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ON THE PERMANENCE OF PREDATOR-PREY MODEL WITH THE BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE IN PERIODIC ENVIRONMENT

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Abstract. This paper studies a predator-prey population system described by a differential equation with the Beddington-DeAngelis functional response in the periodic case. We establish a sufficient criterion for the permanence of the system and for the existence of a positive periodic solution.

1. Introduction

The permanence theory has developed into a mathematically fascinating area for its significance in mathematical models for population dynamics. It formalizes the concepts of non-extinction (uniform persistence) and non-explosion (dissipativity) for the considered species (see [8] and the references therein).

We consider a predator-prey population model described by the following non-autonomous ordinary differential equation

$$
\begin{aligned}
\dot{x} &= x\left[g_1(t,x) - \frac{h_1(t,y)}{k_1(t,x,y)}\right],\\
\dot{y} &= y\left[g_2(t,y) + \frac{h_2(t,x)}{k_2(t,x,y)}\right],
\end{aligned} \tag{1.1}
$$

where $g_i,\, h_i:\ \mathbb{R}\times[0,+\infty)\to\mathbb{R}$ $(i=1,\,2)$ are continuous, $T-$periodic in the first variable $(T>0)$ and continuously differentiable in the second variable; $k_i:\ \mathbb{R}\times[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ $(i=1,\,2)$ are continuous, $T-$periodic in the first variable and continuously differentiable in the second and the third variables; x and y stand for the quantity (or density) of the prey and the predator respectively.

The system (1.1) is a generalization of predator-prey population models with Beddington-DeAngelis functional response considered in [2, 10, 4]. In [2], it is concerned with the case where $g_1 = 1-x,\ g_2=-D,\ h_1 = Ay,\ h_2 = Ex$, and $k_1 = k_2 = 1+Bx+Cy$; $A,\ B,\ C,\ D,\ E$ are positive constants. In [10, 4], the authors deal with $g_1 = a_1 - b_1x,\ g_2 = -a_2 - b_2y,\ h_1 = c_1y,\ h_2 = c_2x$, and $k_1 = k_2 = p+qx+y$, where a_i, b_i, c_i $(i=1,2)$, p and q are positive constants. For the ecological significance of the system (1.1), the reader can refer to [2, 10, 4].

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Our main purpose is to improve the conditions in [2, 4] to study the permanence and the existence of a positive periodic solution of the system (1.1) in the case where its coefficients periodically vary on time t. To do that, the following hypotheses are imposed for the system (1.1):

(H1) $g_1(t,x)$ is strictly decreasing in the second variable; there exists a positive number K such that $g_1(t,K)<0$ for all $t\in[0,T]$; and $\int_0^T g_1(t,0)dt>0$,

(H2) $g_2(t,y)$ is non-increasing in the second variable and $\int_0^T g_2(t,0)dt<0$,

(H3) for each $t\in\mathbb{R}$, the function $h_i(t,\cdot)$ is non-decreasing and $h_i(t,0)=0$ for all $t\in[0,T]$ $(i=1,2)$,

(H4) $k_i(t,0,0)>0$ for all $t\in[0,T]$ and $k_i(t,x,y)$ is non-decreasing in each x and y variable $(i=1,2)$.

(H5) $\lim\limits_{y\to+\infty}\{\inf\limits_{t\in[0,T]}k_2(t,0,y)\}=+\infty$ or $\lim\limits_{y\to+\infty}\{\sup\limits_{t\in[0,T]}g_2(t,y)\}=-\infty$.

The paper is organized as follows: In Section 2, we discuss an equivalence of the persistence between periodic differential equations and discrete semi-dynamical systems corresponding to them and recall some well-known results on the persistence of discrete semi-dynamical systems. In the last section, we prove a sufficient criterion for the permanence and study the existence of a positive periodic solution of the system (1.1), then we consider some special forms of the system (1.1) and finally a numerical example illustrating the obtained result in non-autonomous cases is given.

2. Preliminaries

2.1. Persistence in periodic differential equations

We consider the following system:

$$\dot{v}=f(t,v), \tag{2.1}$$

where $f:\ \mathbb{R}\times\mathbb{R}^m_+\to\mathbb{R}^m$ $(m\geq 1,\ \mathbb{R}^m_+:=\{(v_1,...,v_m):v_i\geq 0,\ i=1,2,...,m\})$ is continuous and $T-$periodic $(T>0)$ in $t-$variable. We assume that:
(F) The Cauchy problem for (2.1) with the initial condition $v(t_0)=v_0\in\mathbb{R}^m_+$ $(t_0\in\mathbb{R})$ has a solution which is unique and continuable for all $t\geq t_0$.

By (F), we may introduce the Cauchy operator $G(t,t_0)$ $(t\geq t_0)$:

$$G(t,t_0):\mathbb{R}^m_+\to\mathbb{R}^m_+,\quad v_0\mapsto v(t),$$

where $v(t)$ is the solution of the system (2.1) at time t with $v(t_0)=v_0$. Note that the assumption of uniqueness implies that the (general) solution $v=G(t,t_0)v_0$ of the system (2.1) is continuous with respect to (t,t_0,v_0) (Theorem 2.1 in [6, p. 96]). Straightforward properties of G are: G is continuous with respect to (t,t_0,v_0) and $t-$differentiable for $t\geq t_0$; $G(t,s)G(s,t_0)=G(t,t_0)$ for $t\geq s\geq t_0$; $G(t+T,t_0+T)=G(t,t_0)$ for $t\geq t_0$; $G(t,t_0)\mathbb{R}^m_+\subset\mathbb{R}^m_+$ for $t\geq t_0$; and $G(t_0,t_0)=Id$ (the identity operator).

