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λ-CENTRAL BMO ESTIMATES FOR MULTILINEAR COMMUTATORS OF MARCINKIEWICZ OPERATOR ON CENTRAL MORREY SPACES

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Abstract. In this paper, we establish λ-central BMO estimates for the multilinear commutator related to the Marcinkiewicz operator in central Morrey spaces.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f) = bT(f) - T(bf).$$

A classical result of Coifman, Rochberb and Weiss (see [2]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. Since $BMO \subset \bigcap_{q>1} CBMO^q$ (see [3]), if we only assume $b \in CBMO^q$, or more generally $b \in CBMO^{q,\lambda}$ with $q > 1$, then $[b, T]$ may not be a bounded operator on $L^p(R^n)$. However, it has some boundedness properties on other spaces. As a matter of fact, Grafakos, Li and Yang (see [4]) considered the commutator with $b \in CBMO^q$ on Herz spaces for the first time. Alvarez, Guzmán-Partida and Lakey (see [1]) and Komori (see [5]) have obtained the λ-central BMO estimates for the commutators of a class of singular integral operators on central Morrey spaces. Motivated by these results, in this paper, we will establish λ-central BMO estimates for the multilinear commutator related to the Marcinkiewicz operator in central Morrey spaces.

2. Preliminaries and theorems

First, let us introduce some notations.

Definition 2.1. Let $0 < \lambda < \delta/n$, $0 < \delta < n$ and $1 < q < \infty$. A function $f \in L^q_{loc}(R^n)$ is said to belong to the λ-central bounded mean oscillation space

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$CBMO^{q,\lambda}(R^n)$ if

$$(2.1)\quad ||f||_{CBMO^{q,\lambda}} = \sup_{r>0}\left(\frac{1}{|B(0,r)|^{1+\lambda q}}\int_{B(0,r)}|f(x)-f_{B(0,r)}|^q dx\right)^{1/q} < \infty,$$

where $B = B(0,r) = \{x \in R^n : |x| < r\}$ and $f_{B(0,r)}$ is the mean value of f on $B(0,r)$.

For $b_j \in CBMO^{p_{j+1},\lambda_{j+1}}(R^n)(j = 1,\dots,m)$, set

$$||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} = \prod_{j=1}^{m} ||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1),\dots,\sigma(j)\}$ of $\{1,\dots,m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1,\dots,m\} \setminus \sigma$. For $\vec{b} = (b_1,\dots,b_m)$ and $\sigma = \{\sigma(1),\dots,\sigma(j)\} \in C_j^m$, set

$$\vec{b}_\sigma = (b_{\sigma(1)},\dots,b_{\sigma(j)}), \qquad b_\sigma = b_{\sigma(1)}\dots,b_{\sigma(j)}$$

and $||\vec{b}_\sigma||_{CBMO^{\vec{p},\vec{\lambda}}} = ||b_{\sigma(1)}||_{CBMO^{p_2,\lambda_2}}\dots||b_{\sigma(j)}||_{CBMO^{p_{j+1},\lambda_{j+1}}}$.

Remark 2.1. If two functions which differ by a constant are regarded as a function then the space $CBMO^{q,\lambda}$ becomes a Banach space. The space $CBMO^{q,\lambda}(R^n)$ when $\lambda = 0$ is just the space $CBMO(R^n)$ defined as follows:

$$||f||_{CBMO_q} = \sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f(x)-f_{B(0,r)}|^q dx\right)^{1/q} < \infty.$$

Apparently, (1) is equivalent to the following condition (see [3]):

$$||f||_{CBMO^{q,\lambda}} = \sup_{r>0}\inf_{c\in\mathbf{C}}\left(\frac{1}{|B(0,r)|^{1+\lambda q}}\int_{B(0,r)}|f(x)-c|^q dx\right)^{1/q} < \infty.$$

