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SOME PROPERTIES OF FUNCTIONS WITH BOUNDED SPECTRUM

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Abstract. In this paper we prove that the results obtained in [3] for L_p-norm are still valid for a norm which is generated by a concave function.

Let $\mathcal{C}$ denote the family of all non-zero concave functions $\Phi : [0, +\infty) \to [0, +\infty]$, which are non-decreasing and satisfy $\Phi(0) = 0$. For an arbitrary measurable function f and $\Phi \in \mathcal{C}$, we define

$$\|f\|_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_f(t)\big)dt,$$

where $\lambda_f(t) = \mu(\{x : |f(x)| > t\}),\ \ t \geq 0$. Let $N_\Phi = N_\Phi(\mathbb{R}^n)$ be the space of all measurable functions f such that $\|f\|_{N_\Phi} < \infty$. Then N_Φ is a Banach space [7, 9].

The following relation between the behaviour of a sequence of norms of the derivatives of a function and the support of its Fourier transform has been shown in [3]:

$$\lim_{|\alpha|\to\infty} \Big(\frac{\|D^\alpha f\|_p}{\sup\limits_{\mathrm{sp}(f)} |\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}} = 1,$$

where $0 < p \leq \infty,\ f \in \mathcal{S}' \cap L_p(\mathbb{R}^n)$ and $\mathrm{sp}(f) = \mathrm{supp}(\hat{f})$ is bounded ($\hat{f}(\xi)$ is the Fourier transform of $f(x)$, $\mathcal{S}'$ is the dual space of the Schwartz space $\mathcal{S}$ of rapidly decreasing infinity differentiable functions). This result has been extended to any Orlicz norm [2, 4].

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In this paper, by using the technique of [3], we prove that the above result still holds for the norm $\|.\|_{\mathbf{N}_\Phi}$. We do that with the help of the following results:

Lemma 1 [9]. *If $f \in N_\Phi$ and $g \in M_\Phi$, then $fg \in L_1$ and*

$$\int\limits_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_{N_\Phi}\|g\|_{M_\Phi}.$$

Lemma 2. *If $f \in N_\Phi$ and $h \in L_1(\mathbb{R}^n)$, then $f * h \in N_\Phi$ and $\|f * h\|_{N_\Phi} \leq \|f\|_{N_\Phi}\|h\|_1$.*

Lemma 3. *If $f \in N_\Phi(\mathbb{R}^n)$ then $f \in \mathcal{S}'$.*

Proof. Let $f \in N_\Phi$ and define $L = L_f$ by letting

$$L(g) = L_f(g) = \int\limits_{\mathbb{R}^n} f(x)g(x)dx \quad \text{for any} \quad g \in \mathcal{S}.$$

It is clear that L is a linear functional on $\mathcal{S}$. To show that it is continuous we prove that

$$|L(g)| \leq C \sup_{\mathbb{R}^n} |(1 + |x|^2)g(x)|,$$

where C is constant. Actually, we have

$$\begin{aligned} |L(g)| &\leq \int\limits_{\mathbb{R}^n} |f(x)g(x)|dx \\ &\leq \sup_{\mathbb{R}^n}(1 + |x|^2)^N|g(x)| \int\limits_{\mathbb{R}^n} (1 + |x|^2)^{-N}|f(x)|dx. \end{aligned}$$

On the other hand, since $f \in N_\Phi$, it follows that $\mu(E) < \infty$, where

$$E = \{\ x \in \mathbb{R}^n \ : |f(x)| > t_0\}, \quad t_0 > 0.$$

Therefore,

$$\begin{aligned} \int\limits_{\mathbb{R}^n} (1 + |x|)^{-N}|f(x)|dx &\leq \int\limits_{\mathbb{R}^n \setminus E} \frac{|f(x)|}{(1 + |x|^2)^N}dx + \int\limits_E \frac{|f(x)|}{(1 + |x|^2)^N}dx \\ &\leq C_1 + C_2 = C. \end{aligned}$$

