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ON THE CONVOLUTIONS OF FOURIER-TYPE TRANSFORMS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. Some convolutions of Fourier-type transforms are introduced. The poly-convolution of the Fourier cosine integral transforms is formulated and its properties are studied. Its application to solving an integral equation and a system of integral equations is presented.

1. Introduction

Generalized convolutions and poly-convolutions of different integral transforms and their applications attract an active research in the last few years. For some recent works and surveys on the subject we refer to [2]–[15], [18], [19] and the references therein.

In this paper, we present some new convolutions of Fourier-type integral transforms which are generalizations of the related classical ones. We study in detail the poly-convolution for the one-dimensional Fourier cosine transforms and apply it to solving an integral equation and a system of integral equations.

This paper is organized as follows. In the next section, we introduce new convolutions and poly-convolutions of the Fourier-type transforms. In Section 3 we study the convolution and poly-convolution for the two-dimensional Fourier cosine transforms. In Section 4, we introduce the poly-convolution for the one-dimensional Fourier cosine transforms and in the last section we apply this notion to solving an integral equation and a system of integral equations.

2. Convolutions of Fourier-type transforms

Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n, n \geq 1$ and A be an $n \times n$-matrix with $\det A \neq 0$. We define $x \cdot y := x_1y_1 + x_2y_2 + \cdots + x_ny_n$, which is xy^T in the matrix theory. Here the upper-index T means the matrix transpose. Similarly, we define Ax as Ax^T in the matrix theory.

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For a function $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform by ([16])

$$F[f](\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int\limits_{\mathbb{R}^n} f(x) e^{-x\cdot\xi} dx.$$

It is well-known that this definition is justified and if $F[f] \in L^1(\mathbb{R}^n)$, then we have the inversion formula

$$f(x) = F^{-1}[F[f]](x) = \frac{1}{\sqrt{2\pi}^n} \int\limits_{\mathbb{R}^n} F[f](\xi) e^{x\cdot\xi} dx.$$

If $f \in L^2(\mathbb{R}^n)$, then $F[f] \in L^2(\mathbb{R}^n)$ and therefore the above inversion formula is justified. In this section, for simplicity, we suppose that all functions are in $L^2(\mathbb{R}^n)$.

For a vector $a \in \mathbb{R}^n$ we can easily prove that

$$F[f(A\cdot -a)](\xi) = \frac{e^{ia\cdot\xi}}{|\det A|} F[f]\Big((A^{-1})^T \xi\Big). \tag{2.1}$$

Hence if there are given two functions f and g in $L^2(\mathbb{R}^n)$, two invertible $n \times n$ matrices A and B and two vectors a and b in $\mathbb{R}^n$, we can define the *generalized convolution* of f and g with respect to the pairs $(A, a), (B, b)$ by

$$f \underset{(A,a),(B,b)}{*} g(x) := \int\limits_{\mathbb{R}^n} f(Ay - a) g(B(x - y) - b) dy, \tag{2.2}$$

which is well defined and belongs to $L^2(\mathbb{R}^n)$.

Taking the Fourier transform of the both sides of (2.2) and using (2.1), we obtain

$$F[f \underset{(A,a),(B,b)}{*} g](\xi) = \frac{e^{i(a+b)\cdot\xi}}{|\det A||\det B|} F[f]\Big((A^{-1})^T\xi\Big) \cdot F[g]\Big((B^{-1})^T\xi\Big), \tag{2.3}$$

which can be considered as the *factorization equality* of the generalized convolution (2.2).

The convolution (2.2) is a slight generalization of the classic convolution when A and B are the identity matrix I, $a = b = 0$, however, it has some interesting consequences. First, we note that the case $A = I, a = 0$ is rather special. In fact, for this case

$$f \underset{(I,0),(B,b)}{*} g(x) := \int\limits_{\mathbb{R}^n} f(y) g(B(x - y) - b) dy, \tag{2.4}$$

which we denote simply by $f \underset{(B,b)}{*} g(x)$. Now, from these definitions we can define the so-called poly-convolutions. In fact, let further be given a function $h \in L^2(\mathbb{R}^n)$, an invertible $n \times n$ matrix C and a vector $c \in \mathbb{R}^n$. We can define

$$(f \underset{(A,b),(B,b)}{*} g) \underset{(C,c)}{*} h(x). \tag{2.5}$$

Taking the Fourier transform of the last expression and using (2.1), we get

$$F[(f \underset{(A,b),(B,b)}{*} g) \underset{(C,c)}{*} h](\xi) = F[(f \underset{(A,b),(B,b)}{*} g)](\xi) \cdot \frac{e^{ic\cdot\xi}}{|\det C|} F[h]\Big((C^{-1})^T\xi\Big)$$

$$= \frac{e^{i(a+b)\cdot\xi}}{|\det A||\det B|} F[f]\Big((A^{-1})^T\xi\Big) \cdot F[g]\Big((B^{-1})^T\xi\Big) \cdot \frac{e^{ic\cdot\xi}}{|\det C|} F[h]\Big((C^{-1})^T\xi\Big)$$