Let $H(\tau) = G(T+\tau, \tau)$ ($\tau \in \mathbb{R}$). We have the discrete semi-dynamical system:

$$\mathbb{N} \times \mathbb{R}_+^m \ni (n, v) \mapsto H^n(\tau)v \in \mathbb{R}_+^m. \tag{2.2}$$

Definition 2.1. System (2.1) is said to be persistent (with respect to $\partial\mathbb{R}_+^m$-the boundary of $\mathbb{R}_+^m$) if $\liminf\limits_{t\to+\infty} d\left(v(t), \partial\mathbb{R}_+^m\right) > 0$ for any solution $v(t)$ of the system (2.1) with initial condition $v(t_0) \in \operatorname{int}\mathbb{R}_+^m$, where $d(v(t), \partial\mathbb{R}_+^m)$ is the Euclid distance from the point $v(t)$ to $\partial\mathbb{R}_+^m$ and $\operatorname{int}\mathbb{R}_+^m$ is the interior of $\mathbb{R}_+^m$. It is uniformly persistent if there exists a positive number δ such that $\liminf\limits_{t\to+\infty} d\left(v(t), \partial\mathbb{R}_+^m\right) \geq \delta$ and δ does not depend on initial condition $v(t_0) = (v_{01}, v_{02}, \dots, v_{0m})$ in $\operatorname{int}\mathbb{R}_+^m$.

Definition 2.2. System (2.1) is said to be dissipative if there exists a positive number M such that $\limsup\limits_{t\to+\infty} \|v(t)\| \leq M$ for any solution $v(t)$ of the system (2.1).

Definition 2.3. System (2.1) is said to be permanent if it is uniformly persistent and dissipative.

Definition 2.4. $H(\tau)$ (or (2.2)) is said to be persistent (with respect to $\partial\mathbb{R}_+^m$) if

$$\liminf_{n\to+\infty} d\left(H^n(\tau)v, \partial\mathbb{R}_+^m\right) > 0 \ \text{ for all } v \in \operatorname{int}\mathbb{R}_+^m.$$

It is uniformly persistent if there exists a positive number δ such that

$$\liminf_{n\to+\infty} d\left(H^n(\tau)v, \partial\mathbb{R}_+^m\right) \geq \delta \text{ for all } v \in \operatorname{int}\mathbb{R}_+^m.$$

Theorem 2.5. (see [1]) *Let* (F) *hold. If the system* (2.1) *is dissipative, then*

(i) *System* (2.1) *is persistent if and only if, for each* $\tau \in [0, T]$, $H(\tau)$ *is persistent,*

(ii) *System* (2.1) *is uniformly persistent if and only if, for each* $\tau \in [0, T]$, $H(\tau)$ *is uniformly persistent.*

2.2. **Persistence for maps**

We now recall some definitions and well-known results on persistence for maps. Let V be a metric space with a metric d and let W be a closed subset of V. Let $F : V \to V$ be continuous such that $F(W) \subset W$ and $F(V\backslash W) \subset V\backslash W$. Denote by $F|_W$ the restriction of F on W. Let us denote by $\mathbb{Z}$ (and $\mathbb{Z}_+$) the set of integers (and the set of non-negative integers, respectively). Recall that a sequence $\{u_n\}_{n\in\mathbb{Z}_+}$ ($\{u_{-n}\}_{n\in\mathbb{Z}_+}$, respectively) of points in V is said to be a positive (negative) orbits through $u \in V$ if $u_0 = u$ and $Fu_n = u_{n+1}$ ($Fu_{-n-1} = u_{-n}$) for all $n \in \mathbb{Z}_+$; a sequence $\{u_n\}_{n\in\mathbb{Z}}$ with $u_0 = u$ and $Fu_n = u_{n+1}$ for all $n \in \mathbb{Z}$ is an orbit through u. A positive (respectively, negative) orbit is said to be compact if the sequence, when considered as a subset of V, is precompact. Denote by $\wedge^+(\{u_n\}_{n\in\mathbb{Z}_+})$ or $\wedge^+(u)$ (respectively, $\wedge^-(\{u_{-n}\}_{n\in\mathbb{Z}_+})$) the omega limit set (the alpha limit set) of positive (negative) orbits through u (see [5]). The map F is said to be dissipative if the set $\Omega(F) = \cup\{\wedge^+(u) : u \in V\}$ is precompact.

The subset M of V is called positively invariant (respectively, invariant) (under F) if $F(M) \subset M$ (respectively, $F(M) = M$).

A non-empty closed invariant subset M is an isolated invariant set if it is the maximal (under the order of inclusion) invariant set in some neighborhood of itself.