Definition 2.2. Let $\lambda \in \mathbf{R}$ and $1 < q < \infty$. The central Morrey space $\dot{B}^{q,\lambda}(R^n)$ is defined by

$$(2.2)\qquad ||f||_{\dot{B}^{q,\lambda}} = \sup_{r>0}\left(\frac{1}{|B(0,r)|^{1+\lambda q}}\int_{B(0,r)}|f(x)|^q dx\right)^{1/q} < \infty.$$

Remark 2.2. It follows from (1) and (2) that $\dot{B}^{q,\lambda}(R^n)$ is a Banach space continuously included in $CBMO^{q,\lambda}(R^n)$. We denote by $CMO^{q,\lambda}(R^n)$ and $B^{q,\lambda}(R^n)$ the inhomogeneous versions of the λ-central bounded mean oscillation space and the central Morrey space by taking the supremum over $r \ge 1$ in Definition 2.1 and Definition 2.2 instead of $r > 0$ there. Obviously, $CBMO^{q,\lambda}(R^n) \subset CMO^{q,\lambda}(R^n)$ for $\lambda < \delta/n$ and $1 < q < \infty$, and $\dot{B}^{q,\lambda}(R^n) \subset B^{q,\lambda}(R^n)$ for $\lambda \in \mathbf{R}$ and $1 < q < \infty$.

Remark 2.3. When $\lambda_1 < \lambda_2$, it follows from the property of monotone functions that $B^{q,\lambda_1}(R^n) \subset B^{q,\lambda_2}(R^n)$ and $CMO^{q,\lambda_1}(R^n) \subset CMO^{q,\lambda_2}(R^n)$ for $1 < q < \infty$. If $1 < q_1 < q_2 < \infty$, then by Hölder's inequality, we know that $\dot{B}^{q_2,\lambda}(R^n) \subset$

$\dot{B}^{q_1,\lambda}(R^n)$ for $\lambda \in \mathbf{R}$ and $CBMO^{q_2,\lambda} \subset CBMO^{q_1,\lambda}$, $CMO^{q_2,\lambda}(R^n) \subset CMO^{q_1,\lambda}(R^n)$ for $0 < \lambda < \delta/n$.

Definition 2.3. Let $0 < \delta < n$, $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n (that is $\Omega(\alpha x) = \Omega(x)$ for any $0 \neq \alpha \in R$ and $x \in R^n$) such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x-y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu^{\vec{b}}_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F^{\vec{b}}_{t,\delta}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F^{\vec{b}}_{t,\delta}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] f(y)dy.$$

Set

$$F_{t,\delta}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy,$$

we also define

$$\mu_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F_{t,\delta}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator (see [12]).

Let H be the space $H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3} \right)^{1/2} < \infty \right\}$. Then, it is clear that

$$\mu_{\Omega,\delta}(f)(x) = ||F_{t,\delta}(f)(x)|| \text{ and } \mu^{\vec{b}}_{\Omega,\delta}(f)(x) = ||F^{\vec{b}}_{t,\delta}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $\mu^{\vec{b}}_{\Omega,\delta}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [3, 7-9]). The purpose of this paper is to study the boundedness properties for the multilinear commutator $\mu^{\vec{b}}_{\Omega,\delta}$ in central Morrey spaces.

Now we state our theorems as follows.

Theorem 2.1. *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$. If $\lambda_1 < -\delta/n$ and $\lambda_2 = \lambda_1 + \delta/n$, then $\mu_{\Omega,\delta}$ is bounded from $\dot{B}^{p,\lambda_1}(R^n)$ to $\dot{B}^{q,\lambda_2}(R^n)$.*

