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THE PROPERTIES (Ω) AND $(\underline{DN})$ OF SPACES OF ENTIRE FUNCTIONS OF BOUNDED TYPE

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Abstract. The first goal of this paper is to establish properties (Ω) and $(\underline{DN})$ for spaces of entire functions of bounded type on (DF)-spaces. Next we show the connection between the property $(\underline{DN})$ fulfilled on $H(E'_b)$ and the existence of a non-pluripolar compact set in E'_b under the assumption that E is a Frechet-Montel space.

1. Introduction

Let E be a Frechet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}$ defining the topology of E. For each subset B of E define

$$\|\cdot\|_B^* : E' \longrightarrow [0, +\infty]$$

given by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}$$

where $u \in E'$, the topological dual space of E.

Instead of $\|\cdot\|_{U_q}^*$ we write $\|\cdot\|_q^*$ where

$$U_q = \Big\{x \in E : \|x\|_q \le 1\Big\}.$$

Using the above notations we consider the following property of E:

$$(\underline{DN})\ \exists p\ \forall q\ \exists k, C > 0, d > 0 : \ \|x\|_q^{1+d} \le C\|x\|_k\|x\|_p^d,\ x \in E,$$

$$(DN)\ \exists p\ \forall q, d > 0\ \exists k, C > 0 : \ \|x\|_q^{1+d} \le C\|x\|_k\|x\|_p^d,\ x \in E,$$

$$(\Omega)\ \forall p\ \exists q\ \forall k\ \exists d, C > 0 : \quad \|u\|_q^{*1+d} \le C\|u\|_k^*\|u\|_p^{*d},\ \forall u \in E'.$$

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The above properties were introduced and investigated by Vogt (see [14], [15], [16]). In the case E has property (DN) (resp. $(\underline{DN})$) the semi-norm p in the above definition is a norm on E and is called to be a (DN)-norm (resp. a $(\underline{DN})$-norm)

For a complex locally convex space E let $H(E)$ denote the vector space of all entire functions on E, i.e. of all continuous complex-valued functions on E which are Gâteaux-holomorphic.

An entire function $f : E \to \mathbf{C}$ is said to be of bounded type if f is bounded on every bounded subset of E. By $H_b(E)$ we denote the vector space of all entire functions of bounded type on E. It is endowed with the topology τ_b of uniform convergence on bounded subsets. It is known [9] that if E is a bornological (DF)-space then $(H_b(E), \tau_b)$ is a Frechet space.

In [7] Meise and Vogt investigated the properties (DN) and (Ω) for $H_b(E'_b)$ in the case where E is a nuclear Frechet space having the property (DN) (respectively (Ω)).

The first aim of this paper is to establish the property (Ω) for $H_b(E'_b)$ in the case E is a non-nuclear Frechet space. We prove the following

Theorem A. *Let E be a Frechet space having the property (Ω). Then $H_b(E'_b)$ also has the property (Ω) if one of the following holds*

(i) *E is Hilbertisable,*

(ii) *E is a Montel space with an absolute basis.*

Next we establish the property $(\underline{DN})$ by the following theorem.

Theorem B. *Let E be a Frechet space such that E has property $(\underline{DN})$ and E'_b has an absolute basis. Then $H_b(B\widehat{\otimes}_\pi E'_b)$ has property $(\underline{DN})$ for every Banach space B.*

Another characterization of a nuclear Frechet space E having property $(\underline{DN})$ has been established by Dineen-Meise-Vogt. In [3] they have proved that a nuclear Frechet space E has property $(\underline{DN})$ if and only if there exists a non-pluripolar bounded set B in E'_b. Here a subset B of a locally convex space E is said to be pluripolar if there exists a plurisubharmonic function φ on E, $\varphi \not\equiv -\infty$ such that

$$B \subset \big\{x \in E : \varphi(x) = -\infty\big\}.$$

The second section of the paper is devoted to the relation between property $(\underline{DN})$ of $H_b(E'_b)$ and the existence of a non-pluripolar compact subset in E'_b in the case E is not assumed to be nuclear.