Let M be an isolated invariant set. A compact positive orbit $\{u_n\}_{n\in\mathbb{Z}_+}$ is said to be in the stable set of M (under F) (in notation $\{u_n\}_{n\in\mathbb{Z}_+} \in W^+(M)$) if $\wedge^+\{u_n\}_{n\in\mathbb{Z}_+} \subset M$; a compact negative orbit $\{u_{-n}\}_{n\in\mathbb{Z}_+}$ is said to be in the unstable set of M (in notation $\{u_{-n}\}_{n\in\mathbb{Z}_+} \in W^-(M)$) if $\wedge^-(\{u_{-n}\}_{n\in\mathbb{Z}_+}) \subset M$.

For two isolated sets M_1 and M_2 we say that M_1 is chained to M_2, in notation $M_1 \to M_2$, if there exists an orbit $\{u_n\}_{n\in\mathbb{Z}}$ with $u_k \notin M_1 \cup M_2$ for some $k \in \mathbb{Z}$ such that $\{u_{-n}\}_{n\in\mathbb{Z}_+} \in \wedge^-(M_1)$ and $\{u_n\}_{n\in\mathbb{Z}_+} \in \wedge^+(M_2)$. A finite sequence $M_1, \ldots, M_k$ of isolated invariant sets will be called a chain if $M_1 \to M_2 \to \cdots \to M_k$ ($M_1 \to M_1$ if $k = 1$). The chain is a cycle if $M_k = M_1$. A covering $\Pi = \{M_1, \ldots, M_k\}$ of $\bar{\Omega}(F|_W)$ (the closure of $\Omega(F|_W)$) is called an isolated covering of $F|_W$ if $M_1, \ldots, M_k$ are pairwise disjoint, compact and isolated invariant (under F); the isolated covering Π is called an acyclic covering if no subsets of Π form a cycle for $F|_W$ in W.

Theorem 2.6. (see [7]) *Suppose that*

(i) *F is dissipative,*

(ii) *$F|_W$ has an acyclic covering $\Pi = \{M_1, \ldots, M_k\}$.*

Then F is uniformly persistent with respect to W (i.e. there exists a positive number ϵ such that $\liminf_{n\to+\infty} d(F^n u, W) \geq \epsilon$ for all $u \in V\backslash W$) if and only if the following condition holds:

(H) *There is no positive orbit $\{u_n\}_{n\in\mathbb{Z}_+}$ in $V\backslash W$ such that $\{u_n\}_{n\in\mathbb{Z}_+} \in W^+(M_i)$ for some $i \in 1, 2, \ldots, k$.*

Theorem 2.7. (see [9]) *Suppose $V = \mathbb{R}^m_+$, $W = \partial\mathbb{R}^m_+$ and F is uniformly persistent and dissipative. Then F has a fixed point in int$(\mathbb{R}^m_+)$.*

3. Permanence for the system (1.1)

In this part, we study the permanence and the existence of positive periodic solutions of the system (1.1).

It is easy to see that the axes Ox, Oy and int$\mathbb{R}^2_+$ are invariant with respect to the system (1.1). The origin $O(0,0)$ is an equilibrium point of the system (1.1).

On the $y-$axis, the following equation presents the growth rate of predator population in the absence of the prey:

$$\dot{y} = yg_2(t, y). \tag{3.1}$$

Lemma 3.1. *Let* (H2) *hold. Then for any solution $Y(t)$ of Equation* (3.1) *with the initial condition $Y(t_0) > 0$ we have $\lim_{t\to+\infty} Y(t) = 0$.*

Proof. By (H2), we have $\dot{Y} \leq Yg_2(t, 0)$. This implies that

$$Y(t) \leq Y(t_0) \exp \int_{t_0}^{t} g_2(s, 0)ds$$

for all $t \geq t_0$. Since $g_2(t,0)$ is $T-$periodic in t and $\int\limits_{t_0}^{T} g_2(s,0)ds < 0$, it follows that $\lim\limits_{t\to+\infty} Y(t) = 0$. □

On the $x-$axis, the following equation presents the growth rate of prey population in the absence of the predator:

$$\dot{x} = xg_1(t,x). \tag{3.2}$$

Lemma 3.2. (see [11]) *Let* (H1) *hold. Then Equation* (3.2) *has a unique positive* $T-$*periodic solution* $X^0(t)$*. Furthermore, this solution is asymptotically globally stable on* $(0,+\infty)$.

Lemma 3.3. *Let* (H1), (H2), (H3) *and* (H4) *hold. Then for each* $(t_0,x_0,y_0) \in \mathbb{R}\times\mathbb{R}^2_+$, *the system* (1.1) *has a unique solution* $(x(t),y(t))$ *defined on* $[t_0,+\infty)$, *satisfying* $x(t_0) = x_0$, $y(t_0) = y_0$.

