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INTEGRAL TRANSFORMS RELATED TO THE FOURIER COSINE CONVOLUTION WITH A WEIGHT FUNCTION

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Abstract. We study a new class of integral transforms from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$, $1 \leqslant p \leqslant 2$, $p^{-1} + q^{-1} = 1$, related to the Fourier cosine convolution with a weight function. We obtain necessary and sufficient conditions under which the new transforms are unitary in $L_2(\mathbb{R}_+)$. A Plancherel type theorem and the boundedness of these integral operators are obtained. We also give several examples of the new transforms kernels.

1. Introduction

Convolutions and integral transforms of convolution type have been studied since the last century and have given many useful applications in various fields. In recent years, based on a constructive method for defining convolutions with a weight function for arbitrary integral transforms [5], several convolutions with a weight function have found and studied [5, 6, 10]. The purpose of this paper is to propose new integral transforms related to the Fourier cosine convolution with a weight function [10] and to study their properties.

Let f be a function defined on $\mathbb{R}_+$. By F_c we denote its Fourier cosine transform [2]

$$(F_c f)(y) \equiv F_c[f](y) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(x) \cos xy \, dx,$$

if $f \in L_1(\mathbb{R}_+)$, and

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \frac{d}{dy} \int\limits_0^\infty f(x) \frac{\sin xy}{x} dx,$$

if $f \in L_2(\mathbb{R}_+)$. These two definitions are equivalent if $f \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$, and the integrals are understood as improper integrals. With these notations, the

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convolution of two functions f and g in $L_1(\mathbb{R}_+)$ for the Fourier cosine transform has been defined in [9] by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)[g(|x-u|) + g(x+u)]du, \quad x > 0, \tag{1.1}$$

which satisfies the factorization equality

$$F_c[f * g](y) = (F_c f)(y)(F_c g)(y) \qquad \forall y > 0. \tag{1.2}$$

For f and $g \in L_2(\mathbb{R}_+)$, the definition (1.1) has been proved to be correct and the Parseval identity holds ([14])

$$(f * g)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty (F_c f)(y)(F_c g)(y) \cos xy \, dy. \tag{1.3}$$

Similarly, the convolution with the weight function $\gamma(y) = \cos y$ of two functions f and g in $L_1(\mathbb{R}_+)$ for the Fourier cosine transform has been defined in [10] by

$$\begin{aligned}(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty & f(y)[g(x+u+1) + g(|x-u+1|) \\ & + g(|x+u-1|) + g(|x-u-1|)]du, \ x > 0.\end{aligned} \tag{1.4}$$

This convolution satisfies the factorization property

$$F_c[f \overset{\gamma}{*} g](y) = \cos y \, (F_c f)(y)(F_c g)(y), \ y > 0. \tag{1.5}$$

In any convolution of two functions f and g, if we fix one function, say g, as the kernel, and allow the other function f vary in a certain function space, we get an integral transform of convolution type. The most famous integral transforms constructed in that way are the Watson transforms which are related to the Mellin convolution and the Mellin transform.

Recently, several classes of integral transforms related to convolutions and generalized convolutions have been investigated in [3, 7, 8, 11, 12, 14, 15]. In this paper, following [3, 14], we consider a class of integral transforms related to the convolutions (1.1) and (1.4). Namely, we study the transforms of the form

$$\begin{aligned} g(x) = & \Big(\sum_{j=0}^{n} (-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int_0^\infty f(u)\Big[k_1(x+u+1) + k_1(|x-u+1|) \\ & + k_1(|x+u-1|) + k_1(|x-u-1|)\Big]du \\ & + \int_0^\infty f(u)[k_2(|x-u|) + k_2(x+u)]du\Big), \ \ x > 0, \end{aligned} \tag{1.6}$$

where $a_0 = 1, a_j \in \mathbb{R}$ such that $1/(\sum_{j=0}^{n} a_j y^{2j}) \in L_2(\mathbb{R}_+)$. We obtain necessary and sufficient conditions on the functions $k_1, k_2 \in L_2(\mathbb{R}_+)$ ensuring the transformation (1.6) to be unitary on $L_2(\mathbb{R}_+)$, and define the inverse transformation. A Plancherel type theorem is also obtained. Furthermore, the boundedness of the transformations (1.6) on $L_p(\mathbb{R}_+)$ for $1 \leqslant p \leqslant 2$ is studied. Finally, we present several examples of these transformations kernels.