2. The properties (Ω) and ($\underline{DN}$)

To prove Theorem A we need some auxiliary lemmas.

Lemma 2.1. *Let E be a Hilbert-Frechet space having property* (Ω). *Then there exists an index set I such that E'_b is a subspace of $\ell^2(I)\widehat{\otimes}_\pi s'$, where s is the space of rapidly decreasing sequences.*

Proof. By the hypothesis and [8] E is quasi-normable. Let $\{\|\cdot\|_k\}$ be a system of Hilbert semi-norms defining the topology of E and satisfying the condition: $\forall k \geq 1$ $\forall \varepsilon > 0$ $\exists$ a bounded set $M_k \subset E$ such that

$$U_{k+1} \subset M_k + \varepsilon U_k.$$

(i) Let us consider the exact sequence of Palamodov [10]

$$0 \longrightarrow E \xrightarrow{e} \prod_{k\geq 1} E_k \xrightarrow{q} \prod_{k\geq 1} E_k \longrightarrow 0 \tag{1}$$

where

$$\begin{aligned} q(x_k) &= \big(\pi_{k+1,k}x_{k+1} - x_k\big), \\ e(x) &= (\omega_k x), \\ \pi_{k+1,k} &: E_{k+1} \longrightarrow E_k, \\ \omega_k &: E \longrightarrow E_k \end{aligned}$$

are the canonical maps and E_k are Hilbert spaces associated to $\|\cdot\|_k$. Now we prove that every bounded set in $\prod_{k\geq 1} E_k$ is an image of a bounded set in $\prod_{k\geq 1} E_k$ under q. Indeed, by virtue of [10] it is enough to check that for any index set I the space $\ell^\infty(I,E)$ is dense in $\ell^\infty(I,E_{k+1})$ with respect to the norm of $\ell^\infty(I,E_k)$.

Given $\sigma \in \ell^\infty(I,E_{k+1})$ and $\varepsilon > 0$. Choose a bounded set M_k in E such that

$$U_{k+1} \subset M_{k+1} + \frac{\varepsilon}{\|\sigma\|_{k+1}} U_k.$$

Since $\left\{\dfrac{\sigma(t)}{\|\sigma\|_{k+1}} : t \in I\right\} \subset U_{k+1}$, it implies that there exists $\beta \in \ell^\infty(I,E)$ such that

$$\Big\|\frac{\sigma(t)}{\|\sigma\|_{k+1}} - \beta(t)\Big\|_k < \frac{\varepsilon}{\|\sigma\|_{k+1}}$$

for $t \in I$. Put $\gamma(t) = \|\sigma\|_{k+1}\beta(t) \in \ell^\infty(I, E)$. Then we have $\|\sigma - \gamma\|_k < \varepsilon$.

(ii) Adapting [14] we put

$$F = \Big\{x = (x_k) \in \prod_{k\geq 1} E_k : \|x\|^2 = \sum_{k=1}^{\infty} \|x_k\|_k^2 < +\infty\Big\}.$$

For each k let F_k be the topological complement of E_k in F, i.e. $F = E_k \oplus F_k$. Taking the direct sum of the resolution (1) above with the exact sequence

$$0 \longrightarrow 0 \longrightarrow \prod_{k\geq 1} F_k \xrightarrow{id} \prod_{k\geq 1} F_k \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow E \longrightarrow F^{\mathbf{N}} \xrightarrow{\tilde{q}} F^{\mathbf{N}} \longrightarrow 0,$$

in which every bounded set in $F^{\mathbf{N}}$ is an image of a bounded set in $F^{\mathbf{N}}$ under the map $\tilde{q}$. Using the same argument as in [14] we infer that E is isomorphic to a quotient space of $\ell^2(I)\widehat{\otimes}_\pi s$ for some index I and s such that every bounded set in E is an image of a bounded set in $\ell^2(I)\widehat{\otimes}_\pi s$. It follows that E'_b is isomorphic to a subspace of $\big[\ell^2(I)\widehat{\otimes}_\pi s\big]'_b = \ell^2(I)\widehat{\otimes}_\pi s'$. The lemma is proved.