Proof. Since the coefficients are continuously differentiable, the system (1.1) is locally and uniquely solvable. Moreover, $x(t) > 0$ and $y(t) > 0$ if $x(t_0) > 0$ and $y(t_0) > 0$ by virtue of invariant property of $\mathrm{int}\mathbb{R}^2_+$. On the other hand, we have $\dot{x}(t) \leq x(t)g_1(t,x(t))$ for all $t > t_0$. Therefore, by the comparison theorem we have $x(t) \leq X(t)$ for all $t \in [t_0,\omega)$, where $X(t)$ is the solution of the equation (3.2) with $X(t_0) = x(t_0)$ and ω is the right maximal interval of the existence of the solution $(x(t),y(t))$. This inequality and the boundedness of $X(t)$ on $[t_0,+\infty)$ (by Lemma 3.2) imply that $x(t)$ is bounded from above. Therefore, $\dfrac{h_2(t,x(t))}{k_2(t,x(t),y(t))}$ is bounded from above by a constant M. Hence, $\dot{y}(t) = y(t)\Big(g_2(t,y(t)) + \dfrac{h_2(t,x(t))}{k_2(t,x(t),y(t))}\Big) \leq y(t)(g(t,0)+M)$. This relation says that $y(t)$ is not exploded. Therefore, any solution of the system (1.1) with $x(t_0) = x_0 > 0$, $y(t_0) = y_0 > 0$ is defined on $[t_0,\infty)$. □

Lemma 3.4. *Let* (H1), (H2), (H3) *and* (H4) *hold. Then the characteristic multipliers of the linear variational system corresponding to the trivial solution of the system* (1.1) *have moduli different from* 1.

Proof. The linear variational system at $O(0,0)$ of the system (1.1) is

$$\begin{aligned} \dot{z}_1 &= g_1(t,0)z_1, \\ \dot{z}_2 &= g_2(t,0)z_2. \end{aligned} \tag{3.3}$$

Let $Z(t)$ be the matrix solution of the system (3.3) with $Z(0) = \mathbb{I}-$ the identity matrix. Then

$$Z(t) = \mathrm{diag}\Big(\exp\{\int_0^{\mathrm{t}} \mathrm{g}_1(\mathrm{s},0)\mathrm{ds}\}, \exp\{\int_0^{\mathrm{t}} \mathrm{g}_2(\mathrm{s},0)\mathrm{ds}\}\Big).$$

Thus, two eigenvalues of $Z(T)$ are

$$\lambda_1 = \exp\{\int_0^T g_1(s,0)ds\} > 1 \quad \text{(by (H1))},$$
$$\lambda_2 = \exp\{\int_0^T g_2(s,0)ds\} < 1 \quad \text{(by (H2))}.$$

□

Lemma 3.5. *Let* (H1), (H2), (H3) *and* (H4) *hold. If*

$$\int_0^T \left[g_2(t,0) + \frac{h_2(t,X^0(t))}{k_2(t,X^0(t),0)}\right] dt \neq 0, \tag{3.4}$$

where $X^0(t)$ is the unique positive $T-$periodic solution of the equation (3.2)*, then the characteristic multipliers of the linear variational system corresponding to the $T-$periodic solution $(X^0(t),0)$ of the system* (1.1) *have moduli different from* 1*.*

Proof. The linear variational system corresponding to the $T-$periodic solution $(X^0(t),0)$ of the system (1.1) is

$$\begin{aligned} \dot z_1 &= [g_1(t,X^0(t)) + g'_{1x}(t,X^0(t))X^0(t)]z_1 - \frac{h'_{1y}(t,0)X^0(t)}{k_1(t,X^0(t),0)}z_2,\\ \dot z_2 &= [g_2(t,0) + \frac{h_2(t,X^0(t))}{k_2(t,X^0(t),0)}]z_2. \end{aligned} \tag{3.5}$$

Let $Z(t)$ be the matrix solution of the system (3.5) with $Z(0) = \mathbb{I}$. Some entries of $Z(t)$ are

$$z_{11} = \exp\{\int_0^t \left[g_1(s,X^0(s)) + g'_{1x}(s,X^0(s))X^0(s)\right] ds\},$$

$$z_{21} = 0, z_{22} = \exp\{\int_0^t \left[g_2(s,0) + \frac{h_2(s,X^0(s))}{k_2(s,X^0(s),0)}\right] ds\}.$$

Hence, two eigenvalues of $Z(T)$ are

$$\lambda_1 = \exp\{\int_0^T \left[g_1(s,X^0(s)) + g'_{1x}(s,X^0(s))X^0(s)\right] ds\},$$

$$\lambda_2 = \exp\{\int_0^T \left[g_2(s,0) + \frac{h_2(s,X^0(s))}{k_2(s,X^0(s),0)}\right] ds\}.$$

According to (3.4), we have $|\lambda_2| \neq 1$. Since $X^0(t)$ is a $T-$periodic solution of the equation (3.2), $\int_0^T g_1(t,X^0(t))dt = 0$. Thus, since $g_1(t,x)$ is strictly decreasing in x, $\lambda_1 = \exp\{\int_0^T \left[g'_{1x}(s,X^0(s))X^0(s)\right] ds\} < 1$. □

Theorem 3.6. *Let* (H1)*,* (H2)*,* (H3)*,* (H4) *and* (H5) *hold. Then the system* (1.1) *is dissipative.*

Proof. Let $(x(t), y(t))$ be a solution of the system (1.1) with initial condition $(x(t_0), y(t_0)) \in \mathbb{R}^2_+$. Let $\triangle$ be a positive number such that $\triangle > \max\limits_{0 \le t \le T} X^0(t)$, where $X^0(t)$ is the unique positive $T-$periodic solution of the equation (3.2). Clearly that $\dot{x}(t) \le x(t) g_1(t, x(t))$ for $t \ge t_0$. By the comparison theorem, we can conclude that $x(t) \le X(t)$ for all $t \ge t_0$, where $X(t)$ is the solution of the equation (3.2) with $X(t_0) = x(t_0)$. By Lemma 3.2, there exists $t_1 > t_0$ such that $x(t) < \triangle$ for all $t \ge t_1$. By (H5), we now consider the following two cases:
Case 1: $\lim\limits_{y \to +\infty} \{ \sup\limits_{t \in [0,T]} g_2(t, y)\} = -\infty$.
Then

$$\dot{y}(t) \le y(t) \left[g_2(t, y(t)) + \frac{h_2(t, \triangle)}{k_2(t, 0, 0)} \right] \text{ for } t \ge t_1.$$