Theorem 2.2. *Let $0 < \delta < n$, $1 < p_u < n/\delta$ $(1 \leq u \leq m+1)$, $1/p_1 + 1/p_2 + \cdots 1/p_{m+1} < 1$ and $1/q = 1/p_1 + 1/p_2 + \cdots + 1/p_{m+1} - \delta/n$. Suppose $0 < \lambda_i < \delta/n$ $(i = 2, 3, \ldots, m+1)$, $\lambda_1 < -\lambda_2 - \lambda_3 - \cdots - \lambda_{m+1} - \delta/n$ and $\lambda = \lambda_1 + \lambda_2 + \cdots \lambda_{m+1} + \delta/n$. If $b_j \in CBMO^{p_{j+1},\lambda_{j+1}}(R^n)$, for $j = 1, \ldots, m$, then $\mu^{\vec{b}}_{\Omega,\delta}$ is bounded from $\dot{B}^{p_1,\lambda_1}(R^n)$ to $\dot{B}^{q,\lambda}(R^n)$, and the following inequality holds:*

$$||\mu^{\vec{b}}_{\Omega,\delta}(f)||_{\dot{B}^{q,\lambda}} \leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}}.$$

3. Proof of theorems

To prove the theorems, we need the following lemmas.

Lemma 3.1. (see [8]). *Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then $\mu_{\Omega,\delta}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Proof. By $\Omega(x) \leq C$ and Minkowski's inequality, we have

$$\begin{aligned}\mu_{\Omega,\delta}(f)(x) &\leq \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}|f(y)|\left(\int_{|x-y|}^{\infty}\frac{dt}{t^3}\right)^{1/2} dy \\ &\leq C\int_{R^n}\frac{1}{|x-y|^{n-\delta}}|f(y)|dy.\end{aligned}$$

Thus, the lemma follows from the boundedness of the fractional integral operator from $L^p(R^n)$ to $L^q(R^n)$ (see [11]). □

Lemma 3.2. *Let $0 < \delta < n$, $1 < p < n/\delta$, $\lambda > 0$. Suppose $b \in CBMO^{p,\lambda}(R^n)$, then*

$$|b_{2^{k+1}B} - b_B| \leq C||b||_{CBMO^{p,\lambda}} k|2^{k+1}B|^{\lambda} \text{ for } k \geq 1.$$

Proof.

$$\begin{aligned}|b_{2^{k+1}B} - b_B| &\leq \sum_{j=0}^{k}|b_{2^{j+1}B} - b_{2^jB}| \leq \sum_{j=0}^{k}\frac{1}{|2^jB|}\int_{2^jB}|b(y) - b_{2^{j+1}B}|dy \\ &\leq C\sum_{j=0}^{k}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^p dy\right)^{1/p} \\ &\leq C||b||_{CBMO^{p,\lambda}}\sum_{j=0}^{k}|2^{j+1}B|^{\lambda} \\ &\leq C||b||_{CBMO^{p,\lambda}}(k+1)|2^{k+1}B|^{\lambda} \leq C||b||_{CBMO^{p,\lambda}}k|2^{k+1}B|^{\lambda}.\end{aligned}$$

□

Proof of Theorem 2.1. Let f be a function in $\dot{B}^{p,\lambda_1}(R^n)$. For fixed $r > 0$, set $B = B(0,r)$ and $B^c = R^n \setminus B(0,r)$. We write

$$\begin{aligned}\left(\frac{1}{|B|^{1+\lambda_2 q}}\int_B |\mu_{\Omega,\delta}(f)(x)|^q dx\right)^{1/q} &\leq \left(\frac{1}{|B|^{1+\lambda_2 q}}\int_B |\mu_{\Omega,\delta}(f\chi_B)(x)|^q dx\right)^{1/q} + \\ &+ \left(\frac{1}{|B|^{1+\lambda_2 q}}\int_B |\mu_{\Omega,\delta}(f\chi_{B^c})(x)|^q dx\right)^{1/q} \\ &= I + II.\end{aligned}$$