2. A Watson type Theorem

Lemma 2.1. *Let $f, g \in L_2(\mathbb{R}_+)$. Then the Parseval formula*

$$
\begin{aligned}
(2.1)\quad & \int_0^\infty f(u)[g(x+u+1) + g(|x-u+1|) + g(|x+u-1|) + g(|x-u-1|)]du \\
&= 2\sqrt{2\pi} F_c\big[\cos y (F_c f)(y)(F_c g)(y)\big](x), x > 0,
\end{aligned}
$$

holds.

Proof. Since $f, g \in L_2(\mathbb{R}_+)$, we have $fg \in L_2(\mathbb{R}_+)$. Since the Fourier cosine transform is isomorphic from $L_2(\mathbb{R}_+)$ onto itself (Theorem 50, p. 70 in [13]) and $\cos y$ is bounded, we have $\cos y(F_c f)(y)(F_c g)(y)$ belongs to $L_2(\mathbb{R}_+)$. Using the formula (see formula 1.1.3 in [4])

$$
(2.2)\qquad F_c[h(x)\cos x](y) = \frac{1}{\sqrt{2\pi}}\big[(F_c h)(y+1) + (F_c h)(y-1)\big], \quad h \in L_2(\mathbb{R}_+),
$$

and the Parseval identity (1.3) for the Fourier cosine convolution on $L_2(\mathbb{R}_+)$, we have

$$
\begin{aligned}
2\sqrt{2\pi} F_c[\cos y (F_c f)(y)(F_c g)(y)](x) &= 2\sqrt{2\pi} F_c\big[\cos y F_c[f * g](y)\big](x) \\
&= 2\big((f * g)(x+1) + (f * g)(x-1)\big).
\end{aligned}
$$

From the above identity, we obtain the relation (2.1). It also shows the existence of the convolution (1.4) for $f, g \in L_2(\mathbb{R}_+)$. The proof is complete. □

Theorem 2.2. *Let $k_1, k_2 \in L_2(\mathbb{R}_+)$ and $a_0 = 1, a_j \in \mathbb{R}$ such that $1/(\sum\limits_{j=0}^{n} a_j y^{2j}) \in L_2(\mathbb{R}_+)$. Then the condition*

$$
(2.3)\qquad |2\cos y (F_c k_1)(y) + (F_c k_2)(y)| = \frac{1}{\sqrt{2\pi}\sum\limits_{j=0}^{n} a_j y^{2j}}
$$

is necessary and sufficient to ensure that the integral transform $f \longmapsto g$:

(2.4)

$$g(x) = \Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int_0^\infty f(u)\Big[k_1(x+u+1) + k_1(|x-u+1|) + k_1(|x+u-1|) + k_1(|x-u-1|)\Big]du + \int_0^\infty f(u)[k_2(|x-u|) + k_2(x+u)]du\Big), \quad x>0,$$

is unitary on $L_2(\mathbb{R}_+)$. *Reciprocally,*

(2.5)

$$f(x) = \Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int_0^\infty g(u)\Big[\overline{k}_1(x+u+1) + \overline{k}_1(|x-u+1|) + \overline{k}_1(|x+u-1|) + \overline{k}_1(|x-u-1|)\Big]du + \int_0^\infty g(u)[\overline{k}_2(|x-u|) + \overline{k}_2(u+x)]du\Big), \quad x>0.$$

The integrals are understood in mean.

