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A METHOD OF EXTENDING BY PARAMETER FOR APPROXIMATE SOLUTIONS OF OPERATOR EQUATIONS

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Abstract. A method of extending by parameter has been researched in the works of V. A. Trenoghin [4]-[6], A. A. Fonarov [1] and Y. L. Gaponenco [2]. In this paper, the author presents an application of this method for approximating solutions of operator equations with any growth. Some results on the existence of solutions of operator equations are derived.

1. About the operator equation $A(x) + B(x) = f$

Let us consider the equation

$$A(x) + B(x) = f. \tag{1.1}$$

Assume that X is a metric space with a distance $\rho(,)$, Y a Banach space and A, B are operators mapping from X to Y, f is given in Y. We pose the following conditions:
I) The operator A maps one-to-one from X onto Y,
II) $\|B(x) - B(y)\| \leq L \|A(x) - A(y)\|, L = \text{const}; x, y \in X$,
III) There exists a number $\gamma > 0$ such that $\forall \lambda \in [0,1]$, $\forall x, y \in X$, the following inequality holds

$$\|A(x) - A(y) + \lambda [B(x) - B(y)]\| \geq \gamma \|A(x) - A(y)\|,$$

IV) $\rho(x, y) \leq \alpha \|A(x) - A(y)\|, \quad \alpha = \text{const}, \quad x, y \in X.$

In view of [5, p. 457] if conditions I) - IV) are satisfied, then equation (1.1) has a unique solution. Assuming that conditions I) - IV) are satisfied, we take a minimal natural number N such that

$$q = \max\{L/N, L/\gamma N\} < 1.$$

Denote $\varepsilon = 1/N$, then equation (1.1) has the form

$$A(x) + N\varepsilon \ B(x) = f.$$

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Consider the following system of equations

$$(1.2)\qquad \begin{cases} A(x) & = u \\ A_1(x) & \equiv A(x) + \varepsilon B(x) = u_1 \\ A_2(x) & \equiv A_1(x) + \varepsilon B(x) = u_2 \\ & ... \\ A_{N-1}(x) & \equiv A_{N-2}(x) + \varepsilon B(x) = u_{N-1} \\ A_N(x) & \equiv A_{N-1}(x) + \varepsilon B(x) = u_N, \end{cases}$$

where $u_N = f$.

We prove that the operators $A_i, i = 1, 2, \ldots, N$ are one-to-one maps from X onto Y. From condition I), it follows that the operator A is invertible and the inverse operator A^{-1} of A is a map from Y onto X. To prove the existence of operator A_1^{-1}, we consider the following system of equations

$$(1.3)\qquad \begin{cases} A(x) = u \\ u + \varepsilon B A^{-1}(u) = u_1, \end{cases}$$

where u_1 is given in Y. Let u and w', be two any elements of Y and $x = A^{-1}(u)$, $x' = A^{-1}(w')$. Using conditions I), III) and IV), we get

$$\rho(x, x') = \rho(A^{-1}(u), A^{-1}(w')) \leq \alpha \left\| u - w' \right\|,$$

$$\begin{aligned} \left\| \varepsilon B A^{-1}(u) - \varepsilon B A^{-1}(w') \right\| &= \varepsilon \left\| B(x) - B(x') \right\| \\ &\leq \varepsilon L \left\| A(x) - A(x') \right\| = L/N \left\| u - w' \right\| \\ &\leq q \left\| u - w' \right\|. \end{aligned}$$

Therefore, the operator $\varepsilon B A^{-1}$ is contractive. This means that system (1.3) has a unique solution and the operator A_1 is invertible. Further, to prove the existence of operator A_2^{-1}, we consider the equation

$$(1.4)\qquad u_1 + \varepsilon B A_1^{-1}(u_1) = u_2,$$

where u_2 is given in Y. Let u and w' be two any elements of Y, $x = A_1^{-1}(u)$, $x' = A_1^{-1}(w')$. We have

$$\begin{aligned} \left\| \varepsilon B A_1^{-1}(u) - \varepsilon B A_1^{-1}(w') \right\| &= \varepsilon \left\| B(x) - B(x') \right\| \\ &\leq \varepsilon L \left\| A(x) - A(x') \right\| \\ &\leq q \left\| A(x) - A(x') + \varepsilon [B(x) - B(x')] \right\| \\ &= q \left\| A_1(x) - A_1(x') \right\| = q \left\| u - w' \right\|. \end{aligned}$$