For I, denoting $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundedness of $\mu_{\Omega,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$, we have

$$\begin{aligned} I &\le C|B|^{-1/q-\lambda_2}\left(\int_{2B}|f(x)|^p dx\right)^{1/p} \\ &\le C|B|^{-1/q-\lambda_2}|B|^{1/p+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}} \\ &\le C||f||_{\dot{B}^{p,\lambda_1}}. \end{aligned}$$

For II, by $\Omega(x) \le C$ and Minkowski's inequality, we have

$$\begin{aligned} \mu_{\Omega,\delta}(f\chi_{B^c})(x) &\le \left(\int_0^\infty |\int_{|x-y|\le t}\frac{\Omega(x-y)f(y)}{|x-y|^{n-1-\delta}}dy|^2\frac{dt}{t^3}\right)^{1/2} \\ &\le C\int_{B^c}\frac{1}{|x-y|^{n-1-\delta}}f(y)\left(\int_{|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy \\ &\le \int_{B^c} f(y)|x-y|^{-n+\delta}dy, \end{aligned}$$

then, using Hölder's inequality and noticing $x \in B$ and $y \in B^c$, we get

$$\begin{aligned} |\mu_{\Omega,\delta}(f\chi_{B^c})(x)| &\le \sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB}|x-y|^{-n+\delta}|f(y)|dy \\ &\le C\sum_{k=1}^\infty |2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|f(y)|^p dy\right)^{1/p}|2^{k+1}B|^{1-1/p} \\ &\le C\sum_{k=1}^\infty |2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}}|2^{k+1}B|^{1-1/p} \\ &\le C||f||_{\dot{B}^{p,\lambda_1}}\sum_{k=1}^\infty 2^{k(\delta+n\lambda_1)}|B|^{\delta/n+\lambda_1} \\ &\le C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta/n+\lambda_1}, \end{aligned}$$

thus

$$II \le C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta/n+\lambda_1}|B|^{-1/q-\lambda_2}|B|^{1/q} \le C||f||_{\dot{B}^{p,\lambda_1}}.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let f be a function in $\dot{B}^{p_1,\lambda_1}(R^n)$. When $m = 1$, set $(b_1)_B = |B|^{-1}\int_B b_1(x)dx$ and note that

$$\mu^{b_1}_{\Omega,\delta}(f)(x) = (b_1(x) - (b_1)_B)\mu_{\Omega,\delta}(f)(x) - \mu_{\Omega,\delta}((b_1 - (b_1)_B)f)(x),$$

we have

$$\begin{aligned}
&\left(\frac{1}{|B|^{1+\lambda q}}\int_B|\mu_{\Omega,\delta}^{b_1}(f)(x)|^q dx\right)^{1/q}\\
&\le\left(\frac{1}{|B|^{1+\lambda q}}\int_B|(b_1(x)-(b_1)_B)(\mu_{\Omega,\delta}(f\chi_B))(x)|^q dx\right)^{1/q}\\
&\quad+\left(\frac{1}{|B|^{1+\lambda q}}\int_B|(b_1(x)-(b_1)_B)(\mu_{\Omega,\delta}(f\chi_{B^c}))(x)|^q dx\right)^{1/q}\\
&\quad+\left(\frac{1}{|B|^{1+\lambda q}}\int_B|\mu_{\Omega,\delta}((b_1-(b_1)_B)f\chi_B)(x)|^q dx\right)^{1/q}\\
&\quad+\left(\frac{1}{|B|^{1+\lambda q}}\int_B|\mu_{\Omega,\delta}((b_1-(b_1)_B)f\chi_{B^c})(x)|^q dx\right)^{1/q}\\
&=J_1+J_2+J_3+J_4.
\end{aligned}$$