Proof. Necessity. It is well-known that $h(y)$, $yh(y)$, $y^2h(y) \in L_2(\mathbb{R})$ if and only if $(Fh)(x)$, $\frac{d}{dx}(Fh)(x)$, $\frac{d^2}{dx^2}(Fh)(x) \in L_2(\mathbb{R})$ (Theorem 68, [13]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = F\big[(-iy)^2h(y)\big](x). \tag{2.6}$$

In particular, if h is an even function such that $\big(\sum\limits_{j=0}^{n} a_j y^{2j}\big)h(y) \in L_2(\mathbb{R}_+)$, the following equality holds

$$\Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)(F_c h)(x) = F_c\Big[\big(\sum_{j=0}^{n} a_j y^{2j}\big)h(y)\Big](x). \tag{2.7}$$

Suppose that the functions $k_1, k_2 \in L_2(\mathbb{R}_+)$ satisfy condition (2.3). Using Lemma 2.1 and the factorization equalities for the convolutions (1.5), (1.2), we have

$$\begin{aligned} g(x) =&\Big(\sum_{j=0}^{n}(-1)^j a_j\frac{d^{2j}}{dx^{2j}}\Big)F_c\Big[2\sqrt{2\pi}\cos y(F_ck_1)(y)(F_cf)(y) \\ &\qquad\qquad\qquad + \sqrt{2\pi}(F_ck_2)(y)(F_cf)(y)\Big](x) \\ =&F_c\left[\sqrt{2\pi}\big(\sum_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_ck_1)(y)+(F_ck_2)(y)\big)(F_cf)(y)\right](x). \end{aligned}$$

Condition (2.3) shows that $\sqrt{2\pi}\big(\sum\limits_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_ck_1)(y)+(F_ck_2)(y)\big)(F_cf)(y)\in L_2(\mathbb{R}_+)$. Therefore, g also belongs to $L_2(\mathbb{R}_+)$. Furthermore, by virtue of the Parseval identity for the Fourier cosine transform, $\|f\|_{L_2(\mathbb{R}_+)} = \|F_cf\|_{L_2(\mathbb{R}_+)}$, and from condition (2.3), we get

$$\begin{aligned} \|g\|_{L_2(\mathbb{R}_+)} =&\Big\|\sqrt{2\pi}\big(\sum_{j=0}^{n}a_jy^{2j}\big)\big[2\cos y(F_ck_1)(y)+(F_ck_2)(y)\big](F_cf)(y)\Big\|_{L_2(\mathbb{R}_+)} \\ =&\|F_cf\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}. \end{aligned}$$

It shows that the transformation (2.4) is unitary.
Besides, since $\sqrt{2\pi}\big(\sum\limits_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_ck_1)(y)+(F_ck_2)(y)\big)(F_cf)(y)\in L_2(\mathbb{R}_+)$, we have

$$(F_cg)(y) = \sqrt{2\pi}\big(\sum_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_ck_1)(y)+(F_ck_2)(y)\big)(F_cf)(y).$$

Therefore, $(F_cf)(y) = \sqrt{2\pi}\big(\sum\limits_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_c\overline{k}_1)(y)+(F_c\overline{k}_2)(y)\big)(F_cg)(y)$.
Again, condition (2.3) shows that
$\sqrt{2\pi}\big(\sum\limits_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_c\overline{k}_1)(y)+(F_c\overline{k}_2)(y)\big)(F_cg)(y)$ belongs to $L_2(\mathbb{R}_+)$. Then formula (2.7) yields

$$\begin{aligned} &f(x) = F_c\big[\sqrt{2\pi}\big(\sum_{j=0}^{n}a_jy^{2j}\big)\big(2\cos y(F_c\overline{k}_1)(y)+(F_c\overline{k}_2)(y)\big)(F_cg)(y)\big](x) \\ &=\Big(\sum_{j=0}^{n}(-1)^j a_j\frac{d^{2j}}{dx^{2j}}\Big)F_c\big[2\sqrt{2\pi}\cos y(F_c\overline{k}_1)(y)(F_cg)(y)+\sqrt{2\pi}(F_c\overline{k}_2)(y)(F_cg)(y)\big](x). \end{aligned}$$

Using formula (2.7) and Lemma 2.1, we have

$$
\begin{aligned}
f(x) =&\Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int_0^\infty g(u)\Big[\overline{k}_1(x+u+1)+\overline{k}_1(|x-u+1|)\\
&+\overline{k}_1(|x+u-1|)+\overline{k}_1(|x-u-1|)\Big]du\\
&+\int_0^\infty g(u)[\overline{k}_2(|x-u|)+\overline{k}_2(u+x)]du\Big),\ x>0.
\end{aligned}
$$

Therefore the transformation (2.4) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation is defined by (2.5).