Lemma 2.2. *Let B be a Banach space. Then $H_b(B\widehat{\otimes}_\pi s')$ has property (Ω).*

Proof. Let $\{e_j\}_{j\geq 1}$ be the canonical basis of s and $\{e_j^*\}_{j\geq 1}$ the basis of s' given by

$$e_k^*\Big(\{\xi_j\}_{j=1}^{\infty}\Big) = \xi_k$$

for every $\xi = \{\xi_j\}_{j=1}^{\infty} \in s$.

Since $\|e_j\|_p = j^p$, it is easy to check that the topology of $H_b\big(B\widehat{\otimes}_\pi s'\big)$ defined by the system of semi-norms $\big\{|||\cdot|||_p\big\}_{p\geq 1}$ given by

$$|||f|||_p = \sup\Big\{p^n \sum_{j_1,\dots,j_n\geq 1} \big|\widehat{P_n f}\big(u_1\otimes e_{j_1}^*, \dots, u_n\otimes e_{j_n}^*\big)\big|(j_1\dots j_n)^p : \\ u_1,\dots,u_n \in W,\ n\geq 0\Big\}$$

where W is the unit ball of B,

$$f(w) = \sum_{n=0}^{\infty} P_n f(w)$$

with

$$w = \sum_{k=1}^{\infty} u_k \otimes v_k \in B\widehat{\otimes}_\pi s'$$

is the Taylor expansion of f at $0 \in B\widehat{\otimes}_\pi s'$,

$$P_n f(w) = \frac{1}{2\pi i} \int\limits_{|t|=\rho} \frac{f(tw)}{t^{n+1}} dt$$

and $\widehat{P_n f}$ is the continuous symmetric n-linear map associated to $P_n f$.

Put

$$V_p = \Big\{ f \in H_b\big(B\widehat{\otimes}_\pi s'\big) : |||f|||_p \le 1 \Big\}.$$

By [15] in order to prove $H_b\big(B\widehat{\otimes}_\pi s'\big)$ has property (Ω) it suffices to show

$$(*)\ \forall p\ \exists q \ge p\ \forall k\ \exists d > 0 : V_q \subset r^d V_k + \frac{1}{r} V_p \quad \text{for all } r > 0.$$

Now let $p > 1$, choose $q > ep$ and take $k > 0$. Obviously (*) holds for $0 < r \le 1$ and $d > 0$. Let $f \in V_q$ and $r > 1$. We have

$$\begin{aligned}
&\Big|\Big|\Big| \sum_{n \ge N} P_n f \Big|\Big|\Big|_p \\
&\le \sup \Big\{ p^n \sum_{j_1, \dots, j_n \ge 1} \big|\widehat{P_n f}\big(u_1 \otimes e_{j_1}^*, \dots, u_n \otimes e_{j_n}^*\big)\big| (j_1 \dots j_n)^p : \\
&\qquad u_1, \dots, u_n \in W,\ n \ge N \Big\} \\
&\le \sup \Big\{ \Big(\frac{p}{q}\Big)^n q^n \sum_{j_1, \dots, j_n \ge 1} \big|\widehat{P_n f}\big(u_1 \otimes e_{j_1}^*, \dots, u_n \otimes e_{j_n}^*\big)\big| (j_1 \dots j_n)^q : \\
&\qquad u_1, \dots, u_n \in W,\ n \ge N \Big\} \\
&\le \Big(\frac{1}{e}\Big)^N \le \frac{1}{r}
\end{aligned}$$

if $N = [\log r] + 1$.

