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# ON A PROPERTY OF INFINITELY DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, the existence of $\lim_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n}$ for an arbitrary function $f \in C^\infty(IR)$ such that $f^{(n)} \in N_\Phi$, $n=0,1,\ldots$ and the concrete calculation of $\lim_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n}$ are shown.

## 1. Introduction

Ha Huy Bang has proved the following result [1]: Let $1 \le p \le \infty$ and $f \in C^\infty(\mathbb{R})$ such that $f^{(n)} \in L^p(\mathbb{R})$, $n = 0, 1, \ldots$. Then there always exists the limit

$$d_f = \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n},$$

and moreover

$$d_f = \sigma_f = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\},$$

where the last equality is the definition of $\sigma_f$ and $\hat{f}(\xi)$ is the Fourier transform of the function $f(x)$. This result has been extended to any Orlicz norm by techniques special for convex functions [2].

In this paper, modifying the methods of [1, 2] we prove this result for another norm generated by concave functions. Note that the Orlicz norm is generated by convex functions and here we must overcome some essential difficulties arising by the difference between convex and concave functions.

## 2. Results.

Let $\mathcal{L}$ denote the family of all non-zero concave functions $\Phi(t) : [0, \infty) \to [0, \infty]$ which are non-decreasing and satisfy $\Phi(0) = 0$. For an arbitrary measurable function $f, \Phi \in \mathcal{L}$ we define

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$$\|f\|_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_f(y)\big)dy,$$

where $\lambda_f(y) = \text{mes}\{x : |f(x)| > y\}$ $(y \geq 0)$. If the space $N_\Phi = N_\Phi(\mathbb{R})$ consists of measurable functions $f(x)$ such that $\|f\|_{N_\Phi} < \infty$, then $N_\Phi$ is a Banach space. Denote by $M_\Phi = M_\Phi(\mathbb{R})$, the space of measurable functions $g$, such that

$$\|g\|_{M_\Phi} = \sup\Big\{\frac{1}{\Phi(\text{mes}\,\Delta)}\int_\Delta |g(x)|dx : \ \Delta \subset \mathbb{R},\ 0 < \text{mes}\ \Delta < \infty\Big\} < \infty.$$

Then $M_\Phi$ is also a Banach space [3, 4].

We need the following results:

**Lemma 1** [3]. *If $f \in N_\Phi$ and $g \in M_\Phi$, then $fg \in L_1$ and*

$$\int_{-\infty}^{\infty} |f(x)g(x)|dx \leq \|f\|_{N_\Phi}\|g\|_{M_\Phi}.$$

**Lemma 2**. *If $f \in N_\Phi$, then*

$$\|f(\cdot - y)\|_{N_\Phi} = \|f\|_{N_\Phi}, \quad \forall y \in \mathbb{R}.$$

*Proof.* By virtue of Theorem 4.3 of [3], it is clear that $N_\Phi^* = M_\Phi$, and if $f \in N_\Phi$ and $g \in M_\Phi$, then

$$< f, g >= J(g)(f) = \int_{-\infty}^{\infty} f(x)g(x)dx.$$

Therefore, since $\|x\|_X = \|x\|_{X^{**}}$ for any normed space $X$ [7, p. 113], we have

$$\|f\|_{N_\Phi} = \sup_{\|g\|_{M_\Phi}=1} | < f, g > | =$$

$$= \sup_{\|g\|_{M_\Phi}=1} \Big|\int_{-\infty}^{\infty} f(x)g(x)dx\Big|.$$

Hence

$$\|f(\cdot - y)\|_{N_\Phi} = \sup_{\|v\|_{M_\Phi}=1} \Big| \int_{-\infty}^{\infty} f(x-y)v(x)dx \Big| = $$
$$= \sup_{\|v(t+y)\|_{M_\Phi}=1} \Big| \int_{-\infty}^{\infty} f(t)v(t+y)dt \Big| = $$
$$= \sup_{\|v_1\|_{M_\Phi}=1} \Big| \int_{-\infty}^{\infty} f(t)v_1(t)dt \Big| = \|f\|_{N_\Phi}$$

because of $\|v(t+y)\|_{M_\Phi} = \|v\|_{M_\Phi}$. The lemma is proved.

Now, we state the main theorem of this paper.