There exist positive numbers M and α such that $g_2(t, M) + \dfrac{h_2(t, \triangle)}{k_2(t, 0, 0)} < -\alpha$ for all $t \in [0, T]$. Thus, $\dot{y}(t) \le -\alpha y(t)$ for all $t \ge t_1$ whenever $y(t) \ge M$. Hence, $\limsup\limits_{t \to +\infty} y(t) \le M$.
Case 2: $\lim\limits_{y \to +\infty} \{ \inf\limits_{t \in [0,T]} k_2(t, 0, y)\} = +\infty$.
Let $\bar{g}_2 = \frac{1}{T} \int\limits_0^T g_2(s, 0) ds$. Then $-\bar{g}_2 t + \int\limits_0^t g_2(s, 0) ds$ is $T-$periodic. By the change of variables $y(t) = u(t) \exp\{-\bar{g}_2 t + \int\limits_0^t g_2(s, 0) ds\}$, from the system (1.1), we obtain

$$\dot{u} = u[\bar{g}_2 - g_2(t, 0) + \tilde{g}_2(t, u) + \frac{h_2(t, x)}{\tilde{k}_2(t, x, u)}],$$

where

$$\tilde{g}_2(t, u) = g_2(t, u \exp\{-\bar{g}_2 t + \int\limits_0^t g_2(s, 0) ds\})$$

and

$$\tilde{k}_2(t, x, u) = k_2(t, x, u \exp\{-\bar{g}_2 t + \int\limits_0^t g_2(s, 0) ds\}).$$

This implies that

$$\dot{u}(t) \le u(t)[\bar{g}_2 + \frac{h_2(t, \triangle)}{\tilde{k}_2(t, 0, u)}] \text{ for } t \ge t_1.$$

Since $\lim\limits_{u \to +\infty} \{ \inf\limits_{t \in [0,T]} \tilde{k}_2(t, 0, u)\} = +\infty$ and $\bar{g}_2 < 0$, there exist positive numbers M and γ such that $\bar{g}_2 + \dfrac{h_2(t, \triangle)}{\tilde{k}_2(t, 0, M)} < -\gamma$ for all $t \in [0, T]$. Thus, $\dot{u}(t) \le -\gamma u$ for all

$t \geq t_1$ whenever $u \geq M$. This implies $\limsup\limits_{t\to+\infty} u(t) \leq M$. Hence

$$\limsup_{t\to+\infty} y(t) \leq \bar{M} := \max_{t\in[0,T]} M \exp\{-\bar{g}_2 t + \int_0^t g_2(s,0)ds\}.$$

The lemma is proved. □

Theorem 3.7. *Let* (H1), (H2), (H3), (H4) *and* (H5) *hold. If*

$$\int_0^T \left[g_2(t,0) + \frac{h_2(t, X^0(t)}{k_2(t, X^0(t), 0)} \right] dt > 0, \tag{3.6}$$

where $X^0(t)$ is the unique positive $T-$periodic solution of the equation (3.2)*, then the system* (1.1) *is permanent. Moreover, the system* (1.1) *has at least one positive $T-$periodic solution, whose components are strictly positive.*

Proof. By Theorem 3.6, the system (1.1) is dissipative. We need to show that the system (1.1) is uniformly persistent. By Lemma 3.3, the system (1.1) satisfies hypothesis (F). By Theorem 2.5, it is enough to show that for each $\tau \in [0,T]$, $H(\tau)$ is uniformly persistent. Let $Q_\tau = (X^0(\tau), 0)$. It is easy to see that $H(\tau)(\partial\mathbb{R}^2_+) \subset \partial\mathbb{R}^2_+$ and $H(\tau)(\operatorname{int}\mathbb{R}^2_+) \subset \operatorname{int}\mathbb{R}^2_+$. On the other hand, $H(\tau)$ is dissipative (by Theorem 3.6), thus $H(\tau)$ satisfies hypothesis (i) in Theorem 2.6. Furthermore, by Lemmas 3.1 and 3.2, $\Omega(H(\tau)|_{\partial\mathbb{R}^2_+}) = \{0,\ Q_\tau\}$. It follows from Lemmas 3.4 and 3.5 that $\{0\}$ and $\{Q_\tau\}$ are isolated invariant sets under $H(\tau)$. Thus $\Pi = \{\{0\},\ \{Q_\tau\}\}$ is an isolated covering of $H(\tau)|_{\partial\mathbb{R}^2_+}$. By Lemma 3.2, Π is acyclic. Thus, $H(\tau)$ satisfies hypothesis (ii) in Theorem 2.6. We shall prove that $H(\tau)$ satisfies hypothesis (H) of Theorem 2.6.