Therefore, the operator $\varepsilon B A_1^{-1}$ is contractive and equation (1.4) has a unique solution. This proves that the operator A_2^{-1} exists. In the same way, we prove that the operators A_i^{-1}; $i = 3, 4, ..., N - 1$ exist and $\varepsilon B A_i^{-1}$ is contractive with the coefficient q. Now we present system (1.2) in the following form

$$(1.5)\qquad \begin{cases} A(x) &= u \\ u &= -\varepsilon BA^{-1}(u) + u_1 \\ u_1 &= -\varepsilon BA_1^{-1}(u_1) + u_2 \\ & ... \\ u_{N-2} &= -\varepsilon BA_{N-2}^{-1}(u_{N-2}) + u_{N-1} \\ u_{N-1} &= -\varepsilon BA_{N-1}^{-1}(u_{N-1}) + f. \end{cases}$$

The approximate solution of system (1.5) is constructed as follows

$$(1.6)\qquad \begin{cases} A(x_k) &= u^{(k)} \\ u^{(k+1)} &= -\varepsilon BA^{-1}(u^{(k)}) + u_1^{(l)}, k = 0, 1, 2, ... \\ u_1^{(l+1)} &= -\varepsilon BA_1^{-1}(u_1^{(l)}) + u_2^{(p)}, l = 0, 1, 2, ... \\ & ... \\ u_{N-2}^{(m+1)} &= -\varepsilon BA_{N-2}^{-1}(u_{N-2}^{(m)}) + u_{N-1}^{(n)}, m = 0, 1, 2, ... \\ u_{N-1}^{(n+1)} &= -\varepsilon BA_{N-1}^{-1}(u_{N-1}^{(n)}) + f, n = 0, 1, 2, ... \end{cases}$$

From condition I), it follows that there exists an element $x_0 \in X$ such that $A(x_0) = 0$. Put $B_1(x) = B(x) - B(x_0)$, $f_1 = f - B(x_0)$, $(x \in X)$. It is clear that equation (1.1) is equivalent to the following equation

$$A(x) + B_1(x) = f_1,$$

where the operator B_1 satisfies conditions II), III) and $B_1(x_0) = 0$. Therefore without loss of generality it can be assumed that $A(x_0) = 0$, $B(x_0) = 0$. It can be assumed that the first approximation in each iteration in (1.6) equals 0 and the number of steps in each iteration equals s. Assume that $(x, u, u_1, u_2, , u_N - 1)$ is an exact solution of system (1.5). At first, we assume that the value of $A_i^{-1}(u_i^{(s)})$ is calculated exactly. We have

$$\rho(x_s, x) \le \alpha \|A(x_s) - A(x)\| = \alpha \left\| u^{(s)} - u \right\|,$$

$$(1.7)\qquad \begin{aligned} \left\| u^{(s)} - u \right\| &\le \frac{q^s}{1-q} \left\| u_1^{(s)} \right\|, \\ \left\| u_1^{(s)} - u_1 \right\| &\le \frac{q^s}{1-q} \left\| u_2^{(s)} \right\|, \\ & ... \\ \left\| u_{N-1}^{(s)} - u_{N-1} \right\| &\le \frac{q^s}{1-q} \|f\|. \end{aligned}$$

Now we estimate $\left\| u_1^{(s)} \right\|, ..., \left\| u_{N-1}^{(s)} \right\|$. Let us consider the operators

$$F_i(v) = v + \varepsilon BA_i^{-1}(v), v \in Y; \quad i = 1, 2, ..., N-1.$$

Let v and v' be two any elements of Y, we denote $F_i(v) = g$, and $F_i(v') = g'$. Then we have $v = F_i^{-1}(g)$, $v' = F_i^{-1}(g')$,

$$\begin{aligned}\left\|F_i^{-1}(g) - F_i^{-1}(g')\right\| &= \|v - v'\| \\ &\leq \frac{1}{\gamma}\left\|v - v' + \varepsilon\left[BA_i^{-1}(v) - BA_i^{-1}(v')\right]\right\| \\ &= \frac{1}{\gamma}\left\|(v + \varepsilon BA_i^{-1}(v)) - (v' + \varepsilon BA_i^{-1}(v'))\right\| \\ &= \frac{1}{\gamma}\|g - g'\|.\end{aligned}$$

Then F_i^{-1} satisfies the Lipschitz condition with the coefficient $1/\gamma$.