Sufficiency. Suppose that the transform (2.4) is unitary on $\mathbb{R}_+$. Then the isometric property on $L_2(\mathbb{R}_+)$ of the Fourier cosine transform $\|f\|_{L_2(\mathbb{R}_+)}=\|F_cf\|_{L_2(\mathbb{R}_+)}$ yields

$$
\begin{aligned}
\|g\|_{L_2(\mathbb{R}_+)} =&\|\sqrt{2\pi}\Big(\sum_{j=0}^{n} a_j y^{2j}\Big)[2\cos y(F_ck_1)(y)+(F_ck_2)(y)](F_cf)(y)\|_{L_2(\mathbb{R}_+)}\\
=&\|F_cf\|_{L_2(\mathbb{R}_+)}=\|f\|_{L_2(\mathbb{R}_+)}.
\end{aligned}
$$

Here, k_1, k_2 are in $L_2(\mathbb{R}_+)$.

Therefore, the multiplication operator $M_\theta[\cdot]$ with

$$
\theta(y)=\sqrt{2\pi}\Big(\sum_{j=0}^{n} a_j y^{2j}\Big)[2\cos y(F_ck_1)(y)+(F_ck_2)(y)]
$$

is unitary on $L_2(\mathbb{R}_+)$. This is equivalent to $|\theta(y)|\equiv 1$ on $\mathbb{R}_+$. Namely,

$$
\Big|\sqrt{2\pi}\Big(\sum_{j=0}^{n} a_j y^{2j}\Big)[2\cos y(F_ck_1)(y)+(F_ck_2)(y)]\Big|=1,\ \forall y>0.
$$

It shows that k_1 and k_2 satisfy (2.3). The proof of Theorem 2.2 is completed. □

We note that the condition $1/(\sum\limits_{j=0}^{n} a_j y^{2j})\in L_2(\mathbb{R}_+^2)$ is satisfied, if $P(x)=\sum\limits_{j=0}^{n} a_j y^{2j}$ has no real zero.

We now show the existence of functions k_1 and k_2 satisfying condition (2.3). Let $a_j, j=1,\dots,n$ be real numbers such that the polynomial $\sum\limits_{j=0}^{n} a_j y^{2j}$ has no real zero, and $h_1,h_2\in L_2(\mathbb{R}_+)$ satisfy the following identity

$$
|(F_ch_1)(y)(F_ch_2)(y)|=\frac{1}{(1+\cos^2 y)\sum\limits_{j=0}^{n} a_j y^{2j}}. \tag{2.8}
$$

The existence of functions h_1, h_2 satisfying (2.8) is always guaranteed. For instance, we can take

$$h_1(x) = F_c\left[\frac{e^{iv_1(y)}}{\left\{(1+\cos^2 y)\sum\limits_{j=0}^{n} a_j y^{2j}\right\}^{\alpha}}\right](x),$$
$$h_2(x) = F_c\left[\frac{e^{iv_2(y)}}{\left\{(1+\cos^2 y)\sum\limits_{j=0}^{n} a_j y^{2j}\right\}^{\beta}}\right](x),$$

where v_1, v_2 are arbitrary real-valued functions defined on $\mathbb{R}_+$, α, β are positive numbers such that $\alpha + \beta = 1$.

Put

$$k_1(x) = \frac{1}{2\sqrt{2\pi}}(h_1 \overset{\gamma}{*} h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}}(h_1 * h_2)(x).$$

One can easily prove that $k_1, k_2 \in L_2(\mathbb{R}_+)$. Moreover

$$\begin{aligned}|2\cos y(F_c k_1)(y) + (F_c k_2)(y)| &= \left|\frac{2}{2\sqrt{2\pi}}\cos^2 y(F_c h_1)(y)(F_c h_2)(y)\right. \\ &\quad \left. + \frac{1}{\sqrt{2\pi}}(F_c h_1)(y)(F_c h_2)(y)\right| \\ &= \frac{1}{\sqrt{2\pi}}|(F_c h_1)(y)(F_c h_2)(y)|(1+\cos^2 y) \\ &= \frac{1}{\sqrt{2\pi}\sum\limits_{j=0}^{n} a_j y^{2j}}.\end{aligned}$$

Thus k_1 and k_2 satisfy condition (2.3).