By Theorem 3.11 of [10], it follows that L is continuous on $\mathcal{S}$. Hence $f \in \mathcal{S}'$. □

Lemma 4. *Let $f \in N_\Phi(\mathbb{R}^n)$. If $\operatorname{sp}(f)$ is bounded, then f is bounded.*

Proof. Since the spectrum of f is bounded, we can choose $\hat{\psi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $\hat{\psi} = 1$ in some neighbourhood of $\operatorname{sp}(f)$. Then we obtain

$$\begin{aligned}\|(F^{-1}\hat{f})(x)\|_\infty &= \|(F^{-1}(\hat{\psi}\hat{f}))(x)\|_\infty \\ &= \|\psi * f\|_\infty \\ &\le \|\psi\|_1 \|f\|_{N_\Phi} < \infty.\end{aligned}$$

□

Applying Lemma 4 and using the techniques of the proof of Theorem 1 [5] for spaces N_Φ, we obtain the following result:

Lemma 5. *Let $\Phi(t) \in \mathcal{C}$, $f \in N_\Phi, f(x) \not\equiv 0$ and let $\xi^0 \in \operatorname{sp}(f)$ be an arbitrary point. Then the restriction of $\hat{f}$ to any neighbourhood of ξ^0 can not concentrate on any finite number of hyperplanes.*

Proof. Without loss of generality, we will prove the theorem for functions f with bounded spectrum and $\xi^0 = 0$.

By contradiction, we assume that there exists a neighbourhood $V \ni 0$ such that the restriction of $\hat{f}(\xi)$ to V concentrates on hyperplanes H_j, $j = 1, 2 \dots, m$.

We put for each $j \in I = \{1, \dots, m\}$,

$$G_j = \mathbb{R}^n \setminus \bigcup_{i \neq j} H_i.$$

Then G_j is open. For any $\psi(\xi) \in C_0^\infty(G_j)$, the distribution $\psi(\xi)\hat{f}(\xi)$ concentrates on the hyperplane H_j. By a transformation of coordinates we can chose such a way that the hyperplane H_j will be transformed into the hyperplane $\xi_j = 0$.

Put $g(x) = F^{-1}\psi * f(x)$. Then $\|g(x)\|_{N_\Phi} = \|F^{-1}\psi * f(x)\|_{N_\Phi}$. Further, the Fourier transform of $g(x)$ will concentrate on the hyperplane $\xi_j = 0$. Therefore, taking into account of a remark on Theorem 2.3.5 mentioned in Example 5.1.2 [6], we get

$$g(x) = \sum_{\ell=0}^{N} g_\ell(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)(-ix_j)^\ell, \tag{1}$$

where N is the order of the distribution $\hat{f}(\xi)$ ($N < \infty$ because $\operatorname{supp}\hat{f}$ is compact) and $\hat{g}_\ell(\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n), 0 \le \ell \le N$, are distributions with compact supports.

By virtue of Lemma 4, the equality (1) is possible only if $N = 0$. So the function $g(x)$ does not depend on x_j.

Further, since $g \in N_\Phi$ we get

$$\int_0^\infty \Phi(\lambda_g(t))dt < \infty. \tag{2}$$

We will show that $g(x) \equiv 0$. Actually, assume to the contrary that $g(x^0) \neq 0$ for some point x^0. Without loss of generality we may assume that $g(x^0) > 0$. Since $g(x) = F^{-1}(\psi\hat{f})(x)$ is continuous, there is a number $\varepsilon > 0$ and a neighbourhood U of x^0 such that $g(x) > \varepsilon$ for every $x \in U$. Since $g(x)$ does not depend on x_j (as seen above), and by the definition [9], we get

$$\lambda_g(\varepsilon) = \mu\{\, x \in \mathbb{R}^n : \ |g(x)| > \varepsilon \,\} = \infty.$$

Since $\lambda_g(t)$ is a nonincreasing function, $\lambda_g(t) = +\infty$ on the interval $[0, \varepsilon]$. Since $\Phi(t)$ is nondecreasing, it follows that $\Phi(\lambda_g(t)) = +\infty$ on the interval $[0, \varepsilon]$, which contradicts (2). Thus, we get $g(x) \equiv 0$, i.e. $\psi(\xi)\hat{f}(\xi) \equiv 0$. Since $\psi(\xi) \in C_0^\infty(G_j)$ is arbitrarily chosen, we get $\hat{f}(\xi) \equiv 0$ on the hyperplane H_j. So $\hat{f}(\xi)$ must concentrate on the planes $H_i \cap H_j$, $i, j \in I$, $i \neq j$.