(2.6)

$$= \frac{e^{i(a+b+c)\cdot\xi}}{|\det A||\det B||\det C|} F[f]\Big((A^{-1})^T\xi\Big) \cdot F[g]\Big((B^{-1})^T\xi\Big) \cdot F[h]\Big((C^{-1})^T\xi\Big).$$

From the last equality we see that

$$(f \underset{(A,b),(B,b)}{*} g) \underset{(C,c)}{*} h(x) = f \underset{(A,b)}{*} (g \underset{(B,b),(C,c)}{*} h)(x). \tag{2.7}$$

Some consequences of this poly-convolution can be derived, however, the calculations are boing and we do not present them here. We now show that the above notion of generalized convolution can generate many other types of convolution. Let f_1 and f_2 be two functions in $L^2(\mathbb{R}^n)$, A_1, A_2 and a_1, a_2 be respectively invertible $n \times n$ matrices and vectors in $\mathbb{R}^n$. Consider the expression

$$G(x) = f_1 \underset{(A_1,a_1),(B,b)}{*} g(x) + f_2 \underset{(A_2,a_2),(B,b)}{*} g(x). \tag{2.8}$$

We have

$$F[G](\xi) = \frac{e^{ib\cdot\xi}}{|\det b|} \left(\frac{e^{ia_1\cdot\xi}}{|\det A_1|} F[f_1]\Big((A_1^{-1})^T\Big) + \frac{e^{ia_2\cdot\xi}}{|\det A_2|} F[f_2]\Big((A_2^{-1})^T\Big) \right). \tag{2.9}$$

From this, we will see that different choices of the functions f_1 and f_2, A_1, A_2 and a_1, a_2 lead to various types of convolution.

For example, let $f_1 = f_2 = f, A_1 = A_2 = A$, and $a_1 = -a_2 = a$, we have

$$F[G](\xi) = 2\frac{\cos(a\cdot\xi)e^{ib\cdot\xi}}{|\det A||\det B|} F[f]\Big((A^{-1})^T\xi\Big) \cdot F[g]\Big((B^{-1})^T\xi\Big). \tag{2.10}$$

Thus, we defined a new convolution

$$f \overset{1}{\underset{(A,a),(B,b)}{*}} g(x) = \int_{\mathbb{R}^n} \Big(f(Ay-a) + f(Ay+a)\Big) g(B(x-y)-b)dy \tag{2.11}$$

which takes (2.10) as the factorization identity.

If $f_1 = f_2 = f, A_1 = A = -A_2$ and $a_1 = a_2 = a$, then

(2.12)

$$F[G](\xi) = \frac{2}{\sqrt{2\pi}^n} \frac{e^{i(a+b)\cdot\xi}}{|\det A||\det B|} \int_{\mathbb{R}^n} \cos\Big(x\cdot(A^{-1})^T\xi\Big) f(x)dx F[g]\Big((B^{-1})^T\xi\Big)$$

which can be considered as the factorization identity of the convolution

$$f \overset{2}{\underset{(A,a),(B,b)}{*}} g(x) = \int_{\mathbb{R}^n} \Big(f(Ay-a) + f(-Ay-a)\Big) g(B(x-y)-b)dy. \tag{2.13}$$

Here

$$\frac{1}{\sqrt{2\pi}^n}\frac{1}{|\det A|}\int_{\mathbb{R}^n}\cos\Big(x\cdot(A^{-1})^T\xi\Big)f(x)dx$$

can be considered as a type of Fourier cosine transform. So, the convolution (2.13) can be considered as that of two different Fourier-type transforms.

Similarly, if $f_1 = -f_2 = f, A_1 = A_2 = A$ and $a_1 = -a_2 = a$, then

$$F[G](\xi) = 2i\sin(a\cdot\xi)\frac{e^{ib\cdot\xi}}{|\det A||\det B|}F[f]\Big((A^{-1})^T\xi\Big)\cdot F[g]\Big((B^{-1})^T\xi\Big), \tag{2.14}$$

which is the factorization identity for the convolution

$$f\underset{(A,a),(B,b)}{\overset{3}{*}}g(x) = \int_{\mathbb{R}^n}\Big(f(Ay-a) - f(Ay+a)\Big)g(B(x-y)-b)dy. \tag{2.15}$$

And with $f_1 = -f_2 = f, A_1 = -A_2 = A$ and $a_1 = a_2 = a$, we have

$$F[G](\xi) = \frac{2i}{\sqrt{2\pi}^n}\frac{e^{i(a+b)\cdot\xi}}{|\det A||\det B|}\int_{\mathbb{R}^n}\sin\Big(x\cdot(A^{-1})^T\xi\Big)f(x)dx\cdot F[g]((B^{-1})^T\xi) \tag{2.16}$$

which is the factorization identity for the convolution

$$f\underset{(A,a),(B,b)}{\overset{4}{*}}g(x) = \int_{\mathbb{R}^n}\Big(f(Ay-a) - f(-Ay-a)\Big)g\Big(B(x-y)-b\Big)dy. \tag{2.17}$$

Again here

$$\frac{1}{\sqrt{2\pi}^n}\frac{1}{|\det A|}\int_{\mathbb{R}^n}\sin\Big(x\cdot(A^{-1})^T\xi\Big)f(x)dx$$

can be considered as a type of Fourier sine transform. So, the convolution (2.17) can be considered as that of the Fourier sine-type transform and the Fourier-type transform.