3. A Plancherel type Theorem

Theorem 3.1. *Let k_1 and k_2 be twice differentiable functions in $L_2(\mathbb{R}_+)$ satisfying condition* (2.3) *and suppose that* $K_1(x) = \Big(\sum\limits_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)k_1(x)$ *and* $K_2(x) = \Big(\sum\limits_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)k_2(x)$ *are locally bounded. Let $f \in L_2(\mathbb{R}_+)$, and for* *each positive integer N, put*

(3.1)

$$g_N(x) = \int\limits_0^N f(u)[K_1(x+u+1) + K_1(|x-u+1|) + K_1(|x+u-1|)$$

$$+ K_1(|x-u-1|)]du + \int\limits_0^N f(u)[K_2(|x-u|) + K_2(x+u)]du, \quad x > 0.$$

Then

1) $g_N \in L_2(\mathbb{R}_+)$ and, as $N \to \infty$, g_N converges in $L_2(\mathbb{R}_+)$ to a function $g \in L_2(\mathbb{R}_+)$. Moreover, $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$.

2) Reciprocally,

$$\begin{aligned} f_N(x) = \int\limits_0^N g(u)\Big[&\overline{K}_1(x+u+1) + \overline{K}_1(|x-u+1|) \\ &+ \overline{K}_1(|x+u-1|) + \overline{K}_1(|x-u-1|)\Big]du \\ &+ \int\limits_0^N g(u)[\overline{K}_2(|x-u|) + \overline{K}_2(u+x)]du, \ x > 0, \end{aligned} \tag{3.2}$$

belongs to $L_2(\mathbb{R}_+)$ and converges in $L_2(\mathbb{R}_+)$ to f as $N \to \infty$.

Remark 3.2. Due to the definitions of g_N and f_N, the integrals (3.1) and (3.2) are over finite intervals and therefore converge.

Proof of Theorem 3.1. Put $f^N = f.\chi_{(0\,;\,N)}$. Changing variables, we have

$$\begin{aligned} g_N(x) = &\int\limits_0^\infty K_1(u)\Big[f^N(x+u+1) + f^N(|x-u+1|) \\ &+ f^N(|x+u-1|) + f^N(|x-u-1|)\Big]du \\ &+ \int\limits_0^\infty K_2(u)[f(|x-u|) + f(x+u)]du \\ = &\Big(\sum_{j=0}^n (-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int\limits_0^\infty k_1(u)\Big[f^N(x+u+1) + f^N(|x-u+1|) \\ &+ f^N(|x+u-1|) + f^N(|x-u-1|)\Big]du \\ &+ \int\limits_0^\infty k_2(u)[f^N(|x-u|) + f^N(x+u)]du\Big). \end{aligned}$$

It is legitimate to interchange the order of integration and differentiation since the integrals are actually over finite intervals. Again, changing variables, we obtain

$$g_N(x) = \Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int_0^N f(u)[k_1(x+u+1|) + k_1(|x-u+1|)$$

$$+ k_1(|x+u-1|) + f^N(|x-u-1|)]du + \int_0^N f(u)[k_2(|x-u|) + k_2(x+u)]du\Big).$$

From this and in view of Theorem 2.2, we conclude that $g_N \in L_2(\mathbb{R}_+)$. Let g be the image of f under the transformation (2.4). Then Theorem 2.2 implies that $g \in L_2(\mathbb{R}_+)$ and $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$. Furthermore, the reciprocal formula (2.5) holds. We have

$$(g-g_N)(x) = \Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)\Big(\int_0^\infty (f-f^N)(u)\big[k_1(x+u+1) + k_1(|x-u+1|)$$

$$+ k_1(|x+u-1|) + k_1(|x-u-1|)\big]du$$

$$+ \int_0^\infty (f-f^N)(u)[k_2(|x-u|) + k_1(x+u)]du\Big).$$

Again by Theorem 2.2, $g - g_N \in L_2(\mathbb{R}_+)$ and

$$\|g-g_N\|_{L_2(\mathbb{R}_+)} = \|f-f^N\|_{L_2(\mathbb{R}_+)}.$$

Since $\|f-f^N\|_{L_2(\mathbb{R}_+)} \to \infty$ as $N \to \infty$, g_N converges in $L_2(\mathbb{R}_+)$ to g as $N \to \infty$. The second part of the theorem can be similarly proved. □