For J_1, taking $1<p_1<n/\delta$ and t such that $1/t=1/p_1-\delta/n$ and $1/q=1/p_2+1/t$, by Hölder's inequality and the boundedness of $\mu_{\Omega,\delta}$ from $L^{p_1}(R^n)$ to $L^t(R^n)$, we know

$$\begin{aligned}
J_1&\le|B|^{-1/q-\lambda}\left(\int_B|b_1(x)-(b_1)_B|^{p_2}dx\right)^{1/p_2}\left(\int_B|\mu_{\Omega,\delta}(f\chi_B)(x)|^t\right)^{1/t}\\
&\le C|B|^{-1/q-\lambda}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\left(\int_B|f(x)|^{p_1}dx\right)^{1/p_1}\\
&\le C|B|^{-1/q-\lambda}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For J_2, using the fact $|\mu_{\Omega,\delta}(f\chi_{B^c})(x)|\le C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta+\lambda_1}$ from the proof of Theorem 2.1 and by Hölder' inequality, we get

$$\begin{aligned}
J_2&\le C|B|^{-1/q-\lambda}|B|^{\delta/n+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}}\left(\int_B|b_1(x)-(b_1)_B|^{p_2}dx\right)^{1/p_2}|B|^{1/q-1/p_2}\\
&\le C|B|^{-1/q-\lambda}|B|^{\delta/n+\lambda_1}||f||_{\dot{B}^{p,\lambda_1}}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}|B|^{1/q-1/p_2}\\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For J_3, taking $1<l<n/\delta$ and q such that $1/q=1/l-\delta/n$ and $1/l=1/p_1+1/p_2$, by the boundedness of $\mu_{\Omega,\delta}$ from L^l to L^q and Hölder's inequality, we have

$$\begin{aligned}
J_3&\le C|B|^{-1/q-\lambda}\left(\int_B|(b_1(x)-(b_1)_Bf(x)|^l dx\right)^{1/l}\\
&\le C|B|^{-1/q-\lambda}\left(\int_B|b_1(x)-(b_1)_B|^{p_2}dx\right)^{1/p_2}\left(\int_B|f(x)|^{p_1}dx\right)^{1/p_1}\\
&\le C|B|^{-1/q-\lambda}|B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{1/p_1+\lambda_1}\\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For J_4, using Hölder's inequality, Lemma 3.2 and noticing that $x \in B$, $y \in B^c = R^n \setminus B$, $\lambda_2 > 0$ and $\lambda_1 < -\lambda_2 - \delta/n$, we have

$$\begin{aligned}
&|\mu_{\Omega,\delta}((b_1-(b_1)_B)f\chi_{B^c})(x)| \\
\le& C\sum_{k=1}^{\infty}|2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_B|^{p_2}dy\right)^{1/p_2} \\
&\times\left(\int_{2^{k+1}B}|f(y)|^{p_1}dy\right)^{1/p_1}|2^{k+1}B|^{1-1/p_1-1/p_2} \\
\le& C||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}|2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p_1+\lambda_1}|2^{k+1}B|^{1-1/p_1-1/p_2} \\
&\times\left[\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_{2^{k+1}B}|^{p_2}dy\right)^{1/p_2}+|(b_1)_{2^{k+1}B}-(b_1)_B||2^{k+1}B|^{1/p_2}\right] \\
\le& C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}k|2^kB|^{\lambda_1+\lambda_2+\delta/n} \\
\le& C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}2^{k(n\lambda_1+n\lambda_2+\delta)}|B|^{\lambda_1+\lambda_2+\delta/n} \\
\le& C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n},
\end{aligned}$$

thus

$$\begin{aligned}
J_4 &\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n}|B|^{-1/q-\lambda}|B|^{1/q} \\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