For each positive integer $s > 0$

$$P_s\Big(\sum_{k\geq 1} u_k \otimes v_k\Big) =$$
$$\sum_{0\leq n\leq N-1}\ \sum_{k_1,\dots,k_n\geq 1}\ \sum_{j_1\dots j_n\leq s} \widehat{P_n f}\big(u_{k_1}\otimes e^*_{j_1},\dots,u_{k_n}\otimes e^*_{j_n}\big)v_{k_1}(e_{j_1})\dots v_{k_n}(e_{j_n})$$

and

$$Q_s\Big(\sum_{k\geq 1} u_k \otimes v_k) =$$
$$\sum_{0\leq n\leq N-1}\ \sum_{k_1,\dots,k_n\geq 1}\ \sum_{j_1\dots j_n> s} \widehat{P_n f}\big(u_{k_1}\otimes e^*_{j_1},\dots,u_{k_n}\otimes e^*_{j_n}\big)v_{k_1}(e_{j_1})\dots v_{k_n}(e_{j_n})$$

It is easy to see that P_s and Q_s are defined correctly because if $\sum\limits_{k\geq 1} u_k\otimes v_k$ $= \sum\limits_{k\geq 1} x_k \otimes y_k$ then

$$\sum_{k=1}^{\infty} v_k(e_j)u_k = \sum_{k=1}^{\infty} y_k(e_j)x_k \quad \text{for all } j\geq 1.$$

We have

$$\begin{aligned}\big|\big\|Q_s\big\|\big|_p = \sup\Big\{\Big(\frac{p}{q}\Big)^n \sum_{j_1\dots j_n>s} q^n\big|\widehat{P_n f}\big(u_1\otimes e^*_{j_1},\dots,u_n\otimes e^*_{j_n}\big)\big|\times\\ (j_1\dots j_n)^q (j_1\dots j_n)^{p-q} : 0\leq n\leq N,\ u_1\dots u_n\in W\Big\}\\ \leq s^{p-q} < \frac{1}{r}\end{aligned}$$

if $r = s$. At the same time,

$$\begin{aligned}&\big|\big\|P_s\big\|\big|_k =\\ &\sup\Big\{\Big(\frac{k}{q}\Big)^n \sum_{j_1\dots j_n\leq s} q^n\big|\widehat{P_n f}\big(u_1\otimes e^*_{j_1},\dots,u_n\otimes e^*_{j_n}\big)\big|(j_1\dots j_n)^q\cdot(j_1\dots j_n)^{k-q} :\\ &\quad 0\leq n\leq N-1,\ u_1,\dots,u_n\in W\Big\} \leq \Big(\frac{k}{q}\Big)^{N-1} s^{k-q}\leq r^d\end{aligned}$$

if $(N-1)k + k\log s \le d\log r$ or $k\log r + k\log r \le d\log r$ or $d \ge 2k$. Hence

$$f = P_s + Q_s + \sum_{n>N} P_n f \in r^d V_k + \frac{2}{r} V_p.$$

The lemma is proved. $\square$

Lemma 2.3. *Let E be a Frechet-Montel space with an absolute basis. Then for every continuous semi-norm ρ on E_b' there exists a continuous semi-norm $\rho_1 \ge \rho$ on E_b' such that the canonical map*

$$\omega_{\rho_1,\rho} : \big(E_b'\big)_{\rho_1} \longrightarrow \big(E_b'\big)_{\rho}$$

can be factorized through the space ℓ^∞.

Proof. Since E has an absolute basis, it follows that E is the Köthe space $\Lambda(A)$ for some matrix $A = \big(a_{j,k}\big)_{j,k\ge1}$,

$$\Lambda(A) = \Big\{ x = (x_j) \in \omega : \sum_{j\ge1} |x_j| a_{j,k} < +\infty \ \forall\ k \ge 1 \Big\}.$$

Given ρ a continuous semi-norm on $E_b' = \Lambda'(A)$. By [13] we can assume that ρ is of the form

$$\rho(u) = \sup \Big\{ \Big| \sum_{j\ge1} x_j u_j \Big| : (x_j) \in B \Big\}$$

for $u = \big(u_j\big)_{j\ge1} \in \Lambda'(A)$, where B is a bounded set in $\Lambda(A)$ of the form

$$B = \Big\{ (x_j) \in \Lambda(A) : \sum_{j\ge1} |x_j| \lambda_j \le 1 \Big\}$$

for some sequence of positive numbers $\big(\lambda_j\big)_{j\ge1}$.