Suppose in the contrary that it is false. Then at least one of the following two cases is met:

(a) There exists $\{H^n(\tau)u\}_{n=0}^{+\infty} \subset \operatorname{int}(\mathbb{R}^2_+)$ such that $\lim\limits_{n\to+\infty} \|H^n(\tau)u\| = 0$;

(b) There exists $\{H^n(\tau)u\}_{n=0}^{+\infty} \subset \operatorname{int}(\mathbb{R}^2_+)$ such that $\lim\limits_{n\to+\infty} \|H^n(\tau)u - Q_\tau\| = 0$.

If (a) holds, then by Arzela-Ascoli theorem, the sequence of continuous functions $\{x(t+\tau+nT), y(t+\tau+nT)\}_{n=0}^{+\infty}$ on $[0,T]$ converges uniformly to $(0,0)$ as $n \to +\infty$, where $(x(t), y(t))$ is the solution of the system (1.1) with $(x(\tau), y(\tau)) = u$. This implies that $\lim\limits_{t\to+\infty}(x(t), y(t)) = (0,0)$. By (H1), (H3) and (H4), there exists a positive number ϵ such that

$$\int_0^T \left[g_1(t,\epsilon) - \frac{h_1(t,\epsilon)}{k_1(t,0,0)} \right] dt > 0.$$

Let t_1 be a number ($t_1 \geq \tau$) such that $x(t) < \epsilon,\ y(t) < \epsilon$ for all $t > t_1$. Then

$$\dot{x}(t) \geq x(t) \left[g_1(t,\epsilon) - \frac{h_1(t,\epsilon)}{k_1(t,0,0)} \right] \text{ for } t > t_1.$$

This implies that

$$x(t) \geq x(t_1) \exp \int_{t_1}^{t} \left[g_1(s,\epsilon) - \frac{h_1(s,\epsilon)}{k_1(s,0,0)} \right] ds \text{ for } t > t_1.$$

Hence, $\lim_{t\to+\infty} x(t) = +\infty$. This contradicts the fact that $\lim_{t\to+\infty} x(t) = 0$. Thus, (a) cannot happen.

If (b) holds, then by the same argument as given above, we can conclude that $\lim_{t\to+\infty} |x(t) - X^0(t)| = 0$ and $\lim_{t\to+\infty} y(t) = 0$, where $(x(t), y(t))$ is the solution of the system (1.1) with $(x(\tau), y(\tau)) = u$. By (3.6), there exists a positive number ϵ such that

$$\int_0^T \left[g_2(t,\epsilon) + \frac{h_2(t, X^0(t) - \epsilon)}{k_2(t, X^0(t) + \epsilon, \epsilon)} \right] dt > 0.$$

Let t_1 be a number ($t_1 \geq \tau$) such that $X^0(t) - \epsilon < x(t) < X^0(t) + \epsilon$ and $y(t) < \epsilon$ for all $t \geq t_1$. Then

$$\dot{y}(t) \geq y(t) \left[g_2(t,\epsilon) + \frac{h_2(t, X^0(t) - \epsilon)}{k_2(t, X^0(t) + \epsilon, \epsilon)} \right] \text{ for } t \geq t_1.$$

Thus,

$$y(t) \geq y(t_1) \exp \int_{t_1}^{t} \left[g_2(s,\epsilon) + \frac{h_2(s, X^0(s) - \epsilon)}{k_2(s, X^0(s) + \epsilon, \epsilon)} \right] ds \text{ for } t \geq t_1.$$

This implies $\lim_{t\to+\infty} y(t) = +\infty$, which contradicts $\lim_{t\to+\infty} y(t) = 0$. Thus, case (b) does not happen. Hence, for each $\tau \in [0, T]$, $H(\tau)$ is uniformly persistent. By Theorem 2.5, the system (1.1) is uniformly persistent. Thus, the system (1.1) is permanent.

Since $H(0)$ is permanent, it follows from Theorem 2.7 that $H(0)$ has at least one equilibrium (x_0^*, y_0^*) in $\operatorname{int} \mathbb{R}_+^2$. It is easy to see that $(x^0(t), y^0(t)) = G(t,0)(x_0^*, y_0^*)$ is a T-periodic solution of the system (1.1). Since the system (1.1) is uniformly persistent, it follows that $(x^0(t), y^0(t)) \in \operatorname{int} \mathbb{R}_+^m$ for all $t \in \mathbb{R}$. The theorem is proved. $\square$

The following two theorems are extinction results for the predator.

Theorem 3.8. *Let* (H1)*,* (H2)*,* (H3) *and* (H4) *hold. If*

$$\int_0^T \left[g_2(t,0) + \frac{h_2(t, X^0(t))}{k_2(t,0,0)} \right] dt < 0, \tag{3.7}$$

then $\lim_{t\to+\infty} y(t) = 0$ *for any solution* $(x(t), y(t))$ *of the system* (1.1) *with* $x(t_0) > 0$ *and* $y(t_0) > 0$.