Remark 3.3. Theorem 2.2 and Theorem 3.1 show that the integral transform (2.4) is unitary in $L_2(\mathbb{R}_+)$ and the inverse transform is defined by formula (2.5). Moreover, integral operators (2.4) and (2.5) can be approximated in the $L_2(\mathbb{R}_+)$ norm by operators (3.1) and (3.2), respectively.

If we assume in addition that

$$K_1(x) = \Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)k_1(x) \quad \text{and} \quad K_2(x) = \Big(\sum_{j=0}^{n}(-1)^j a_j \frac{d^{2j}}{dx^{2j}}\Big)k_2(x)$$

are bounded on $\mathbb{R}_+$, then the transformation (2.4) is a bounded operator from $L_1(\mathbb{R}_+)$ to $L_\infty(\mathbb{R}_+)$.
On the other hand, Theorem 3.1 shows that the transformation (2.4) is bounded on $L_2(\mathbb{R}_+)$. Then Riesz's interpolation theorem yields the following result.

Theorem 3.4. *Let k_1, k_2 satisfy condition* (2.3) *and suppose that $K_1(x)$ and $K_2(x)$, defined as in Theorem 2, are bounded on $\mathbb{R}_+$. Let $1 \leqslant p \leqslant 2$ and q be its* *conjugate exponent, i.e.* $\frac{1}{p}+\frac{1}{q}=1$. *Then the transformation*

(3.3)
$$f(x)\longmapsto g(x)=\lim_{N\to\infty}\Big(\int_0^N f(u)[K_1(x+u+1)+K_1(|x-u+1|)+K_1(|x+u-1|)$$
$$+K_1(|x-u-1|)]du+\int_0^N f(u)[K_2(|x-u|)+K_2(x+u)]du\Big),$$

is a bounded operator from $L_p(\mathbb{R}_+)$ *into* $L_q(\mathbb{R}_+)$. *Here the limits are understood in the* $L_q(\mathbb{R}_+)$ *norm.*

4. Examples

We now present some examples of the kernels k_1, k_2 satisfying condition (2.3), and hence, the corresponding transforms (2.4).

Example 4.1. Let

$$(F_ck_1)(y)=\frac{\cos y}{2\sqrt{2\pi}(1+y^2)}\quad ;\qquad (F_ck_2)(y)=\frac{\sin^2 y}{\sqrt{2\pi}(1+y^2)}. \tag{4.1}$$

It is obvious that k_1 and k_2 defined by (4.1) satisfy condition (2.3). Moreover,

$$\begin{aligned}k_1(x)=&F_c\Big[\frac{\cos y}{2\sqrt{2\pi}(1+y^2)}\Big](x)\\ =&\frac{1}{2\pi}\int_0^{+\infty}\frac{\cos y\cos(xy)}{1+y^2}dy\\ =&\frac{1}{4\pi}\int_0^{+\infty}\frac{\cos(x+1)y+\cos(x-1)y}{1+y^2}dy.\end{aligned}$$

In view of formula (1.4.1) from [4], we get

$$k_1(x)=\frac{1}{2}\big(e+\frac{1}{e}\big)e^{-x}. \tag{4.2}$$

On the other hand,

$$\begin{aligned}k_2(x)=&F_c\Big[\frac{\sin^2 y}{\sqrt{2\pi}(1+y^2)}\Big](x)=\frac{1}{\pi}\int_0^{+\infty}\frac{\sin^2 y\cos(xy)}{1+y^2}dy\\ =&\frac{1}{4\pi}\int_0^{+\infty}\frac{2\cos(xy)-\cos(x+2)y-\cos(x-2)y}{1+y^2}dy.\end{aligned}$$