Since E_b' is Schwartz we can find a continuous semi-norm $\rho_1 \ge \rho$ on E_b' such that the canonical map $\pi_{\rho_1\rho} : \Lambda'(A)_{\rho_1} \longrightarrow \Lambda'(A)_\rho$ is compact. Again we can assume that ρ_1 is defined by a bounded subset B_1 of $\Lambda(A)$ of the form as B:

$$B_1 = \Big\{ (x_j) \in \Lambda(A) : \sum_{j\ge1} |x_j| \lambda_j^1 \le 1 \Big\}$$

and $B \subset B_1$.

The compactness of $\pi_{\rho_1 \rho}$ yields $\lim\limits_{j\to\infty} \dfrac{\lambda_j^1}{\lambda_j} = 0$. Define the continuous linear maps

$$T \;:\; \lambda'(A)_{\rho_1} \longrightarrow \ell^\infty,$$
$$S \;:\; \ell^\infty \longrightarrow \big(\lambda(A)[B]\big)'$$

by

$$T\big((u_j)\big) = \Big(\frac{u_j}{\lambda_j^1}\Big) \quad \text{for} \quad (u_j) \in \Lambda'(A)_{\rho_1}$$

and

$$S\big((v_j)\big) = (\lambda_j^1 v_j) \quad \text{for} \quad (v_j) \in \ell^\infty.$$

From the equality $\lim\limits_{j\to\infty} \dfrac{\lambda_j^1}{\lambda_j} = 0$ we infer that $\mathrm{Im} S \subset \Lambda'(A)_\rho$. Obviously $\pi = S_0 T$. $\square$

Proof of Theorem A.

(i) By Lemma 2.1 E'_b is a subspace of $\ell^2(I)\widehat{\otimes}_\pi s'$. Since s' is nuclear, it follows that $\ell^2(I)\widehat{\otimes}_\pi s'$ has a fundamental system of Hilbert semi-norms. Combining this together with the fact that every entire function of bounded type on a (DF)-space can be factorized through a Banach space [4] we infer that the restriction map

$$R : H_b\big(\ell^2(I)\widehat{\otimes}_\pi s'\big) \longrightarrow H_b(E'_b)$$

is surjective. From the Lemma 2.2 we deduce that $H_b(E'_b)$ has property (Ω).

(ii) Since E has property (Ω), by [14] E is a quotient space of $B\widehat{\otimes}_\pi s$, where B is a Banach space. Let $Q : B\widehat{\otimes}_\pi s \longrightarrow E$ be the projection. By the Monteless of E every bounded set of E is an image of a bounded set of $B\widehat{\otimes}_\pi s$ under the map Q. Hence E'_b is a subspace of $\big(B\widehat{\otimes}_\pi s\big)'_b = B'\widehat{\otimes}_\pi s'$. As in (i) every entire function of bounded type on E'_b can be factorized through $\big(E'_b\big)_\rho$ for some continuous semi-norm ρ on E'_b and by using Lemma 2.3 it implies that $H_b(E'_b)$ is a quotient space of $H_b(B'\widehat{\otimes}_\pi s')$. By Lemma 2.2 this yields that $H_b(E'_b)$ has property (Ω). $\square$

Proof of Theorem B.