Multiplying the both sides of (2.17) by $-i$ and then adding the obtained expression to (2.13), we get the new convolution

$$\begin{aligned} f\underset{(A,a),(B,b)}{\overset{5}{*}}g(x) = &\int_{\mathbb{R}^n}\Big(f(Ay-a) + f(-Ay-a) - if(Ay-a) + if(-Ay-a)\Big) \\ &\times g(B(x-y)-b)dy \end{aligned} \tag{2.18}$$

which takes

$$F[G](\xi) = \frac{2}{\sqrt{2\pi}^n}\frac{e^{i(a+b)\cdot\xi}}{|\det A||\det B|}\int_{\mathbb{R}^n}\operatorname{cas}\Big(x\cdot(A^{-1})^T\xi\Big)f(x)dx\cdot F[g]\Big((B^{-1})^T\xi\Big) \tag{2.19}$$

as the factorization identity. Here, we used the standard notation of the Hartley transform (see, e.g., [7]) $\operatorname{cas}(x\cdot y) = \cos(x\cdot y) + \sin(x\cdot y)$. The list of new convolutions generated in this way can be added, however, it does not seem interesting.

3. Convolution of the two-dimensional Fourier cosine transforms

In this section, we introduce the convolution of the two-dimensional Fourier cosine transforms. We note that the two-dimensional Fourier cosine transforms are frequently used in multi-dimensional signal processing (see, e.g. [17]), however, up to now the convolution for them has not been discussed.

Let $f \in L^1(\mathbb{R}^2_+)$. We define the two-dimensional Fourier cosine transform as follows

$$F_c[f](\xi) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(x_1, x_2) \cos(x_1\xi_1) \cos(x_2\xi_2) dx_1 dx_2. \tag{3.1}$$

We extend the function f to a function defined in the whole plane $\mathbb{R}^2$ as follows

$$\mathfrak{F}(x_1, x_2) = \begin{cases} f(x_1, x_2), & x_1, x_2 \geq 0, \\ f(-x_1, x_2), & x_1 < 0, x_2 \geq 0, \\ f(x_1, -x_2), & x_1 > 0, x_2 < 0, \\ f(-x_1, x_2), & x_1 < 0, x_2 < 0. \end{cases} \tag{3.2}$$

We see that $\mathfrak{F}(x_1, x_2)$ is an even function with respect to x_1 and x_2. It is clear that $\mathfrak{F} \in L^1(\mathbb{R}^2)$. Similarly, we extend a function $g \in L^1(\mathbb{R}^2_+)$ to the function $\mathfrak{G} \in L^1(\mathbb{R}^2)$. Thus, we can define the convolution

$$\mathfrak{H}(x_1, x_2) = \mathfrak{F} * \mathfrak{G}(x_1, x_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty \mathfrak{F}(y_1, y_2) \mathfrak{G}(x_1 - y_1, x_2 - y_2) dy_1 dy_2, \tag{3.3}$$

which is well defined and belongs to $L^1(\mathbb{R}^2)$. Since, $\mathfrak{F}$ and $\mathfrak{G}$ are even functions with respect to x_1 and x_2, their convolution is so. In fact,

$$\begin{aligned} \mathfrak{H}(-x_1, x_2) = \mathfrak{F} * \mathfrak{G}(-x_1, x_2) &= \int_{-\infty}^\infty \int_{-\infty}^\infty \mathfrak{F}(y_1, y_2) \mathfrak{G}(-x_1 - y_1, x_2 - y_2) dy_1 dy_2 \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \mathfrak{F}(y_1, y_2) \mathfrak{G}(x_1 + y_1, x_2 - y_2) dy_1 dy_2 \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \mathfrak{F}(y_1, y_2) \mathfrak{G}(x_1 - y_1, x_2 - y_2) dy_1 dy_2 \\ &= \mathfrak{H}(x_1, x_2). \end{aligned}$$

Thus, $\mathfrak{H}(x_1, x_2)$ is even with respect to x_1. Similarly, it is even with respect to x_2. From (3.3), we have

$$\widehat{\mathfrak{F}}(\xi_1, \xi_2) \cdot \widehat{\mathfrak{G}}(\xi_1, \xi_2) = \widehat{\mathfrak{H}}(\xi_1, \xi_2).$$

Since $\mathfrak{F}, \mathfrak{G}$ and $\mathfrak{H}$ are even functions with respect to x_1 and x_2, from the last equality, we obtain

$$F_c[f] \cdot F_c[g] = F_c[h], \tag{3.4}$$

where h is the restriction of $\mathfrak{H}$ in the first quarter of $\mathbb{R}^2$.

