each $\epsilon > 0$, $L_\varphi(u_\epsilon)$ is finite dimensional, where*

$$u_\epsilon = \{\omega \in \Omega : u(\omega) \geq \epsilon \text{ a.e. } \mu T^{-1}\}$$

and

$$L^{\varphi}(u_\epsilon) = \{f\chi_{u_\epsilon} : f \in L_\varphi\}.$$

(2) $W_{(u,T)}$ *has closed range if and only if there exists* $\delta > 0$ *such that* $u(\omega) \geq \delta$ *a.e. on the support of* u*, with respect to the measure* μT^{-1}*.*

(3) $W_{(u,T)}$ *is Fredholm if and only if there exists* $\delta > 0$ *such that* $u(\omega) \geq \delta$ *a.e. on* Ω*, with respect to the measure* μT^{-1}*.*

Proof. The proof follows by applying the ideas used in Theorem 2.1, Theorem 2.3 and the results [9, Theorems 3.1, 4.1, 4.2] which are extended on Lorentz spaces in [3]. □

Let φ, ψ be two complementary Orlicz functions. Now for each $g \in L_\psi$, define a bounded linear functional F_g on L_φ given by

$$F_g(f) = \int f \cdot g d\mu$$

for each $f \in L_\varphi$. Then the mapping $g \mapsto F_g$ is an isometry from L_ψ onto L_φ^* and hence the norm dual of L_φ can be identified with L_ψ (see [11]).

Theorem 2.5. *Let* $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$*. Then the adjoint* $W^*_{(u,T)}$ *of* $W_{(u,T)}$ *is given by*

$$W^*_{(u,T)} g \;=\; f_T \cdot u \cdot E(g) \circ T^{-1}$$

for each $g \in L_\psi$*.*

Proof. Let $A \in \mathcal{A}$ be such that $\mu(A) < \infty$, then for $g \in L_\psi$,

$$\begin{aligned}
(W^*_{(u,T)} F_g)(\chi_A) &= F_g(W_{(u,T)}\chi_A) \\
&= \int (W_{(u,T)}\chi_A) \cdot g d\mu \\
&= \int u \circ T \cdot \chi_A \circ T g d\mu \\
&= \int E(u \circ T \cdot g) \cdot \chi_A \circ T d\mu \\
&= \int u \circ T \cdot E(g) \cdot \chi_A \circ T d\mu \\
&= \int f_T \cdot u \cdot E(g) \circ T^{-1} \cdot \chi_A d\mu \\
&= (F_{f_T \cdot u \cdot E(g) \circ T^{-1}})(\chi_A).
\end{aligned}$$

Hence, $(W^*_{(u,T)} F_g) \;=\; F_{f_T \cdot u \cdot E(g) \circ T^{-1}}$. On identifying g with F_g, we find that $(W^*_{(u,T)} g) \;=\; f_T \cdot u \cdot E(g) \circ T^{-1}$, for each $g \in L_\psi$. □

Corollary 2.6. *Let* Ω *be a non-atomic measure space and* $W_{(u,T)} \in \mathfrak{B}(L_\psi)$*. Then the kernel of* $W_{(u,T)}$ *is either zero dimensional or infinite dimensional.*

Proof. For $f \in L_\psi$, on replacing g by $W_{(u,T)}f$ in the above theorem, we have

$$W^*_{(u,T)}(W_{(u,T)}f) \;=\; f_T \cdot u \cdot E(W_{(u,T)}f) \circ T^{-1} \;=\; f_T \cdot u^2 \cdot f \;= M_h f,$$

where $h \;=\; f_T \cdot u^2$. Hence, $W^*_{(u,T)}W_{(u,T)} \;=\; M_h$.
Also $\mathrm{Ker}W_{(u,T)} \;=\; L_\psi(S_{f_T \cdot u^2}) \;=\; L_\psi(S_{f_T \cdot u})$, where $S_{f_T \cdot u}$ denotes the support of $f_T \cdot u$ and

$$L_\psi(S_{f_T \cdot u}) = \{f \chi_{S_{f_T \cdot u}} : f \in L_\psi\}.$$

Thus, since Ω is non-atomic, if $\mu(S_{f_T \cdot u}) = 0$ then the kernel of $W_{(u,T)}$ is zero dimensional and if $\mu(S_{f_T \cdot u}) > 0$ then the kernel of $W_{(u,T)}$ is infinite dimensional. □

Using the characterization known for the Fredholm multiplication operators on Orlicz spaces, we have the following:

Corollary 2.7. *Let Ω be a non-atomic measure space and $W_{(u,T)} \in \mathfrak{B}(L_\psi)$. Then $W^*_{(u,T)}W_{(u,T)}$ is a Fredholm operator if and only if there exists $\delta > 0$ such that $|f_T \cdot u^2| \geq \delta$ a.e. on Ω.*

The only compact multiplication operator on the Orlicz space when the measure is non-atomic, is the zero operator. Using this observation, our next result which follows as a corollary to the Theorem 2.5, characterizes the compactness of the weighted composition operators on Orlicz spaces when the measure space under consideration is non-atomic.