This completes the proof of the case $m=1$.
When $m>1$, set $\vec{b}_B=((b_1)_B,\ldots,(b_m)_B)$, where $(b_j)_B=|B|^{-1}\int_B b_j(x)dx$, $1\le j\le m$, we have

$$\begin{aligned}
&F_{t,\delta}^{\vec{b}}(f)(x) \\
=&\int_{|x-y|\le t}\left[\prod_{j=1}^{m}((b_j(x)-(b_j)_B)-(b_j(y)-(b_j)_B))\right]f(y)\Omega(x-y)|x-y|^{1-n+\delta}dy \\
=&\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma\int_{|x-y|\le t}(b(y)-b_B)_{\sigma^c}f(y)\Omega(x-y)|x-y|^{1-n+\delta}dy \\
=&\prod_{j=1}^{m}(b_j(x)-(b_j)_BF_{t,\delta}(f)(x)+(-1)^mF_{t,\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_B)f)(x) \\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma F_{t,\delta}^{b_{\sigma^c}}(f)(x),
\end{aligned}$$

then

$$\begin{aligned}
&\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = ||F_{t,\delta}^{\vec{b}}(f)(x)|| \\
\leq & || \prod_{j=1}^{m}(b_j(x)-(b_j)_B)F_{t,\delta}(f)(x)|| + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_B)_\sigma F_{t,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|| \\
&+ ||F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)f)(x)|| \\
\leq & \prod_{j=1}^{m}\Big(b_j(x)-(b_j)_B\Big)\mu_{\Omega,\delta}(f)(x) + (-1)^m\mu_{\Omega,\delta}\Big(\prod_{j=1}^{m}\Big(b_j-(b_j)_B\Big)_B\Big)f)(x) \\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_B)_\sigma\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f)(x),
\end{aligned}$$

thus

$$\begin{aligned}
&\left(\frac{1}{|B|^{1+\lambda q}}\int_B|\mu_{\Omega,\delta}^{\vec{b}}(f)(x)|^q dx\right)^{1/q} \\
\leq & \left(\frac{1}{|B|^{1+\lambda q}}\int_B|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)(\mu_{\Omega,\delta}(f\chi_B))(x)|^q dx\right)^{1/q} \\
&+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)(\mu_{\Omega,\delta}(f\chi_{(B)^c}))(x)|^q dx\right)^{1/q} \\
&+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B|\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)f\chi_B)(x)|^q dx\right)^{1/q} \\
&+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B|\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)f\chi_{(B)^c})(x)|^q dx\right)^{1/q} \\
&+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-b_B)_\sigma\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f\chi_B)(x)\right|^q dx\right)^{1/q} \\
&+ \left(\frac{1}{|B|^{1+\lambda q}}\int_B\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-b_B)_\sigma\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f\chi_{(B)^c})(x)\right|^q dx\right)^{1/q} \\
= & \nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_6.
\end{aligned}$$

For ν_1, taking $1 < p_1 < n/\delta$ and t such that $1/t = 1/p_1 - \delta/n$ and $1/q = 1/p_2 + \cdots + 1/p_{m+1} + 1/t$, by Hölder's inequality and the boundedness of $\mu_{\Omega,\delta}$

from $L^{p_1}(R^n)$ to $L^t(R^n)$, we have

$$\begin{aligned}\nu_1 \leq & |B|^{-1/q-\lambda}\prod_{j=1}^{m}\left(\int_B |b_j(x)-(b_j)_B|^{p_{j+1}}dx\right)^{1/p_{j+1}} \\ & \times\left(\int_B |(\mu_{\Omega,\delta}(f\chi_B))(x)|^t dx\right)^{1/t} \\ \leq & C|B|^{-1/q-\lambda}\prod_{j=1}^{m}|B|^{1/p_{j+1}+\lambda_{j+1}}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}\left(\int_B |f(x)|^{p_1}dx\right)^{1/p_1} \\ \leq & C|B|^{-1/q-\lambda}\prod_{j=1}^{m}|B|^{1/p_{j+1}+\lambda_{j+1}}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}} \\ \leq & C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.\end{aligned}$$