Now we find an analytic expression for h. We have

$$\begin{aligned}
H(x_1, x_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathfrak{F}(y_1, y_2)\mathfrak{G}(x_1 - y_1, x_2 - y_2)dy_1 dy_2 \\
&= \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} \mathfrak{F}(y_1, y_2)\mathfrak{G}(x_1 - y_1, x_2 - y_2)dy_1 \\
&= \int_{-\infty}^{\infty} dy_2 \Big(\int_{-\infty}^{0} \mathfrak{F}(y_1, y_2)\mathfrak{G}(x_1 - y_1, x_2 - y_2)dy_1 \\
&\quad + \int_{0}^{-\infty} \mathfrak{F}(y_1, y_2)\mathfrak{G}(x_1 - y_1, x_2 - y_2)dy_1\Big) \\
&= \int_{-\infty}^{\infty} dy_2 \Big(\int_{0}^{\infty} f(y_1, y_2)g(x_1 + y_1, x_2 - y_2)dy_1 \\
&\quad + \int_{0}^{\infty} f(y_1, y_2)g(|x_1 - y_1|, x_2 - y_2)dy_1\Big) \\
&= \int_{0}^{\infty} f(y_1, y_2)\Big(g(x_1 + y_1, x_2 + y_2) + g(x_1 + y_1, |x_2 - y_2|) \\
&\quad + g(|x_1 - y_1|, x_2 + y_2) + g(|x_1 - y_1|, x_2 + y_2)\Big)dy_1 dy_2.
\end{aligned}$$

Thus, we defined the convolution of the two-dimensional Fourier cosine transforms of the two functions f and g in $L^1(\mathbb{R}^2_+)$ by

$$\begin{aligned}
f \underset{F_c}{*} g(x_1, x_2) &= \int_{0}^{\infty}\int_{0}^{\infty} f(y_1, y_2)\Big(g(x_1 + y_1, x_2 + y_2) + g(x_1 + y_1, |x_2 - y_2|) \\
&\quad + g(|x_1 - y_1|, x_2 + y_2) + g(|x_1 - y_1|, x_2 + y_2)\Big)dy_1 dy_2,
\end{aligned} \tag{3.5}$$

which takes (3.4) as the factorization identity.

As $f \underset{F_c}{*} g \in L^1(\mathbb{R}^2_+)$, if there is given a function $r \in L^1(\mathbb{R}^2_+)$, we can define the convolution

$$(f \underset{F_c}{*} g) \underset{F_c}{*} r,$$

which takes

$$F_c[f] \cdot F_c[g] \cdot F_c[r]$$

as the factorization identity.

4. Poly-convolution the one-dimensional Fourier cosine transforms

We recall that the one-dimensional Fourier cosine transform of a function $f \in L(\mathbb{R}_+)$ is defined as ([16])

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x)\cos(yx)dx.$$

and the convolution of two functions f and g in $L(\mathbb{R}_+)$ for the Fourier cosine is defined by ([16])

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\big[g(|x-y|) + g(x+y)\big]dy, \quad x > 0,$$

with the factorization property

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0. \tag{4.1}$$

In this section we considered the poly-convolution of the one-dimensional cosine transforms by a different method rather than in the previous one, although it can be treated by the same method.

Definition 4.1. The poly-convolution for the Fourier cosine integral transforms of the functions f, g and h is defined by

$$\begin{aligned} *(f,g,h)(x) = \frac{1}{2\pi} \int_0^{+\infty}\int_0^{+\infty} f(u)g(v)\Big(&h(x+u+v) + h(|x+u-v|) \\ &+ h(|x-u+v|) + h(|x-u-v|)\Big)dudv, \quad x > 0. \end{aligned} \tag{4.2}$$

Theorem 4.2. *Let f, g and h be functions in $L(\mathbb{R}_+)$, then the poly-convolution* (4.2) *for the Fourier cosine integral transforms of the functions f, g and h belongs to $L(\mathbb{R}_+)$ and the factorization property holds*

$$F_c\big[*(f,g,h)\big](y) = (F_c f)(y)(F_c g)(y)(F_c h)(y), \quad \forall y > 0. \tag{4.3}$$

Proof. First we prove that $*(f,g,h)(x) \in L(\mathbb{R}_+)$. Indeed

$$\begin{aligned} \int_0^{+\infty} \big|*(f,g,h)(x)\big|dx \le \frac{1}{2\pi} \int_0^{+\infty} |f(u)|du \int_0^{+\infty} |g(v)|du \int_0^{+\infty} \Big(&|h(|x+u+v|) \\ &+ |h(|x+u-v|)| + |h(|x-u+v|)| + |h(|x-u-v|)|\Big)dx. \end{aligned}$$

It is easy to see that

$$\int_0^{+\infty} \big[\big|h(x+u+v)\big] + \big|h(|x+u-v|)\big| + \big|h(|x-u+v|)\big| + \big|h(|x-u-v|)\big|\big]dx$$
$$= 4\int_0^{+\infty} |h(t)|dt.$$

Hence

$$\int_0^{+\infty} \big|*(f,g,h)(x)\big|dx \leq \frac{2}{\pi}\int_0^{+\infty} |f(u)|du \int_0^{+\infty} |g(v)|du \int_0^{+\infty} |h(t)|dt < +\infty.$$

So $*(f,g,h)(x)$ belongs to $L(\mathbb{R}_+)$.