We get for $i, j \in I, i \neq j$,

$$G_{ij} = \mathbb{R}^n \setminus \bigcup \{H_k \cap H_\ell : (k, \ell) \neq (i, j), k \neq \ell\}.$$

Then G_{ij} is open. For any $\psi(\xi) \in C_0^\infty(G_{ij})$, the distribution $\psi(\xi)\hat{f}(\xi)$ concentrates on the plane $H_i \cap H_j$.

By an analogous argument we obtain $\psi(\xi)\hat{f}(\xi) \equiv 0$. Since $\psi \in C_0^\infty(G_{ij})$ is arbitrarily chosen, we see that $\hat{f}(\xi)$ must concentrate on $H_i \cap H_j \cap H_\ell$, $i, j, \ell = 1, \ldots, m$, $i \neq j \neq \ell$.

Repeating the above arguments $(k-3)$ times, we deduce that the distribution $\hat{f}(\xi)$ concentrates on $\bigcap_{i=1}^m H_i$ and then, by the same way, we get $\hat{f}(\xi) \equiv 0$, which contradicts $f(x) \not\equiv 0$. The proof is complete. □

Theorem 1. *Let* $\Phi \in \mathcal{C}, f(x) \not\equiv 0,\ f \in N_\Phi,\ D^\alpha f \in N_\Phi,\ |\alpha| = 1, 2, \ldots$ *and* $\operatorname{sp}(f)$ *is bounded. Then*

$$\lim_{|\alpha| \to \infty} \Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_{\operatorname{sp}(f)} |\xi^\alpha|} \Big)^{\frac{1}{|\alpha|}} = 1. \tag{3}$$

Proof. We first show that $\sup_{\mathrm{sp}(f)} |\xi^\alpha| \neq 0$, for all $\alpha \geq 0$. Assume to the contrary that there exists $\alpha \geq 0$ such that $\sup_{\mathrm{sp}(f)} |\xi^\alpha| = 0$. Then for all $\xi \in \mathrm{sp}(f)$, we have $\xi^\alpha = 0$. Therefore, $\mathrm{sp}(f) \subset \bigcup_{j=1}^{n} \{\xi \in \mathbb{R}^n : \xi_j = 0\}$, which contradicts Lemma 5.

Now we will prove (3). The proof will be divided into three steps.

Step 1. We first establish the inequality

$$\varliminf_{|\alpha|\to\infty} \Big(\frac{\|D^\alpha f\|_{N_\Phi}}{|\xi^\alpha|} \Big)^{\frac{1}{|\alpha|}} \geq 1 \tag{4}$$

for any point $\xi \in \mathrm{sp}(f)$.

Choose $\xi^0 \in \mathrm{sp}(f)$ such that $\xi^0_j \neq 0,\ j = 1, 2, \ldots, n$. For the sake of convenience we assume that $\xi^0_j > 0,\ j = 1, \ldots, n$. We choose $0 < \varepsilon < \frac{1}{2} \min_{1\leq j\leq n} \xi^0_j$ and G is a domain with smooth boundary such that $\xi^0 \in G$ and

$$G \subset \big\{\xi \in \mathbb{R}^n : \ \xi^0_j - \varepsilon \leq \xi_j \leq \xi^0_j + \varepsilon,\ j = 1, \ldots, n\big\}.$$