Corollary 2.8. *Let Ω be a non-atomic measure space and $W_{(u,T)} \in \mathfrak{B}(L_\psi)$. Then $W_{(u,T)}$ is a compact operator if and only if $f_T \cdot u \;=\; 0$ a.e. on Ω.*

Proof. Suppose $W_{(u,T)}f$ is a compact weighted composition operator on L_ψ then $W^*_{(u,T)}W_{(u,T)}$ is a compact operator on L_ψ. So, M_h, where $h \;=\; f_T \cdot u^2$ is a compact operator on L_ψ. Hence, $h \;=\; f_T \cdot u^2 = 0$ a.e. on Ω, equivalently, $f_T \cdot u = 0$ a.e. on Ω. Conversely, if $f_T \cdot u = 0$ a.e. on Ω then $u \circ T \cdot f \circ T \;=\; 0$ for each $f \in L_\psi$ so that $W_{(u,T)}$ is a zero operator. Hence the result. □

Corollary 2.9. *Let Ω be a non-atomic measure space and $W_{(u,T)} \in \mathfrak{B}(L_\psi)$. Then $W = W^*_{(u,T)}W_{(u,T)}$ is a compact operator if and only if $W_{(u,T)}$ is a compact operator.*

Theorem 2.10. *If $W_{(u,T)} \in \mathfrak{B}(L_\varphi)$ has closed range and the co-dimension is finite then $W_{(u,T)}$ is surjective.*

Proof. Suppose on the contrary that, $W_{(u,T)}$ is not surjective and let $f_0 \in L_\varphi \setminus R(W_{(u,T)})$. Since $R(W_{(u,T)})$ is closed, we can find a function $g_0 \in L_\psi$, where ψ is the complementary Orlicz function to φ, such that

$$\int f_0 g_0 d\mu = 1 \quad \text{ and } \quad \int (W_{(u,T)}f) g_0 d\mu = 0$$

for all $f \in L_\varphi$. From the first equality, $\int \mathrm{Re}(f_0 g_0) d\mu = 1$. Hence the set

$$E_\epsilon = \{\omega \in \Omega : \mathrm{Re}(f_0 g_0)(\omega) \geq \epsilon\}$$

must have positive measure for some $\epsilon > 0$. Since μ is non-atomic, we can choose a sequence $\{E_n\}$ of subsets of E_ϵ with $0 < \mu(E_n) < \infty$ and $E_m \cap E_n = \emptyset$ $(m \neq n)$. Let $g_n = \chi_{E_n} g_0$. Then each $g_n \in L_\psi$ and is nonzero because

$$\mathrm{Re} \int f_0 g_n d\mu = \mathrm{Re} \int_{E_n} f_0 g_0 d\mu \ \geq \ \epsilon \mu(E_n) > 0.$$

Furthermore, for each $f \in L_\varphi$, $\chi_{E_n} f$ is in L_φ, and so

$$\begin{aligned}
(W^*_{(u,T)} g_n)(f) &= \int f \cdot f_T \cdot u \cdot E(g_n) \circ T^{-1} d\mu \\
&= \int_{E_n} f \cdot u \cdot E(g_0) \circ T^{-1} \cdot f_T d\mu \\
&= \int_{E_n} f \cdot u \cdot E(g_0) \circ T^{-1} d\mu T^{-1} \\
&= \int_{T^{-1}(E_n)} u \circ T \cdot f \circ T \cdot E(g_0) d\mu \\
&= \int_{T^{-1}(E_n)} E(g_0 \cdot u \circ T \cdot f \circ T) d\mu \\
&= \int_{T^{-1}(E_n)} g_0 \cdot u \circ T \cdot f \circ T d\mu \\
&= \int g_0 \cdot u \circ T \cdot f \circ T \cdot \chi_{E_n} \circ T d\mu \\
&= \int g_0 (W_{(u,T)}(\chi_{E_n} f)) d\mu = 0.
\end{aligned}$$

This implies $g_n \in \mathrm{Ker} W^*_{(u,T)}$. Thus the sequence $\{g_n\}$ forms a linearly independent subset of $\mathrm{Ker} W^*_{(u,T)}$. This contradicts the fact that co-dimension of $W_{(u,T)}$ is finite. Hence $W_{(u,T)}$ is surjective. □

Remark. Theorem 2.5, the corollaries based on this theorem and Theorem 2.10 all hold almost along the same lines of proof, when the spaces under consideration are Lorentz spaces $L(p, q)$, $1 < p < \infty, 1 < q < \infty$ or Lebesgue spaces L^p, $1 < p < \infty$.

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