Now we prove the factorization property (4.3). Since

$$(F_cf)(y)(F_cg)(y)(F_ch)(y) =$$
$$= \Big(\sqrt{\frac{2}{\pi}}\Big)^3 \int_0^{+\infty} du \int_0^{+\infty} dv \int_0^{+\infty} f(u)g(v)h(t)\cos(uy)\cos(vy)\cos(ty)dt,$$

and

$$\cos(uy)\cos(vy)\cos(ty) = \frac{1}{4}\big[\cos y(u+t+v) + \cos y(u+t-v)$$
$$+ \cos y(u-t+v) + \cos y(u-t-v)\big],$$

we get

$$(F_cf)(y)(F_cg)(F_ch)(y)$$
$$= \frac{1}{\pi\sqrt{2\pi}} \int_0^{+\infty} du \int_0^{+\infty} dv \int_0^{+\infty} f(u)g(v)h(t)\big[\cos y(u+t+v)$$
$$+ \cos y(u+t-v) + \cos y(u-t+v) + \cos y(u-t-v)\big]dt. \tag{4.4}$$

With substitution $u+t+v = x$, we get

$$\frac{1}{\pi\sqrt{2\pi}} \int_0^{+\infty} du \int_0^{+\infty} dv \int_0^{+\infty} f(u)g(v)h(t)\cos y(x+u+v)dt$$
$$= \frac{1}{\pi\sqrt{2\pi}} \int_0^{+\infty} du \int_0^{+\infty} dv \int_{u+v}^{+\infty} f(u)g(v)h(|x-u-v|)\cos yxdx$$
$$= \frac{1}{\pi\sqrt{2\pi}} \int_0^{+\infty} du \int_0^{+\infty} dv \int_0^{+\infty} f(u)g(v)h(|x-u-v|)\cos(yx)dx$$

$$(4.5)\qquad -\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_0^{u+v}f(u)g(v)h(|x-u-v|)\cos(yx)dx.$$

Similarly, substituting $u-t+v=-x$, we have

$$\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}\int_0^{+\infty}f(u)g(v)h(t)\cos y(u-t+v)dt$$
$$=\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_{-(u+v)}^{+\infty}f(u)g(v)h(u+v+x)\cos(yx)dx$$
$$=\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_0^{+\infty}f(u)g(v)h(u+v+x)\cos(yx)dx$$
$$(4.6)\qquad +\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_{-(u+v)}^{0}f(u)g(v)h(u+v+x)\cos(yx)dx.$$

Further,

$$\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_{-(u+v)}^{0}f(u)g(v)h(t)h(u+v+x)\cos(yx)dx$$
$$(4.7)\qquad =\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_0^{u+v}f(u)g(v)h(u+v-x)\cos(yx)dx.$$

From (4.5), (4.6) and (4.7), we have

$$\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_0^{+\infty}f(u)g(v)h(t)\big[\cos y(u+t+v)+\cos y(u-t+v)\big]dt$$
(4.8)
$$=\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_0^{+\infty}f(u)g(v)\big[h(|x-u-v|)+h(x+u+v)\big]\cos(yx)dx.$$

Similarly,

$$\frac{1}{\pi\sqrt{2\pi}}\int_0^{+\infty}du\int_0^{+\infty}dv\int_0^{+\infty}f(u)g(v)h(t)\big[\cos y(u+t-v)+\cos y(u-t-v)\big]dt$$

(4.9)

$$= \frac{1}{\pi\sqrt{2\pi}} \int_0^{+\infty} du \int_0^{+\infty} dv \int_0^{+\infty} f(u)g(v)\big[h(|x+u-v|) + h(|x-u+v)\big] \cos(yx)dx.$$

Finally, by (4.4), (4.8), (4.9),

$$(F_c f)(y)(F_c g)(y)(F_c h)(y) = F_c\big[*(f,g,h)\big](y).$$

The proof is complete. □

Theorem 4.3. (Titchmarch-type Theorem) *Let* $f, g, h \in L(e^{-x}, \mathbb{R}_+)$. *If* $\forall x > 0$, $*(f,g,h)(x) \equiv 0$, *then either* $f(x) = 0$, *or* $g(x) = 0$, *or* $h(x) = 0$, $\forall x > 0$.

Proof. The hypothesis $*(f,g,h)(x) \equiv 0$ implies that

$$F_c\big[f,g,h)\big](y) = 0, \quad \forall y > 0.$$

Due to Theorem 4.2, we have

$$(F_c f)(y)(F_c g)(y)(F_c h)(y) = 0, \quad \forall y > 0. \tag{4.10}$$

Consider the Fourier cosine integral transform

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x)\cos(yx)dx, \quad y \in \mathbb{R}^+.$$