We choose $\hat{\psi}(\xi) \in C_0^\infty(G)$ such that $\xi^0 \in \mathrm{supp}(\hat{\psi}\hat{f})$. Then for any $\hat{g}(\xi) \in C_0^\infty(G)$, we have

$$\begin{aligned} \langle \hat{\psi}(\xi)\hat{f}(\xi), \hat{g}(\xi)\rangle &= \langle f(x), \check{\psi} * \check{g}(x)\rangle \\ &= \langle f(x), \varphi(x)\rangle, \end{aligned} \tag{5}$$

where $\varphi(x) = \check{\psi} * \check{g}(x)$ and $\check{g}(x) = g(-x)$. Since $\hat{\psi}(\xi)\hat{f}(\xi)$ is a distribution with compact support, by Theorem 6.27 of [8], it is represented in the form

$$\hat{\psi}(\xi)\hat{f}(\xi) = \sum_{|\alpha|\leq m} D^\alpha h_\alpha(\xi),$$

where $m \in \mathbf{N}$ and $h_\alpha(\xi)$ are continuous functions on G.

By an argument analogous to the proof of Theorem 1 of [3], the Dirichlet problem for the elliptic differential equation

$$L_{2m}\hat{z}(\xi) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha (D^\alpha \hat{z}(\xi)) = \hat{\psi}(\xi)\hat{f}(\xi),$$

has a solution $\hat{z}(\xi) \in W^0_{m,2}(G)$ (the space Sobolev [1]).

Because of (5) we get

$$\langle \hat{z}(\xi), L_{2m}(\xi^\alpha \hat{g}(\xi)\rangle = (-i)^{|\alpha|}\langle D^\alpha f(x), \varphi(x)\rangle, \tag{6}$$

where $\hat{g}(\xi) \in W^0_{m,2}(G)$. Let $\hat{g}_0(\xi) \in W^0_{m,2}(G)$ be the solution of $L_{2m}\hat{g}_0(\xi) = \overline{\hat{z}}(\xi)$. Define

$$\hat{g}_\alpha(\xi) = \prod_{j=1}^{n}(\xi_j^0 - 2\varepsilon)^{\alpha_j}\xi^{-\alpha}\hat{g}_0(\xi).$$

It follows from $0 \notin G$ that

$$L_{2m}(\xi^\alpha \hat{g}_\alpha(\xi)) = \prod_{j=1}^{n}(\xi_j^0 - 2\varepsilon)^{\alpha_j}\overline{\hat{z}}(\xi).$$

By virtue of Lemma 1, we obtain the inequality

$$\begin{aligned}\prod_{j=1}^{n}(\xi_j^0 - 2\varepsilon)^{\alpha_j}\langle \hat{z}(\xi), \overline{\hat{z}}(\xi)\rangle &\le \|D^\alpha f\|_{N_\Phi}\|\psi * g_\alpha\|_{M_\Phi}\\ &\le \|D^\alpha f\|_{N\Phi}\|\psi\|_1\|g_\alpha\|_{M_\Phi}.\end{aligned} \tag{7}$$

As in the proof of Theorem 1 of [3] we get the estimate

$$\sup_{\mathbb{R}^n}(1+x_1^2)\dots(1+x_n^2)|g_\alpha(x)| \le C_3, \quad \alpha \ge 0. \tag{8}$$

It follows that

$$|g_\alpha(x)| \le \frac{C_3}{(1+x_1^2)\dots(1+x_n^2)} \qquad \text{for all } x = (x_1,\dots,x_n) \in \mathbb{R}^n. \tag{9}$$

By the definition of $\|.\|_{M_\Phi}$ in [9] we obtain

$$\|g_\alpha\|_{M_\Phi} = \sup_{0<\mu(E)<\infty}\frac{1}{\Phi(\mu(E))}\int_E |g_\alpha|dx.$$

Let $\delta > 0$. Then for any subset $E \subset \mathbb{R}^n$ such that $0 < \mu(E) < \infty$, there are two cases.