For ν_2, using the fact $|\mu_{\Omega,\delta}(f\chi_{B^c})(x)| \leq C||f||_{\dot{B}^{p,\lambda_1}}|B|^{\delta/n+\lambda_1}$ from the proof of Theorem 2.1 and the Hölder's inequality, we get

$$\begin{aligned}\nu_2 \leq & C|B|^{-1/q-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\delta/n+\lambda_1}\prod_{i=1}^{m}\left(\int_B |b_i(x)-(b_i)_B|^{p_{i+1}}dx\right)^{1/p_{i+1}} \\ & \times|B|^{1/q-1/p_2-\cdots-1/p_{m+1}} \\ \leq & C|B|^{-1/q-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\delta/n+\lambda_1}\prod_{i=1}^{m}|B|^{1/p_{i+1}+\lambda_{i+1}}||b_i||_{CBMO^{p_{i+1},\lambda_{i+1}}} \\ & \times|B|^{1/q-1/p_2-\cdots-1/p_{m+1}} \\ \leq & C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.\end{aligned}$$

For ν_3, taking $1 < l < n/\delta$ and q such that $1/q = 1/l - \delta/n$ and $1/l = 1/p_1 + \cdots + 1/p_{m+1}$, by the boundedness of $\mu_{\Omega,\delta}$ from L^l to L^q and the Hölder's inequality, we have

$$\begin{aligned}\nu_3 \leq & \ C|B|^{-1/q-\lambda}\left(\int_B |\prod_{j=1}^{m}(b_j(x)-(b_j)_B)f(x)|^l dx\right)^{1/l} \\ \leq & \ C|B|^{-1/q-\lambda}\prod_{j=1}^{m}\left(\int_B |b_j(x)-(b_j)_B|^{p_{j+1}}dx\right)^{1/p_{j+1}}\left(\int_B |f(x)|^{p_1}dx\right)^{1/p_1} \\ \leq & \ C|B|^{-1/q-\lambda}\prod_{j=1}^{m}|B|^{1/p_{j+1}+\lambda_{j+1}}||b_i||_{CBMO^{p_{j+1},\lambda_{j+1}}}|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}} \\ \leq & \ C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.\end{aligned}$$

For ν_4, by Hölder's inequality, Lemma 3.2 and noticing that $x \in B$, $y \in B^c$, $\lambda_j > 0$ $(2 \leq j \leq m+1)$ and $\lambda_1 < -\lambda_2 - \cdots - \lambda_{m+1} < \delta/n$, we have

$$
\begin{aligned}
&|\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)f\chi_{(B)^c})(x)| \\
\leq &\sum_{k=0}^{\infty}\int_{2^{k+1}B\backslash 2^kB}|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||x-y|^{-n+\delta}|f(y)|dy \\
\leq &C\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_B|^{p_2}dy\right)^{1/p_2}\cdots \\
&\times\left(\int_{2^{k+1}B}|b_m(y)-(b_m)_B|^{p_{m+1}}dy\right)^{1/p_{m+1}} \\
&\times\left(\int_{2^{k+1}B}|f(y)|^{p_1}dy\right)^{1/p_1}|2^{k+1}B|^{1-1/p_1-1/p_2\cdots-1/p_{m+1}} \\
\leq &C||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=0}^{\infty}|2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p_1+\lambda_1}|2^{k+1}B|^{1-1/p_1-1/p_2-\cdots 1/p_{m+1}} \\
&\times k|2^{k+1}B|^{1/p_2+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\cdots \\
&\times k|2^{k+1}B|^{1/p_{m+1}+\lambda_{m+1}}||b_m||_{CBMO^{p_{m+1},\lambda_{m+1}}} \\
\leq &C||f||_{\dot{B}^{p_1,\lambda_1}}\prod_{j=1}^{m}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}\sum_{k=1}^{\infty}k^m|2^{k+1}B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n} \\
\leq &C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}k^m 2^{kn(\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n)}|B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n} \\
\leq &C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n},
\end{aligned}
$$