Since

$$\begin{aligned}\Big|\frac{d^n}{dy^n}\big[\cos(yx)f(x)\big]\Big| &= \Big|f(x)x^n \cos\Big(yx + n\frac{\pi}{2}\Big)\Big| \le \big|f(x)x^n\big| \\ &= \big|e^{-x}x^n f_1(x)\big| \\ &= \big|e^{-x}x^n\big|\big|f_1(x)\big| \le C\big|f_1(x)\big|\end{aligned}$$

for x large enough, due to Weierstrass' criterion, the integral $\int_0^{+\infty} \frac{d^n}{dy^n}\big[\cos(yx)f(x)\big]dx$ uniformly converges on $\mathbb{R}^+$. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_c f)(y)$ is analytic for $y > 0$. Similarly, $(F_c g)(y)$ and $(F_c h)(y)$ are analytic for $y > 0$. So from (4.10) we have $(F_c f)(y) = 0$, $\forall y > 0$, or $(F_c g)(y) = 0$, $\forall y > 0$, or $(F_c h)(y) = 0$, $\forall y > 0$. It follows that either $f(x) = 0$, $\forall x > 0$, or $g(x) = 0$, $\forall x > 0$, or $h(x) = 0$, $\forall x > 0$.

The theorem is proved. □

In the sequel, for simplicity, we define the norm in the space $L(\mathbb{R}_+)$ by

$$\|f\| = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} |f(x)|dx.$$

Theorem 4.4. *If* f, g *and* h *belong to* $L(\mathbb{R}_+)$, *then the following inequality holds*

$$\big\| *(f,g,h)\big\| \le \|f\|\,\|g\|\,\|h\|.$$

Proof. From the proof of Theorem 4.2, we obtain

$$\int_0^{+\infty} \big| * (f,g,h)(x)\big| dx \leq \frac{2}{\pi} \int_0^{+\infty} |f(u)| du \int_0^{+\infty} |g(v)| dv \int_0^{+\infty} |h(t)| dt.$$

Hence

$$\int_0^{+\infty} \big| * (f,g,h)(x)\big| dx \leq \frac{1}{\sqrt{\pi}} \int_0^{+\infty} |f(x)| dx \frac{1}{\sqrt{\pi}} \int_0^{+\infty} |g(x)| dx \int_0^{+\infty} |2h(t)| dt.$$

Thus,

$$\big\| * (f,g,h)\big\| \leq \|f\|\,\|g\|\,\|h\|.$$

The proof is complete. □

Theorem 4.5. *Let f, g and h be functions in $L(\mathbb{R}_+)$. The poly-convolution for the Fourier cosine integral transforms relates to the known convolutions as follows*

(i) $\displaystyle *(f,g,h)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \big[(g \underset{F_c}{*} h)(u+x) + (g \underset{F_c}{*} h)(|u-x|)\big] f(u) du.$

(ii) $*(f,g,h) = f \underset{F_c}{*} \big(g \underset{F_c}{*} h\big).$

Proof. We first prove the equality (i).

By Definition 4.1 we have

$$\begin{aligned} *(f,g,h)(x) =& \frac{1}{\sqrt{2\pi}}\Big\{ \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u) du \int_0^{+\infty} g(v)\big[h(x+u+v) + h(|x+u-v|)\big] dv \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u) du \int_0^{+\infty} g(v)\big[h(|x-u+v|) + h(|x-u-v|)\big] dv \Big\}. \end{aligned}$$

On the other hand, for any $x > 0$, $u > 0$ and $v > 0$,

$$h(|x-u+v|) + h(|x-u-v|) = h(|x-u|+v) + h\big(\big||x-u|-v\big|\big).$$

Indeed, for $x \geq u$,

$$\begin{aligned} h(|x-u|+v) + h\big(\big||x-u|-v\big|\big) =& h(x-u+v) + h(|x-u-v|) \\ =& h(|x-u+v|) + h(|x-u-v|). \end{aligned}$$

Similarly, for $0 < x \leq u$,

$$\begin{aligned} h(|x-u|+v) + h\big(\big||x-u|-v\big|\big) =& h(|u-x+v|) + h(|u-x-v|) \\ =& h(|x-u-v|) + h(|x-u+v|). \end{aligned}$$

Hence

$$*(f,g,h)(x) = \frac{1}{\sqrt{2\pi}}\Big\{ \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u) du \int_0^{+\infty} g(v)\big[h(x+u+v) + h(|x+u-v|)\big] dv$$

$$+\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} f(u)du\int_0^{+\infty} g(v)\big[h(|x-u|+v)+h(|x-u|-v)\big]dv\Big\}$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\big[(g\underset{F_c}{*}h)(u+x)+(g\underset{F_c}{*}h)(|u-x|)\big]f(u)du.$$

The equality (i) is proved.

We next prove the equality (ii).

From Theorem 4.2, we deduce that

$$F_c\big[*(f,g,h)\big](y)=(F_cf)(y)(F_cg)(y)(F_ch)(y),\quad \forall y>0.$$

On the other hand, using the formula (4.1), we get

$$F_c\big[f\underset{F_c}{*}(g\underset{F_c}{*}h)\big](y)=(F_cf)(y)F_c(g\underset{F_c}{*}h)(y)=(F_cf)(y)(F_cg)(y)(F_ch)(y),\quad \forall y>0.$$

Therefore, $*(f,g,h)=f\underset{F_c}{*}\big(g\underset{F_c}{*}h\big)$, and the equality (ii) is proved.