We take $v = u_i$ and $v' = 0$. From $F_i(u_i) = u_{i+1}, F_i(0) = 0$, we get

$$\|u_i\| \leq \frac{1}{\gamma}\|u_{i+1}\|. \tag{1.8}$$

Applying this inequality several times, we have

$$\|u_i\| \leq \left(\frac{1}{\gamma}\right)^{N-i}\|u_N\| = \left(\frac{1}{\gamma}\right)^{N-i}\|f\|. \tag{1.9}$$

On the other hand, we have

$$\begin{aligned}\left\|u_i^{(s)}\right\| &\leq \left\|u_i^{(s)} - u_i\right\| + \|u_i\| \\ &\leq \frac{q^s}{1-q}\|u_{i+1}\| + \|u_i\|, \quad i = 1, 2, ..., N-2.\end{aligned} \tag{1.10}$$

Combining inequalities (1.8), (1.9) and (1.10), we obtain

$$\begin{aligned}\left\|u_i^{(s)}\right\| &\leq \left\|u_i^{(s)} - u_i\right\| + \|u_i\| \\ &\leq \frac{q^s}{1-q}\|u_{i+1}\| + \frac{1}{\gamma}\|u_{i+1}\| \\ &= \left(\frac{q^s}{1-q} + \frac{1}{\gamma}\right)\|u_{i+1}\|) \quad (i = 1, 2, ..., N-2).\end{aligned} \tag{1.11}$$

From (1.9) and (1.11), we have

$$\left\|u_i^{(s)}\right\| \leq \left(\frac{q^s}{1-q} + \frac{1}{\gamma}\right)\left(\frac{1}{\gamma}\right)^{N-1-i}\|f\|, \quad i = 1, 2, ..., N-1. \tag{1.12}$$

We denote

$$c = max\left\{\left(\frac{\mathrm{q}^{\mathrm{s}}}{1-\mathrm{q}}+\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}\right)^{N-1-i}\|f\|, i=1,2,...,N-1\right\},$$

$$c_1 = max\left\{\left(\frac{\mathrm{q}^{\mathrm{s}}}{1-\mathrm{q}}+\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}\right)^{N-1-i}\|f_1\|, i=1,2,...,N-1\right\},$$

$$\delta = c\frac{q^s}{1-q}.$$

Therefore, from (1.7), we have

$$\left\|u^{(s)}-u\right\| \le \delta. \tag{1.13}$$

Now we evaluate the speed of convergence of iterative processes taking into account of the error of the calculations of $\varepsilon BA_i^{-1}(u_i^{(s)})$, $(i=1,2,...,N-1)$. We consider the following problems:

Problem 1. One step by parameter ($N=1$). In this case, the system (1.5) has the form

$$\begin{cases} A(x_k) &= u^{(k)} \\ u^{(k+1)} &= -BA(u^{(k)})+f, \quad k=1,2,... \end{cases}$$

We have

$$\begin{aligned} &\delta_1 \equiv \left\|u^{(s)}-u\right\| \le \delta, \\ &\eta_1 = \rho(x_s,x) \le \alpha\delta_1 \le \alpha\delta, \\ &\Delta_1 \equiv \eta_1 \le \alpha\delta. \end{aligned} \tag{1.14}$$

Problem 2. Two steps by parameter ($N=2$). In this case, the system (1.5) has the form

$$\begin{cases} A(x_k) &= u^{(k)} \\ u^{(k+1)} &= -\varepsilon BA^{-1}(u^{(k)})+u_1^{(l)}, k=1,2,... \\ u_1^{(l+1)} &= -\varepsilon BA_1^{-1}(u_1^{(l)})+f, l=1,2,... \end{cases} \tag{1.15}$$

The value $u_1^{(l)}$ is calculated with an error δ, on the other hand, the operator $\varepsilon BA_1^{-1}(u_1^{(l)})$ is contractive with the coefficient q. Therefore, $\varepsilon BA_1^{-1}(u_1^{(l)})$ is calculated with the error $q\delta$. The error of an iterative process in the calculation of u_1 equals δ. Then we have

$$\delta_2 \equiv \left\|u_1^{(s)}-u_1\right\| \le \delta + q\delta,$$

$$\Delta_2 \equiv \rho(x_s,x) \le \alpha(\delta + q\delta_1).$$

Problem i. i steps by parameter $(N = i)$

$$(1.16)\qquad \begin{cases} A(x_k) & = u^{(k)} \\ u^{(k+1)} & = -\varepsilon BA^{-1}(u^{(k)}) + u_1^{(l)}, \quad k = 0, 1, 2, ... \\ u_1^{(l+1)} & = -\varepsilon BA_1^{-1}(u_1^{(l)}) + u_2^{(p)}, \quad l = 0, 1, 2, ... \\ & ... \\ u_{i-1}^{(n+1)} & = -\varepsilon BA_{i-1}^{-1}(u_{i-1}^{(n)}) + f, \quad n = 0, 1, 2, ... \end{cases}$$