Again, using formula (1.4.1) from [4], we have

$$k_2(x) = \frac{1}{2}\big(2 - e^{-2} - e^{2}\big)e^{-x}. \tag{4.3}$$

Example 4.2. Let

$$(F_c k_1)(y) = \frac{1}{2\sqrt{2\pi}(1+y^2)} \quad ; \qquad (F_c k_2)(y) = \frac{i\sin y}{\sqrt{2\pi}(1+y^2)}. \tag{4.4}$$

It is clear that $|2\cos y(F_c k_1)(y) + (F_c k_2)(y)| = \dfrac{1}{\sqrt{2\pi}(1+y^2)}$. Moreover, formula (1.2.11) in [4] yields

$$k_1(x) = \frac{1}{2\pi}\int_0^\infty \frac{\cos xy}{1+y^2}dy = \frac{e^{-x}}{4}.$$

Using formula (2.2.14) in [4], we have

$$\begin{aligned} k_2(x) &= \frac{i}{\pi}\int_0^\infty \frac{\sin y \cos xy}{1+y^2}dy = \frac{i}{2\pi}\int_0^\infty \frac{\sin(x+1)y - \sin(x-1)y}{1+y^2}dy \\ &= \frac{i}{4\pi}\big(e^{-x-1}\overline{Ei}(x+1) - e^{x+1}Ei(-x-1) - e^{-x+1}\overline{Ei}(x+1) + e^{x-1}Ei(-x+1)\big), \end{aligned}$$

here Ei is the integral exponential [1,4].

Example 4.3. Let

$$(F_c k_1)(y) = \frac{i\sin y}{\sqrt{2\pi}(1+y^2)} \quad ; \qquad (F_c k_2)(y) = \frac{\cos 2y}{\sqrt{2\pi}(1+y^2)}. \tag{4.5}$$

Obviously, $|2\cos y(F_c k_1)(y) + (F_c k_2)(y)| = \dfrac{1}{\sqrt{2\pi}(1+y^2)}$. Moreover,

$$\begin{aligned} k_1(x) = \frac{i}{4\pi}\Big\{&e^{-x-1}\overline{Ei}(x+1) - e^{x+1}Ei(-x-1) \\ &- e^{-x+1}\overline{Ei}(x+1) + e^{x-1}Ei(-x+1)\Big\}, \end{aligned}$$

and

$$k_2(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos 2y \cos xy}{1+y^2} = \frac{1}{2\pi}\int_0^\infty \frac{\cos(x+2)y + \cos(x-2)y}{1+y^2} = (e^{-2} + e^{2})e^{-x}.$$

Example 4.4. Now we consider a generalization of Example 4.1. Let

$$(F_c k_1)(y) = \frac{\cos y}{2\sqrt{2\pi}(y^2+a^2)^{n+1}} \quad \text{and} \quad (F_c k_2)(y) = \frac{\sin^2 y}{\sqrt{2\pi}(a^2+y^2)^{n+1}}. \tag{4.6}$$

One can easily see that k_1, k_2 defined by (4.6) satisfy condition (2.3). Moreover,

$$k_1(x) = F_c\Big[\frac{\cos y}{2\sqrt{2\pi}(y^2+a^2)^{n+1}}\Big](x) = \frac{1}{2\pi}\int\limits_0^{+\infty}\frac{\cos y\cos(xy)}{(y^2+a^2)^{n+1}}$$
$$= \frac{1}{4\pi}\int\limits_0^{+\infty}\frac{\cos(x+1)y+\cos(x+1)y}{(y^2+a^2)^{n+1}}.$$

Using formula (1.2.28) from [4], we obtain
(4.7)
$$k_1(x) = \left((-1)^n\frac{\pi}{2.n!}\times\frac{d^n}{dz^n}\left(\frac{e^{-(x+1)\sqrt{z}}}{\sqrt{z}}\right)+(-1)^n\frac{\pi}{2.n!}\times\frac{d^n}{dz^n}\big(\frac{e^{-(x-1)\sqrt{z}}}{\sqrt{z}}\big)\right)\Bigg|_{z=a^2}.$$