The proof is complete. □

Theorem 4.6. *In the space $L(\mathbb{R}_+)$, the poly-convolution for the Fourier cosine integral transforms is commutative, associative and distributive.*

Proof. We prove that the poly-convolution for the Fourier cosine integral transforms is commutative, i.e.,

$$*(f,g,h)=*(f,h,g)=*(g,f,h)=*(g,h,f)=*(h,f,g)=*(h,g,f).$$

Indeed,

$$\begin{aligned}F_c\big[*(f,g,h)\big](y)=&(F_cf)(y)(F_cg)(F_ch)(y)=(F_cf)(y)(F_ch)(y)(F_cg)(y)\\=&F_c\big(*(f,h,g)\big)(y),\quad \forall y>0\end{aligned}$$

implies that

$$*(f,g,h)=*(f,h,g).$$

The following equalities are similarly proved.

The associative, distributive properties are similarly proved. □

5. Application to solving an integral equation and a system of integral equations

5.1. Consider the integral equation

$$f(x)+\lambda\int_0^{+\infty}\int_0^{+\infty}\varphi(u)\psi(v)\theta_1(x,u,v)dudv=h(x),\quad x>0. \tag{5.1}$$

Here λ is a complex constant, φ, ψ and h are functions of $L(\mathbb{R}_+)$, f is the unknown function, and

$$\theta_1(x,u,v) = \frac{1}{2\pi}\big[f(x+u+v) + f(|x+u-v|) + f(|x-u+v|) + f(|x-u-v|)\big].$$

Theorem 5.1. *With the condition* $1 + \lambda(F_c\varphi)(y)(F_c\psi)(y) \neq 0$, $\forall y \in \mathbb{R}_+$, *there exists a unique solution in* $L(\mathbb{R}_+)$ *of* (5.1) *which is defined by*

$$f = h - (h \underset{F_c}{*} l).$$

Here, $l \in L(\mathbb{R}_+)$ *and it is determined by the equation*

$$(F_c l)(y) = \frac{\lambda F_c(\varphi \underset{F_c}{*} \psi)(y)}{1 + \lambda F_c(\varphi \underset{F_c}{*} \psi)(y)}.$$

Proof. The equation (5.1) can be rewritten in the form

$$f(x) + \lambda\big[* (\varphi, \psi, f)(x)\big] = h(x).$$

Due to Theorem 4.2,

$$(F_c f)(y) + \lambda(F_c\varphi)(y)(F_c\psi)(y)(F_c f)(y) = (F_c h)(y).$$

It follows that

$$(F_c f)(y)\big[1 + \lambda(F_c\varphi)(y)(F_c\psi)(y)\big] = (F_c h)(y).$$

Since $1 + \lambda(F_c\varphi)(y)(F_c\psi)(y) \neq 0$,

$$(F_c f)(y) = (F_c h)(y)\frac{1}{1 + \lambda(F_c\varphi)(y)(F_c\psi)(y)}.$$

Therefore,

$$\begin{aligned}(F_c f)(y) =& (F_c h)(y)\Big[1 - \frac{\lambda(F_c\varphi)(y)(F_c\psi)(y)}{1 + \lambda(F_c\varphi)(y)(F_c\psi)(y)}\Big]\\ &= (F_c h)(y)\Big[1 - \frac{\lambda F_c(\varphi \underset{F_c}{*} \psi)(y)}{1 + \lambda F_c(\varphi \underset{F_c}{*} \psi)(y)}\Big].\end{aligned}$$

Due to Wiener-Levi's theorem [1], there exists a function $l \in L(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{\lambda F_c(\varphi \underset{F_c}{*} \psi)(y)}{1 + \lambda F_c(\varphi \underset{F_c}{*} \psi)(y)}.$$

It follows that

$$\begin{aligned}(F_c f)(y) =& (F_c h)(y)\big[1 - (F_c l)(y)\big]\\ =& (F_c h)(y) - F_c(h \underset{F_c}{*} l)(y).\end{aligned}$$

Thus,

$$f = h - (h \underset{F_c}{*} l).$$

It is easy to see that $f \in L(\mathbb{R}_+)$. The theorem is proved. □

5.2. Consider the system of integral equations

$$\begin{cases} f(x) + \lambda_1 \int\limits_0^{+\infty} \int\limits_0^{+\infty} \varphi(u)\psi(u)\theta_2(x,u,v)dudv & = h(x), \\ \lambda_2 \int\limits_0^{+\infty} \theta_3(x,u)\eta(u)du + g(x) & = k(x), \quad x > 0, \end{cases} \tag{5.2}$$

where

$$\theta_2(x,u,v) = \frac{1}{2\pi}\big[g(x+u+v) + g(|x+u-v|) + g(|x-u+v|) + g(|x-u-v|)\big],$$

$$\theta_3(x,u) = \frac{1}{2\sqrt{2\pi}}\big[f(|y-x-1|) - f(|y-x+1|) + f(|y+x-1|) - f(y+x+1)\big],$$

φ, ψ, η, h and k are given functions in $L(\mathbb{R}_+)$, λ_1 and λ_2 are complex constants, f and g are the unknown functions.