If $\mu(E) \ge \delta$, then $\Phi(\mu(E)) \ge \Phi(\delta) > 0$ because $\Phi \in \mathcal{C}$. Therefore, by (9) we get

$$\begin{aligned}\frac{1}{\Phi(\mu(E))}\int_E |g_\alpha|dx &\le \frac{1}{\Phi(\mu(E))}\int_E \frac{C_3}{(1+x_1^2)\dots(1+x_n^2)}dx\\ &\le \frac{1}{\Phi(\delta)}\int_{\mathbb{R}^n}\frac{C_3}{(1+x_1^2)\dots(1+x_n^2)}dx\\ &= \frac{C_3\pi^n}{\Phi(\delta)}.\end{aligned}$$

If $\mu(E) < \delta$, then $\dfrac{\Phi(\mu(E))}{\mu(E)} \geq \dfrac{\Phi(\delta)}{\delta}$ (since $\dfrac{\Phi(x)}{x}$ is decreasing [9]). From (9) we have

$$\begin{aligned}\frac{1}{\Phi(\mu(E))}\int\limits_E |g_\alpha|dx &\leq \frac{1}{\Phi(\mu(E))}\int\limits_E \frac{C_3}{(1+x_1^2)\dots(1+x_n^2)}dx\\ &\leq \frac{1}{\Phi(\mu(E))}\int\limits_E C_3 dx\\ &= \frac{C_3\mu(E)}{\Phi(\mu(E))}\\ &\leq \frac{C_3\delta}{\Phi(\delta)}.\end{aligned}$$

Hence

$$\frac{1}{\Phi(\mu(E))}\int\limits_E |g_\alpha|dx \leq C \quad \text{for all} \quad E \ (0 < \mu(E) < \infty).$$

That means

$$\|g_\alpha\|_{M_\Phi} \leq C. \tag{10}$$

Combining (7) with (10) we get

$$\prod_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j} \langle \hat{z}(\xi), \overline{\hat{z}}(\xi)\rangle \leq \|D^\alpha f\|_{N_\Phi} C_1.$$

Therefore

$$1 \leq \varliminf_{|\alpha|\to\infty} \Bigg(\frac{\|D^\alpha f\|_{N_\Phi}}{\prod\limits_{j=1}^n (\xi_j^0 - 2\varepsilon)^{\alpha_j}}\Bigg)^{\frac{1}{|\alpha|}}. \tag{11}$$

On the other hand, we have

$$\Big[\prod_{j=1}^{n}\Big(\frac{\xi_j^0-2\varepsilon}{\xi_j^0}\Big)^{-\alpha_j}\Big]^{\frac{1}{|\alpha|}} \le \max_j \frac{\xi_j^0}{\xi_j^0-2\varepsilon}. \tag{12}$$

From (11) and (12) we obtain

$$1 \le \varliminf_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{|{\xi^0}^\alpha|}\Big)^{\frac{1}{|\alpha|}} \max_j \frac{\xi_j^0}{\xi_j^0-2\varepsilon}. \tag{13}$$

Letting $\varepsilon\to 0$ and $\xi^0\to\xi$, we get

$$1 \le \varliminf_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{|\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}}.$$

Step 2. We will prove that

$$\varliminf_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_{\mathrm{sp}(f)}|\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}} \ge 1. \tag{14}$$

Assume to the contrary that there exists a subsequence $I_1\subset I$ such that

$$(I_1)\quad \varliminf_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_{\mathrm{sp}(f)}|\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}} < 1. \tag{15}$$

As in proof of Theorem 1 of [3], given $\lambda>1$, there exist a number $m\ge 1$ and a subsequence $I_2\subset I_1$ such that $m\xi^\alpha\in\mathrm{sp}(f)$ and

$$(I_2)\quad \varliminf_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_{sp(f)}|\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}} \ge (I_2)\frac{1}{\lambda}\varliminf_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{|m\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}} \ge \frac{1}{\lambda},$$

which contradicts (15) as $\lambda\to 1$.