thus

$$
\begin{aligned}
\nu_4 &\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\cdots+\lambda_{m+1}+\delta/n}|B|^{-1/q-\lambda}|B|^{1/q} \\
&\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For ν_5, taking $1<s<n/\delta$ and t such that $1/t=1/s-\delta/n$ and $1/q=1/p_3+1/t$, $1/s=1/p_1+1/p_2$, by the boundedness of $\mu_{\Omega,\delta}$ from L^t to L^s and Hölder's inequality, we have

$$
\begin{aligned}
\nu_5 \leq &C|B|^{-1/q-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} \\
&\times\left(\int_B|(b(x)-b_B)_\sigma|^{p_3}dx\right)^{1/p_3}\left(\int_B|\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f\chi_B)(x)|^t dx\right)^{1/t}
\end{aligned}
$$

$$\begin{aligned}
&\leq C|B|^{-1/q-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\int_B|(b(x)-b_B)_\sigma|^{p_3}dx\right)^{1/p_3}\left(\int_B|(b-b_B)_{\sigma^c}|^{p_2}dx\right)^{1/p_2}\\
&\times\left(\int_B|f(x)|^{p_1}dx\right)^{1/p_1}\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\Big\{|B|^{-1/q-\lambda}|B|^{1/p_3+\lambda_3}||\vec{b}_\sigma||_{CBMO^{p_3,\lambda_3}}|B|^{1/p_2+\lambda_2}||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}\\
&\times|B|^{1/p_1+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\Big\}\\
&\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}$$

For ν_6, by Hölder's inequality, Lemma 3.2 and noticing that $x\in B$, $y\in B^c$, $\lambda_j>0$ $(2\leq j\leq m+1)$ and $\lambda_1<-\lambda_2-\delta/n$, we have

$$\begin{aligned}
&|\mu_{\Omega,\delta}((b-b_B)_{\sigma^c}f\chi_{(B)^c})(x)|\\
&\leq\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|(b(z)-b_B)_{\sigma^c}||x-y|^{-n+\delta}|f(y)|dy\\
&\leq C\sum_{k=1}^{\infty}|2^kB|^{\delta/n-1}\left(\int_{2^{k+1}B}|(b(y)-b_B)_{\sigma^c}|^{p_2}dz\right)^{1/p_2}\\
&\quad\times\left(\int_{2^{k+1}B}|f(y)|^{p_1}dy\right)^{1/p_1}|2^{k+1}B|^{1-1/p_1-1/p_2}\\
&\leq C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\\
&\quad\sum_{k=1}^{\infty}|2^kB|^{\delta/n-1}|2^{k+1}B|^{1/p_1+\lambda_1}k^m|2^{k+1}B|^{1/p_2+\lambda_2}|2^{k+1}B|^{1-1/p_1-1/p_2}\\
&\leq C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}k^m|2^kB|^{\lambda_1+\lambda_2+\delta/n}\\
&\leq C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}k^m2^{kn(\lambda_1+\lambda_2+\delta/n)}|B|^{\lambda_1+\lambda_2+\delta/n}\\
&\leq C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n},
\end{aligned}$$

thus,

$$\begin{aligned}
\nu_6\leq&C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|B|^{-1/q-\lambda}||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}|B|^{\lambda_1+\lambda_2+\delta/n}\\
&\times\left(\int_B|(b(x)-b_B)_\sigma|^{p_3}dx\right)^{1/p_3}|B|^{1/q-1/p_3}
\end{aligned}$$

$$\leq C||\vec{b}_{\sigma^c}||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}||\vec{b}_\sigma||_{CBMO^{p_3,\lambda_3}}|B|^{-1/q-\lambda}|B|^{\lambda_1+\lambda_2+\delta/n} \times |B|^{1/p_3+\lambda_3}|B|^{1/q-1/p_3}$$
$$\leq C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.$$

This completes the total proof of the Theorem 2.2. □

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