Proof. By (3.7), there exists a positive number ϵ such that

$$\int_0^T \left[g_2(t,0) + \frac{h_2(t, X^0(t)+\epsilon)}{k_2(t,0,0)} \right] dt < 0.$$

Let $(x(t), y(t))$ be a solution of the system (1.1) with $x(t_0) > 0$, $y(t_0) > 0$. We have $\dot{x}(t) \le x(t) g_1(t, x(t))$ for all $t \ge t_0$. By the comparison theorem, it follows that $x(t) \le X(t)$ for all $t \ge t_0$, where $X(t)$ is the solution of the equation (3.2) with $X(t_0) = x(t_0)$. By Lemma 3.2, there exists $t_1 > t_0$ such that $x(t_0) < X^0(t) + \epsilon$ for all $t \ge t_1$. Therefore,

$$\dot{y}(t) \le y(t) \left[g_2(t,0) + \frac{h_2(t, X^0(t)+\epsilon)}{k_2(t,0,0)} \right] \text{ for } t \ge t_1.$$

This implies that

$$y(t) \le y(t_1) \exp \int_{t_1}^t \left[g_2(s,0) + \frac{h_2(s, X^0(s)+\epsilon)}{k_2(s,0,0)} \right] ds \text{ for } t \ge t_1.$$

Thus, $\lim\limits_{t\to+\infty} y(t) = 0$. □

Theorem 3.9. *Let* (H1), (H2), (H3) *and* (H4) *hold. If*

$$\int_0^T \left[g_2(t,0) + \frac{h_2(t, X^0(t))}{k_2(t, X^0(t), 0)} \right] dt < 0 \tag{3.8}$$

and

$$\frac{h_2(t,x)}{k_2(t,x,0)} \text{ is non-decreasing in } x, \tag{3.9}$$

then $\lim\limits_{t\to+\infty} y(t) = 0$ *for any solution* $(x(t), y(t))$ *of the system* (1.1) *with* $x(t_0) > 0$ *and* $y(t_0) > 0$.

Proof. By (3.8) and (3.9), there exists a positive number ϵ such that

$$\int_0^T \left[g_2(t,0) + \frac{h_2(t, X^0(t)+\epsilon)}{k_2(t, X^0(t)+\epsilon, 0)} \right] dt < 0.$$

By the same argument as given in the proof of Theorem 3.8, we can conclude that $\lim\limits_{t\to+\infty} y(t) = 0$. □

Remark 3.10. Hypothesis (3.9) is adapted to the real life of populations, since the quantity of prey population x is increasing then it makes itself useful about the development of predator y.

Consider a special case of the system (1.1), where the right hand side does not depend on t, i.e., $g_1(t,x) = g_1^*(x),\ g_2(t,y) = g^*(y),\ h_1(t,y) = h_1^*(y),\ h_2(t,x) = h_2^*(x),\ k_1(t,x,y) = k_1^*(x,y)$ and $k_2(t,x,y) = k_2^*(x,y)$:

$$\begin{aligned} \dot{x} &= x\left[g_1^*(x) - \frac{h_1^*(y)}{k_1^*(x,y)}\right], \\ \dot{y} &= y\left[g_2^*(y) + \frac{h_2^*(x)}{k_2^*(x,y)}\right]. \end{aligned} \tag{3.10}$$

Corollary 3.11. *Suppose that*

(H1*) $g_1^*(x)$ *is strictly decreasing; there exists a positive numbers* K *such that* $g_1^*(K) < 0$ *and* $g_1^*(0) > 0$,
(H2*) $g_2^*(y)$ *is non-increasing and* $g_2^*(0) < 0$,
(H3*) h_i^* *is non-decreasing and* $h_i^*(0) = 0$ $(i = 1,2)$,
(H4*) $k_i^*(0,0) > 0$ *and* $k_i^*(x,y)$ *is non-decreasing in each* x *and* y *variable* $(i = 1,2)$,
(H5*) $\lim\limits_{y\to+\infty}[k_2^*(0,y) - g_2^*(y)] = +\infty$.

If

$$g_2^*(0) + \frac{h_2^*(X^0)}{k_2^*(X^0,0)} > 0, \tag{3.11}$$

where X^0 *is a unique positive solution of the equation* $g_1^*(X) = 0$, *then the system* (3.10) *is permanent. Moreover, the system* (3.10) *has at least one positive equilibrium.*

Proof. By Theorem 3.7, (3.10) is permanent. Thus, the system (3.10) has at least one positive equilibrium (the permanence implies the existence of a positive equilibrium; see [9]). □

Consider the following system, which is a special case of the system (1.1):

$$\begin{aligned} \dot{x} &= x\left[a_1(t) - b_1(t)x - \frac{c_1(t)y}{p_1(t) + q_1(t)x + r_1(t)y}\right], \\ \dot{y} &= y\left[-a_2(t) - b_2(t)y - \frac{c_2(t)x}{p_2(t) + q_2(t)x + r_2(t)y}\right], \end{aligned} \tag{3.12}$$

where $a_i,\ b_i,\ c_i,\ p_i,\ q_i,\ r_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ are continuous and $T-$periodic. By Theorem 3.7, one can easily reach the following corollary:

Corollary 3.12. *Suppose that*

(H1**) $b_1(t) > 0$ *for all* $t \in [0,T]$ *and* $\int\limits_0^T a_1(t)dt > 0$,
(H2**) $b_2(t) \geq 0$ *for all* $t \in [0,T]$ *and* $\int\limits_0^T a_2(t)dt > 0$,
(H3**) $c_i(t) \geq 0$ $(i = 1,2)$ *for all* $t \in [0,T]$,
(H4**) $p_i(t) > 0,\ q_i(t) \geq 0,\ r_i(t) \geq 0$ $(i = 1,2)$ *for all* $t \in [0,T]$ *and*
(H5**) $r_2(t) > 0$ *for all* $t \in [0,T]$ *or* $b_2(t) > 0$ *for all* $t \in [0,T]$.