In this problem, $u_{i-1}^{(s)}$ is calculated with the error $\delta_1+\delta_2+...+\delta_{i-2}$. On the other hand, the operators $\varepsilon BA_{i-1}^{-1}$ are contractive with the coefficient q. Therefore, $\varepsilon BA_{i-1}^{-1}(u_{i-1}^{(s)})$ is calculated with the error $q(\delta_1 + \delta_2 + ... + \delta_{i-2})$. We have

$$\delta_{i-1} \equiv \left\| u_{i-1}^{(s)} - u_{i-1} \right\| \leq \delta + q(\delta_1 + \delta_2 + ...\delta_{i-2}),$$
$$\Delta_{i-1} \equiv \rho(x_s, x) \leq \alpha(\delta_1 + \delta_2 + ... + \delta_{i-2}).$$

Using Bellman-Gronwall inequality, we obtain

$$\delta_{i-1} \leq \delta \exp\left[(\mathrm{i} - 2)\, q\right]$$
$$\Delta_{i-1} \leq \alpha \sum_{j=1}^{i} \delta \exp\left[(\mathrm{j} - 2)\mathrm{q}\right] = \delta\alpha \frac{\exp[(i-2)q] - 1}{[\exp(q) - 1]}.$$

Substituting N for $i - 1$, we have

$$\Delta_N \equiv \rho(x_s, x) \leq \delta\alpha \frac{\exp[(N-1)q] - 1}{[\exp(q) - 1]} = c\alpha q^s \frac{1}{1-q} \frac{\exp[(\mathrm{N}-1)\mathrm{q}] - 1}{[\exp(\mathrm{q}) - 1]}.$$

In the general case, we have

$$\Delta_N \equiv \rho(x_s, x) \leq c_1 \alpha q^s \frac{1}{1-q} \frac{\exp[(\mathrm{N}-1)\mathrm{q}] - 1}{[\exp(\mathrm{q}) - 1]}.$$

From the above obtained results, we have the following theorem:

Theorem 1. *Suppose that the operators A and B satisfy conditions I) - IV). Then there exists a unique solution x of equation* (1.1) *and that approximated solutions of x constructed by formula* (1.6) *tend to x. Moreover, the speed of the convergence is estimated by the formula*

$$\rho(x_s, x) \leq c_1 \alpha q^s \frac{1}{1-q} \frac{\exp[(\mathrm{N}-1)\mathrm{q}] - 1}{[\exp(\mathrm{q}) - 1]},$$

where

$$c_1 = \max \left\{ \left(\frac{q^s}{1-q} + \frac{1}{\gamma} \right) \left(\frac{1}{\gamma} \right)^{N-1-i} \|f_1\|, \quad i = 1, 2, ..., N-1. \right\}$$

2. About the operator equation $A(x) + B(x) + C(x) = f$

Let us consider the equation

$$A(x) + B(x) + C(x) = f. \tag{2.1}$$

2.1. **The existence of solution.**

Theorem 2. *Suppose that the operators A and B satisfy conditions I) - IV). The operator C maps from X into Y and satisfies the Lipschitz condition:*

$$\|C(x) - C(x')\| \leq K\rho(x, x'),$$

where $\theta \equiv K\frac{\alpha}{\gamma} < 1$. Then the equation (2.1) has a unique solution.

Proof. Let us denote

$$G(x) = A(x) + B(x), \forall x \in X. \tag{2.2}$$

Application of Theorem 1 yields the existence of the inverse operator G^{-1}, that maps from Y into X. By taking $x = G^{-1}(u), \forall u \in Y$, we see that the equation (2.2) has the form

$$u + CG^{-1}(u) = f. \tag{2.3}$$

We show that the operator G^{-1} satisfies the Lipschitz condition. For every $u \in Y$ and $u' \in Y$ we take $x = G^{-1}(u)$ and $x' = G^{-1}(u')$.

Using conditions III) and IV), we get

$$\begin{aligned}\rho\left(G^{-1}(u), G^{-1}(u')\right) &= \rho(x, x') \\ &\leq \alpha \|A(x) - A(x')\| \\ &\leq \frac{\alpha}{\gamma} \|A(x) - A(x') + [B(x) - B(x')]\| \\ &= \frac{\alpha}{\gamma} \|u - u'\|.\end{aligned}$$

Therefore, the operator G^{-1} satisfies the Lipschitz condition with $L_1 = \frac{\alpha}{\gamma}$.