To solve this system we recall the convolution ([11])

$$(f \overset{\gamma_1}{\underset{3}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)\big[g(|y-x-1|) - g(|y-x+1|) + g(|y+x-1|) - g(y+x+1)\big]dy, \quad x > 0,$$

with the factorization property

$$F_c(f \overset{\gamma_1}{\underset{3}{*}} g)(y) = \sin y (F_s f)(y)(F_c g)(y), \quad \forall y > 0.$$

Theorem 5.2. *With the condition*

$$1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big] \neq 0, \quad \forall y > 0,$$

there exists a solution in $L(\mathbb{R}_+)$ *of* (5.2) *which is defined by*

$$f(y) = h(y) + (l \underset{F_c}{*} h)(y) - \lambda_1\big[*(k,\varphi,\psi)(y)\big] - \big[*(k,\varphi,\psi) \underset{F_c}{*} l\big](y) \in L(\mathbb{R}_+),$$

$$g(y) = k(y) + (l \underset{F_c}{*} k)(y) - \lambda_2\big(\eta \overset{\gamma_1}{\underset{3}{*}} h\big)(y) - \lambda_2\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} h\big) \underset{F_c}{*} l\big](y) \in L(\mathbb{R}_+).$$

Here $l \in L(\mathbb{R}_+)$ *and defined by the equation*

$$(F_c l)(y) = \frac{\lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{F_c}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)}{1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)}.$$

Proof. System (5.2) can be written in the form

$$f(x) + \lambda_1\big[*(\varphi,\psi,g)(x)\big] = h(x),$$

$$\lambda_2\big(\eta \overset{\gamma_1}{\underset{3}{*}} f\big)(x) + g(x) = k(x), \quad x > 0.$$

Using the factorization property of the poly-convolution (4.2) and the convolution $\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big)(x)$ we obtain the linear system of algebraic equations with respect to $(F_c f)(y)$ and $(F_c g)(y)$:

$$(F_c f)(y) + \lambda_1 (F_c\varphi)(y)(F_c\psi)(y)(F_c g)(y) = (F_c h)(y),$$
$$\lambda_2 \gamma_1(y)(F_s\eta)(y)(F_c f)(y) + (F_c g)(y) = (F_c k)(y), \quad y > 0.$$

Formally, we have

$$(F_c f)(y) = \frac{(F_c h)(y) - \lambda_1 F_c\big[\,*\,(k,\varphi,\psi)\big](y)}{1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)},$$

$$(F_c g)(y) = \frac{(F_c k)(y) - \lambda_2 F_c\big(\eta \overset{\gamma_1}{\underset{3}{*}} h\big)(y)}{1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)}.$$

We note that

$$\frac{1}{1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)} = 1 + \frac{\lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{F_c}{*}} \varphi\big) \underset{F_c}{*} \big](y)}{1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)}.$$

Further, due to Wiener-Levi's theorem [1], there exists a function $l \in L(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{\lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)}{1 - \lambda_1\lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} \varphi\big) \underset{F_c}{*} \psi\big](y)}.$$

Therefore,

$$\begin{aligned}(F_c f)(y) =& \big[1 + (F_c l)(y)\big]\big[(F_c h)(y) - \lambda_1 F_c\big(\,*\,(k,\varphi,\psi)(y)\big]\\ =& (F_c h)(y) + F_c\big(l \underset{F_c}{*} h\big)(y) - \lambda_1 F_c\big[\,*\,(k,\varphi,\psi)\big](y) - F_c\big[\,*\,(k,\varphi,\psi) \underset{F_c}{*} l\big](y).\end{aligned}$$

Hence

$$f(y) = h(y) + (l \underset{F_c}{*} h)(y) - \lambda_1\big[\,*\,(k,\varphi,\psi)(y)\big] - \big[\,*\,(k,\varphi,\psi) \underset{F_c}{*} l\big](y) \in L(\mathbb{R}_+).$$

We conclude similarly that

$$\begin{aligned}(F_c g)(y) =& \big[1 + (F_c l)(y)\big]\big[(F_c k)(y) - \lambda_2 F_c\big[\eta \overset{\gamma_1}{\underset{3}{*}} h\big)(y)\big]\\ =& (F_c k)(y) + F_c(l \underset{F_c}{*} k)(y) - \lambda_2 F_c\big(\eta \overset{\gamma_1}{\underset{3}{*}} k\big)(y) - \lambda_2 F_c\big[\big(\eta \overset{\gamma_1}{\underset{3}{*}} h\big) \underset{F_c}{*} l\big](y).\end{aligned}$$

Consequently,

$$g(y) = k(y) + \big(l \underset{F_c}{*} k\big)(y) - \lambda_2\big(\eta \overset{\gamma_1}{\underset{3}{*}} h\big)(y) - \lambda_2\big[\big(\eta \overset{\gamma_1}{\underset{F_c}{*}} h\big) \underset{F_c}{*} l\big](y) \in L(\mathbb{R}_+).$$

The proof is complete. $\square$

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