**Theorem 1.** *Let $\Phi \in \mathcal{L}$ and $f \in C^\infty(\mathbb{R})$ such that $f^{(n)} \in N_\Phi, n = 0, 1, \ldots$. Then there always exists the limit*

$$d_f = \lim_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n}.$$

*Moreover, if we put*

$$\sigma_f := \sup\{|\xi| : \xi \in \mathrm{supp}\hat{f}(\xi)\},$$

*where $\hat{f}(\xi)$ is the Fourier transform of the function $f(x)$, then $d_f = \delta_f$.*

*Proof.* We first observe that

$$\overline{\lim_{n\to\infty}} \|f^{(n)}\|_{N_\Phi}^{1/n} \le \sigma_f. \tag{1}$$

It is enough to show (1) for $\sigma_f < \infty$. Using $f \in \mathcal{S}'$ (this follows from $f \in N_\Phi$) and the well-known Paley-Wiener-Schwartz theorem, we obtain that $f$ is an analytic function of exponential type $\le \sigma_f$. It is easily seen that the Bernstein-Nikolsky inequality holds for the norm $\| \, . \, \|_{N_\Phi}$. Therefore, we get

$$\|f^{(n)}\|_{N_\Phi} \le \sigma_f^n \|f\|_{N_\Phi}, \quad n = 0, 1, \ldots,$$

and (1) is an immediate consequence of the last inequalities.

Finally, we claim that

$$\sigma_f \le \underline{\lim_{n\to\infty}} \|f^{(n)}\|_{N_\Phi}^{1/n},$$

from which the statement immediately follows.

Let $\psi_\lambda(x) \in C_0^\infty(R)$, $\psi_\lambda(x) \geq 0$, $\psi_\lambda(x) = 0$ for $|x| \geq \lambda$ and $\int \psi_\lambda(x) = 1$. We put $f_\lambda = f * \psi_\lambda$. Then $f_\lambda \in C^\infty(\mathbb{R})$ because of $f \in L_{1,loc}(\mathbb{R})$. Therefore, $f_\lambda^{(n)} = f^{(n)} * \psi_\lambda$. By virtue of Lemma 2 we get

$$\|f^{(n)} * \psi_\lambda\|_{N_\Phi} \leq \|f^{(n)}(\cdot - y)\|_{N_\Phi} \|\psi_\lambda\|_1 = \|f^{(n)}\|_{N_\Phi}.$$

Hence, $f_\lambda^{(n)} \in N_\Phi$. It is clear that $\psi_\lambda \in M_\Phi$ because of $\psi_\lambda \in C_0^\infty(\mathbb{R})$. Thus, by virtue of Lemma 1,

$$\begin{aligned} |f_\lambda^{(n)}(x)| &\leq \int_{-\infty}^{\infty} \big|f^{(n)}(x-y)\psi_\lambda(y)\big| dy \\ &\leq \|f^{(n)}(\cdot - y)\|_{N_\Phi} \|\psi_\lambda(y)\|_{M_\Phi} = \|f^{(n)}\|_{N_\Phi} \|\psi_\lambda\|_{M_\Phi}. \end{aligned} \tag{2}$$

Therefore, $f_\lambda^{(n)} \in L_\infty(\mathbb{R})$. It follows from (2) and [1] that

$$\sigma_{f_\lambda} = d_{f_\lambda} \leq \varliminf_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n}.$$

Consequently, to complete the proof it remains to show that

$$\sigma_f \leq \varliminf_{\lambda\to 0} \sigma_{f_\lambda},$$

and therefore the problem is now reduced to proving the inequality

$$|\xi| \leq \varliminf_{\lambda\to 0} \sigma_{f_\lambda}, \quad \forall \xi \in \operatorname{supp}\hat{f}(\xi). \tag{3}$$

Assume to the contrary that (3) is not satified. Then there exist a point $\xi_0 \in \operatorname{supp}\hat{f}(\xi)$, a number $\varepsilon > 0$, and a subsequence $\lambda_k$ (for simplicity we assume $\xi_0 > 0$) such that

$$\sigma_{f_{\lambda_k}} \leq \xi_0 - 2\varepsilon, \quad k = 1, 2, \ldots. \tag{4}$$

Assume that for some $\varepsilon_0 > 0$, $g \in M_\Phi$ and a subsequence $\lambda_k \to 0$,

$$\Big|\int_{-\infty}^{\infty} \big(f_{\lambda_k}(x) - f(x)\big) g(x) dx\Big| \geq \varepsilon_0\ , \quad k \geq 1\ . \tag{5}$$

It is known that $f_\lambda \to f, \lambda \to 0$ in $L_{1,loc}(\mathbb{R})$. Therefore, there exists a subsequence $\{k_m\}$ (for simplicity we assume $k_m = m$) such that $f_{\lambda_k}(x) \to f(x)$ a.e.