If

$$\int_0^T \left(-a_2(t) + \frac{c_2(t)X^0(t)}{p_2(t) + q_2(t)X^0(t)} \right) dt > 0, \tag{3.13}$$

where $X^0(t)$ *is the unique positive* $T-$*periodic solution of* $\dot{x} = x[a_1(t) - b_1(t)x]$, *then the system* (3.12) *is permanent. Moreover, the system* (3.12) *has at least one positive* $T-$*periodic solution, whose components are strictly positive.*

Remark 3.13. In [2], the authors considered the system (3.10) with $g_1^* = 1 - x,\ g_2^* = -D,\ h_1^* = Ay,\ h_2^* = Ex$, and $k_1^* = k_2^* = 1 + Bx + Cy$, i.e., the following system

$$\begin{aligned} \dot{x} &= x\left[1 - x - \frac{Ay}{1 + Bx + Cy}\right], \\ \dot{y} &= y\left[-D + \frac{Ex}{1 + Bx + Cy}\right], \end{aligned} \tag{3.14}$$

where $A,\ B,\ C$ and D are positive constants. It is easy to see that hypotheses (H1*)-(H5*) are satisfied. Furthermore, condition (3.11) becomes the inequality $E > (B+1)D$. Thus, from Corollary 3.11, we obtain the sufficient condition for the permanence of the system (3.14) which was given in [2].

In [4], the system (3.10) was considered with $g_1^* = a_1 - b_1 x,\ g_2^* = -a_2 - b_2 y,\ h_1^* = c_1 y,\ h_2^* = c_2 x$ and $k_1^* = k_2^* = p + qx + y$, where a_i, b_i, c_i $(i = 1, 2)$, p and q are positive constants. It is easy to see that Hypotheses (H1*)-(H5*) are satisfied and $X^0 = \dfrac{a_1}{b_1}$. Hence, condition (3.11) becomes the inequality $-a_2 + \dfrac{a_1 c_2}{b_1 p + a_1 q} > 0$. This inequality is the sufficient condition for the permanence of the system (3.10) which was given in [4].

Finally, we present an example to illustrate our result in non-autonomous cases.

Example. Consider the system

$$\begin{aligned} \dot{x} &= x\left[1 - (\frac{3}{2} + \cos t)x - \frac{(1 + \cos^2 t)y}{1 + x + (2 + \sin t)y}\right], \\ \dot{y} &= y\left[-\frac{1}{2} + \cos t - y\cos^2 t + \frac{(2 + \cos^2 t)x}{2[1 + x + (2 + \sin t)y]}\right]. \end{aligned} \tag{3.15}$$

It is easy to see that the system (3.15) satisfies Hypotheses (H1**)-(H5**). Moreover, $X^0(t) = \dfrac{2}{3 + \cos t + \sin t}$ is the unique positive $T-$periodic solution of the equation $\dot{x} = x\left[1 - (\frac{3}{2} + \cos t)x\right]$. Condition (3.13) is satisfied, since

$$\int_0^{2\pi} \left(-\frac{1}{2} + \cos t + \frac{2 + \cos^2 t}{5 + \cos t + \sin t} \right) dt = -\pi + \frac{5}{23}\sqrt{23}\pi > 0.$$

By Corollary 3.12, the system (3.15) is permanent and it has at least one $T-$periodic solution, whose components are strictly positive.

We illustrate the behavior of numerical solutions of the system (3.15) by Fig.1
The behavior of the solutions of the system (3.15) for two initial values: the dash line shows the trajectory of the solution $(x_1(t), y_1(t))$ with $(x_1(0) = 1, y_1(0) = 0.3)$, the solid line corresponds to the solution $(x_2(t), y_2(t))$ with $(x_2(0) = 0.4, y_2(0) = 0.05)$.

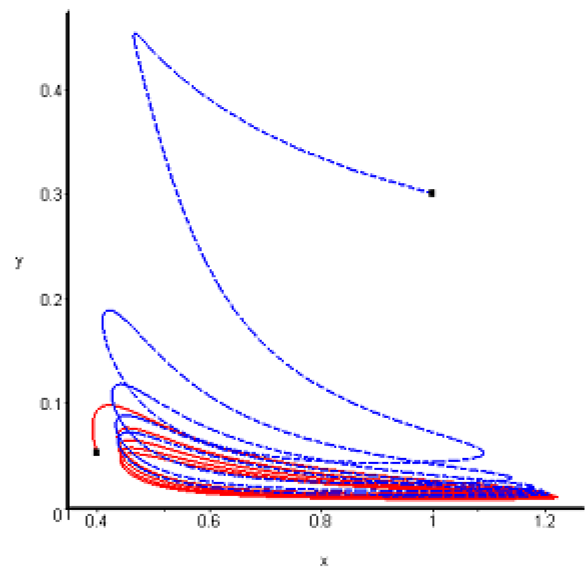

FIGURE 1. The behavior of solutions of the system (3.15) for two initial values.

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