Step 3. Finally we will prove that

$$\varlimsup_{|\alpha|\to\infty}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_{\mathrm{sp}(f)}|\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}} \le 1. \tag{16}$$

We first prove the inequality

$$\overline{\lim_{|\alpha|\to\infty}}\Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_G|\xi^\alpha|}\Big)^{\frac{1}{|\alpha|}}\le 1. \tag{17}$$

for an arbitrary domain $G\supset \mathrm{sp}(f)$.

We choose $\psi(\xi)\in C_0^\infty(G)$ such that $\psi(\xi)=1$ in some neighbourhood of $\mathrm{sp}(f)$ (since $\mathrm{sp}(f)$ is bounded). Put $h_\alpha(\xi)=\psi(\xi)\xi^\alpha,\quad \alpha\ge 0$. Then

$$\begin{aligned}\|F^{-1}h_\alpha\|_1 &= \int\limits_{\mathbb{R}^n}|\hat h_\alpha(\xi)|d\xi\\ &\le \Big(\int\limits_{\mathbb{R}^n}|\hat h_\alpha(\xi)|^2(1+|\xi|^2)^s d\xi\Big)^{1/2}\Big(\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^{-s}d\xi\Big)^{1/2}\\ &\le C_1(s)\|h_\alpha\|_{(s)},\quad s>\frac{n}{2}.\end{aligned}$$

On the other hand, since $H_{(k)}=W_{k,2}(\mathbb{R}^n)$ (see [1]), we obtain

$$\begin{aligned}\|D^\alpha f\|_{N_\phi} &= \|F^{-1}(\psi(\xi)\xi^\alpha)*f\|_{N_\Phi}\\ &\le \|F^{-1}(\psi(\xi)\xi^\alpha)\|_1\|f\|_{N_\Phi}\\ &\le C_1\|\psi(\xi)\xi^\alpha\|_{(k)}\|f\|_{N_\Phi}\\ &\le C_2\|\psi(\xi)\xi^\alpha\|_{k,2}\|f\|_{N_\Phi},\end{aligned} \tag{18}$$

where $k=\dfrac{n}{2}+1$. As in the proof of Theorem 1 of [3] we can shows that

$$\|\psi(\xi)\xi^\alpha\|_{k,2}\le C_3|\alpha|^k\sup\Big\{\sup_G|\xi^{\alpha-\gamma}|:\gamma\le\alpha,\ |\gamma|\le k\Big\}\ \alpha\ge 0, \tag{19}$$

and

$$\lim_{|\alpha|\to\infty}\Big(\sup\big\{\sup_G|\xi^{\gamma-\alpha}|:\gamma\le\alpha,\ |\gamma|\le k\big\}\Big)^{\frac{1}{|\alpha|}}/\big(\sup_G|\xi^\alpha|\big)^{\frac{1}{|\alpha|}}=1. \tag{20}$$

From (18), (19) and (20) we obtain (17).

Now suppose that (16) is false. By an argument analogous to the above proof, there exist $I_1\subset I$ and $\lambda>1$, $0\le\beta_j\le 1$, $j=1,\dots,n$, such that $|\beta|=1$ and

$$(I_1)\quad \lim_{|\alpha|\to\infty}\frac{\alpha_j}{|\alpha|}=\beta_j,\quad j=1,\dots,n.$$

$$(I_1) \quad \lim_{|\alpha|\to\infty} \Big(\frac{\|D^\alpha f\|_{N_\Phi}}{\sup\limits_{\mathrm{sp}(f)} |\xi^\alpha|} \Big)^{\frac{1}{|\alpha|}} = \lambda.$$

Therefore, from the previous results we get

$$\sup_G |\xi^\beta| / \sup_{\mathrm{sp}(f)} |\xi^\beta| \geq \lambda$$

for any domain $G \supset \mathrm{sp}(f)$, which is impossible. The proof is complete. □

Remark 1. Theorem 1 still holds for functions defined on torus $\mathbf{T}^n$.

Remark 2. Theorem 1 still holds without the assumption $D^\alpha f \in N_\Phi$, $|\alpha| = 1, 2, \ldots$.

Remark 3. Equality (3) is not true if $\mathrm{sp}(f)$ is bounded.

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