We prove that the operator CG^{-1} is contractive. Indeed, we have

$$\begin{aligned}\left\|CG^{-1}(u) - CG^{-1}(u')\right\| &\leq K\rho(G^{-1}(u), G^{-1}(u')) \\ &\leq K\frac{\alpha}{\gamma} \|u - u'\| \\ &= \theta \|u - u'\|.\end{aligned}$$

Hence, the equation (2.3) has a unique solution. This completes the proof. □

2.2. **The approximate solution of equation** (2.1)**.** By using the notations in (2.2) and (2.3), we see that, the equation (2.1) is equivalent to the system of equations

$$\begin{cases} A(x) + B(x) = u \\ u + CG^{-1}(u) = f. \end{cases} \tag{2.4}$$

The system (2.4) is solved approximately as follows:

$$\begin{cases} A(x_k) + B(x_k) = u_k \\ u_{k+1} = -CG^{-1}(u_k) + f, \quad k = 0, 1, 2, ..., u_0 = 0. \end{cases} \tag{2.5}$$

In this scheme, the equation $A(x) + B(x) = u$ is solved approximately as in Section 1. The equation $u + CG^{-1}(u) = f$ is solved by method of iteration:

$$u_{k+1} = -CG^{-1}(u_k) + f, k = 0, 1, 2, ..., u_0 = 0.$$

Assume that the number of steps in each iteration process equals s. Then the error ε_s of method of iteration is calculated as follows:

$$\varepsilon_s = \frac{\theta^s}{1-\theta} \|f\| .$$

We see that $G^{-1}(u_s)$ is calculated with the error Δ_N. On the other hand, the operator C has the Lipschitz coefficient K, hence, the error of calculation of iteration (2.5) is $K\Delta_N$.

Therefore, the error η_s of iteration (2.5) is calculated as follows:

$$\eta_s = \varepsilon_s + K\Delta_N = \frac{\theta^s}{1-\theta} \|f\| + K\Delta_N. \tag{2.6}$$

Let us denote $\beta_s = \rho(x_s, x)$.

Using Theorem 1 and the Lipschitz coefficient of the operator G^{-1} , we get

$$\begin{aligned} \beta_s = \rho(x_s, x) &\le \Delta_N + \frac{\alpha}{\gamma}\eta_s = \frac{\alpha}{\gamma}\varepsilon_s + (K\frac{\alpha}{\gamma} + 1)\Delta_N \\ &\le \frac{\alpha}{\gamma}\frac{\theta^s}{1-\theta} \|f\| + (\theta + 1)c\alpha q^s \frac{1}{1-q} \frac{\exp[(\mathrm{N}-1)\mathrm{q}] - 1}{[\exp(\mathrm{q}) - 1]}. \end{aligned} \tag{2.7}$$

In the general case, we get

$$\begin{aligned} \beta_s &= \rho(x_s, x) \\ &\le \frac{\alpha}{\gamma}\frac{\theta^s}{1-\theta} \|f_1\| + (\theta + 1)c_1\alpha q^s \frac{1}{1-q} \frac{\exp[(\mathrm{N}-1)\mathrm{q}] - 1}{[\exp(\mathrm{q}) - 1]}. \end{aligned}$$

Thus, we have proved the following theorem:

Theorem 3. *Assume that the operators A, B, C satisfy conditions in Theorem 2. Then approximated solution x_k tends to the solution x of equation* (2.1) *and the speed of the convergence is calculated by the following formula*

$$\beta_s = \rho(x_s, x)$$

$$\leq \frac{\alpha}{\gamma} \frac{\theta^s}{1-\theta} \|f_1\| + (\theta+1)c_1 q^s \frac{1}{1-q} \frac{\exp[(N-1)q]-1}{[\exp(q)-1]}.$$

2.3. **Some remarks.** 1) If $A = I$, $Y = X$, and X is a Banach space, then equation (1.1) has the form:

$$x + Bx = f.$$

Assume that the operator B satisfies condition II)

$$\|B(x) - B(y)\| \leq L\|x - y\|, L = \text{const}; x, y \in X \tag{2.8}$$

and condition III) with $\gamma = 1$:

$$\|x - y + \lambda[B(x) - B(y)]\| \geq \|x - y\|, \forall\lambda \in [0,1]; \quad x, y \in X. \tag{2.9}$$

Then the application of Theorem 1 yields the result of Y. L. Gaponenco [2].

2) If $A = I$, $Y = X$, and X is a Banach space, then equation (2.1) has the form:

$$x + Bx + Cx = f.$$

Assume that the operator B satisfies conditions (2.8) and (2.9), the operator C satisfies the following condition:

$$\|C(x) - C(y)\| \leq \theta\|x - y\|, \theta < 1; \quad x, y \in X.$$

Then from Theorem 2 and Theorem 3 we get the results in [3].

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