Similarly,

$$k_2(x) = F_c\Big[\frac{\sin^2 y}{\sqrt{2\pi}(a^2+y^2)^{n+1}}\Big](x)$$
$$= \frac{1}{4\pi}\int\limits_0^{+\infty}\frac{2\cos(xy)-\cos(x+2)y-\cos(x-2)y}{(a^2+y^2)^{n+1}}dy.$$

The formula (1.2.28) from [4] again gives us

$$k_2(x) = \Big(2(-1)^n\frac{\pi}{2.n!}\times\frac{d^n}{dz^n}\big(\frac{e^{-x\sqrt{z}}}{\sqrt{z}}\big)-(-1)^n\frac{\pi}{2.n!}\times\frac{d^n}{dz^n}\big(\frac{e^{-(x+2)\sqrt{z}}}{\sqrt{z}}\big)$$
$$-(-1)^n\frac{\pi}{2.n!}\times\frac{d^n}{dz^n}\big(\frac{e^{-(x-2)\sqrt{z}}}{\sqrt{z}}\big)\Big)\Big|_{z=a^2}. \tag{4.8}$$

Example 4.5. Finally, let k_1 and k_2 be functions in $L_2(\mathbb{R}_+)$ defined by

$$(F_ck_1)(y) = \frac{\cos y}{2\sqrt{2\pi}(x^{2n}+a^{2n})}; \qquad (F_ck_2)(y) = \frac{\sin^2 y}{\sqrt{2\pi}(x^{2n}+a^{2n})}. \tag{4.9}$$

It is obvious that k_1, k_2 defined by (4.9) satisfy condition (2.3)

Since k_1 and k_2 are functions in $L_2(\mathbb{R}_+)$, we have

$$k_1(x) = F_c\Big[\frac{\cos y}{2\sqrt{2\pi}(x^{2n}+a^{2n})}\Big](x) = \frac{1}{4\pi}\int\limits_0^{+\infty}\frac{\cos(x+1)y+\cos(x-1)y}{x^{2n}+a^{2n}}dy.$$

Using formula (1.3.29) from [4], we obtain

(4.10)

$$k_1(x) = \frac{1}{8na^{2n-1}}\Bigg\{\Big[\sum_{k=1}^{n} \exp\big[-a(x+1)\sin\frac{(2k-1)\pi}{2n}\big] \times \sin\big[\frac{(2k-1)\pi}{2n} + a(x+1)\cos\frac{(2k-1)\pi}{2n}\big]\Big] + \sum_{k=1}^{n} \exp\big[-a(x-1)\sin\frac{(2k-1)\pi}{2n}\big]\sin\big[\frac{(2k-1)\pi}{2n} + a(x-1)\cos\frac{(2k-1)\pi}{2n}\big]\Bigg\}.$$

Similarily,

$$\begin{aligned} k_2(x) =& F_c\Big[\frac{\sin^2 y}{\sqrt{2\pi}(x^{2n}+a^{2n})}\Big](x) \\ =& \frac{1}{4\pi}\int_0^{+\infty} \frac{2\cos(xy) - \cos(x+2)y - \cos(x-2)y}{a^{2n}+y^{2n}}dy. \end{aligned}$$

Hence, using formula (1.3.29) in [4], we have

(4.11)

$$\begin{aligned} k_2(x) = &\sum_{k=1}^{n} \exp\big[-ax\sin\frac{(2x-1)\pi}{2n}\big] \times \sin\big[\frac{(2k-1)\pi}{2n} + a(x-1)\cos\frac{(2k-1)\pi}{2n}\big] \\ -&\sum_{k=1}^{n} \exp\big[-a(x+2)\sin\frac{(2k-1)\pi}{2n}\big] \times \sin\big[\frac{(2k-1)\pi}{2n} + a(x+2)\cos\frac{(2k-1)\pi}{2n}\big] \\ -&\sum_{k=1}^{n} \exp\big[-a(x-2)\sin\frac{(2k-1)\pi}{2n}\big] \times \sin\big[\frac{(2k-1)\pi}{2n} + a(x-2)\cos\frac{(2k-1)\pi}{2n}\big]. \end{aligned}$$

We note that the results of this work can be applied to solving some convolution type equations in closed form such as in [7, 8, 10–12].

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