On the other hand, $\{f_{\lambda_k}\}$ is bounded in $N_\Phi$ because of $\|f_{\lambda_k}\|_{N_\Phi} \leq \|f\|_{N_\Phi}$. So $\{f_{\lambda_k}\}$ is a weak precompact sequence. Therefore, there exists a subsequence, denoted again by $\{f_{\lambda_k}\}$, and a function $f_* \in N_\Phi$ such that

$$\int_{-\infty}^{\infty} f_{\lambda_k}(x)v(x)dx \to \int_{-\infty}^{\infty} f_*(x)v(x)dx, \quad \forall\, v \in M_\Phi. \tag{6}$$

Let $u$ be an arbitrary function in $C_0^\infty(\mathbb{R})$, then $u \in M_\Phi$. By (6) we get

$$\int_{-\infty}^{\infty} f_{\lambda_k}(y)u(y)dy \to \int_{-\infty}^{\infty} f_*(y)u(y)dy,\ \forall\, u \in C_0^\infty(\mathbb{R}).$$

Because each $u \in C_0^\infty(\mathbb{R})$ has a finite support, it follows from $f_{\lambda_k}(x) \to f(x)$ a.e. that

$$\int_{-\infty}^{\infty} f_{\lambda_k}(y)u(y)dy \to \int_{-\infty}^{\infty} f(y)u(y)dy, \quad \forall\, u \in C_0^\infty(\mathbb{R}). \tag{7}$$

Combining (6) and (7) we get

$$\int_{-\infty}^{\infty} f(y)u(y)dy = \int_{-\infty}^{\infty} f_*(y)u(y)d(y), \quad \forall\, u \in C_0^\infty(\mathbb{R}).$$

It is known [6, p. 15] that

$$f(x) = f_*(x) \text{ a.e.}$$

Therefore,

$$\int_{-\infty}^{\infty} f_{\lambda_k}(x)v(x)dx \to \int_{-\infty}^{\infty} f(x)v(x)dx.$$

because of (6), which contradicts (5). So $f_r$ weakly converges to $f$.

It follows that $\hat{f}_\lambda$ also converges weakly to $\hat{f}$. Now we choose a function $\varphi(x) \in C_0^\infty(\mathbb{R})$ such that $< \hat{f}, \varphi > \neq 0$, $\operatorname{supp} \varphi(x) \subset [\xi_0 - \varepsilon, \xi_0 - \varepsilon]$. Then (4) implies that

$$0 =< \hat{f}_k, \varphi > \to < \hat{f}, \varphi > \neq 0, \quad k \to \infty.$$

So we arrive at a contradiction. The proof of Theorem 1 is complete. □

For periodic functions we have the following result.

**Theorem 2.** *Let $\Phi \in \mathcal{L}$, and suppose that $f(x) \in C^\infty(\mathbb{R})$ is an arbitrary $2\pi$-periodic function. Then there always exists the limit*

$$d_f = \lim_{n \to \infty} \| f^{(n)} \|_{N_\Phi}^{1/n}.$$

*Moreover, if we put*

$$\sigma_f := \sup\{|k| : k \in \operatorname{supp} \hat{f}(\xi)\},$$

*where $\| \cdot \|_{N_\Phi}$ is the $N_\Phi(0, 2\pi)-$ norm, then $d_f = \delta_f$.*

*Proof.* We proof this theorem by an argument similar to [1]. Representing the function $f$ by its Fourier series, we have

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \, \exp(ikx),$$

where

$$f_k = (2\pi)^{-1}(f, \exp(-ikx)), \quad k = 0, \pm 1, \ldots .$$

Therefore,

$$f^{(n)}(x) = \sum_{k=-\infty}^{\infty} f_k (ik)^n \, \exp(ikx), \quad n = 0, 1, \ldots .$$

From the definition of $\|.\|_{M_\Phi}$ we see that $\| \exp(-ik\cdot) \|_{M_\Phi} = \dfrac{2\pi}{\Phi(2\pi)} < \infty$. Then

$$\begin{aligned} |f_k k^n| &= (2\pi)^{-1} |(f^{(n)}, \exp(-ikx))| \\ &\leq \frac{1}{\Phi(2\pi)} \| f^{(n)} \|_{N_\Phi}, \end{aligned}$$

where $n = 0, \pm 1, \dots;\ k = 0, 1, \dots$. Consequently,

$$\lim_{n\to\infty} |f_k k^n|^{1/n} = |k| \leq \varliminf_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n} \tag{8}$$

for any index $k$ such that $f_k \neq 0$. Using

$$\hat{f}(\xi) = \sum_{k=-\infty}^{\infty} f_k \delta(\xi + k)$$

and (8), we get

$$\sigma_f \leq \varliminf_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n}. \tag{9}$$

Further, we shall show that

$$\varlimsup_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n} \leq \sigma_f. \tag{10}$$

It is enough to prove (10) for $\sigma_f < \infty$. By the Paley-Wiener-Schwartz theorem, $f$ is an analytic function of exponential type $\leq \sigma$. Hence, it follows from the Bernstein-Nikolsky inequality for $\|\ .\ \|_{N_\Phi}$ that

$$\|f^{(n)}\|_{N_\Phi} \leq \sigma_f^n \|f\|_{N_\Phi}, \quad n = 0, 1, \dots,$$

and (10) is an immediate consequence of the last inequalities.

Combining (9) and (10) yields

$$\varliminf_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n} = \varlimsup_{n\to\infty} \|f^{(n)}\|_{N_\Phi}^{1/n} = \sigma_f.$$

The proof for Theorem 2 is now complete. □

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