Assume that E is a Frechet space having property ($\underline{DN}$) and E'_b has an absolute basis $\{e_j^*\}_{j=1}^{\infty}$ and B a Banach space. Choose $p \geq 1$ such that

$$\forall q\ \exists k, C, d > 0\ \forall r > 0 : U_q^0 \subseteq Cr^d U_k^0 + \frac{1}{r} U_p^0. \tag{2}$$

(i) From (2) we have

$$\begin{aligned}\|z\|_q &= \sup\Big\{|z(u)| : u \in U_q^0\Big\}\\ &\leq \sup\Big\{\big|z(Cr^d v + \frac{1}{r} w)\big| : v \in U_k^0,\ w \in U_p^0\Big\}\\ &\leq Cr^d \sup\Big\{|z(v)| : v \in U_k^0\Big\} + \frac{1}{r}\sup\Big\{|z(w)| : w \in U_p^0\Big\}\\ &\leq Cr^d \|z\|_k + \frac{1}{r}\|z\|_p \text{ for all } z \in \big(E'', \beta(E'', E')\big),\ \forall r > 0,\end{aligned}$$

and by [17] we infer that $\big(E'', \beta(E'', E')\big)$ has property ($\underline{DN}$).

(ii) Choose an index set I such that B is quotient space of $\ell^1(I)$. Since

$$\begin{aligned}B\widehat{\otimes}_\pi E'_b = \Big\{&\big(x_j\big)_{j\geq 1} :\\ &x_j \in B, \sum_{j\geq 1} \|x_j\|\rho(e_j^*) < +\infty \text{ for all continuous semi-norms } \rho \text{ on } E'_b\Big\}\end{aligned}$$

it follows that $H_b(B\widehat{\otimes}_\pi E'_b)$ is a subspace of $H_b\big(\ell^1(I)\widehat{\otimes}_\pi E'_b\big)$. Thus it remains to shows that $H_b\big(\ell^1(I)\widehat{\otimes}_\pi E'_b\big)$ has property ($\underline{DN}$).

(iii) Since $\ell^1(I)\widehat{\otimes}_\pi E'_b \cong \ell^1(I, E'_b)$ it follows that

$$\begin{aligned}\ell^1(I)\widehat{\otimes}_\pi E'_b = \Big\{&z = (t_{ij}) : (i,j) \in I \times \mathbf{N},\ t_{ij} \in \mathbf{C},\\ &\sum_{\substack{j\geq 1\\ i\in I}} |t_{ij}|\rho(e_j^*) < +\infty \text{ for all continuous semi-norms } \rho \text{ on } E'_b\Big\}\end{aligned}$$

For each $k \geq 1$, put

$$F(k) = \Big\{z = \big(t_{ij}\big)_{i\in I, j\geq 1} : \big|\|z\|\big|_k = \sum_{\substack{j\geq 1\\ i\in I}} |t_{ij}|\|e_j^*\|_k^* < +\infty\Big\}$$

where

$$\left\|e_j^*\right\|_k^* = \sup\left\{|e_j^*(t)| : \|t\|_k \leq 1,\ t \in E\right\}.$$

Since $\left\{e_j^*\right\}_{j\geq 1}$ is an absolute basis of E_b' it implies that for every bounded set A in $\ell^1(I)\widehat{\otimes}_\pi E_b'$ there exist $k \geq 1$ such that A is contained and bounded in $F(k)$. Otherwise, for every k there exists $z^k = \left(t_{ij}^k\right)_{j\geq 1, i\in I} \in A$ such that

$$\left|\left|\left|z^k\right|\right|\right|_k = \sum_{\substack{j\geq 1\\ i\in I}} |t_{ij}^k \| e_j^* \|_k^* = +\infty.$$

Hence, for each k we can find $u_k^j \in U_k$, $J_k \subset \mathbf{N}$, $I_k \subset I$ are finite such that

$$\sum_{j\in J_k, i\in I_k} |t_{ij}^k|\ |e_j^*(u_k^j)| > k.$$

Put $M = \{u_k^j : k \geq 1, j \in J_k\}$ and consider the semi-norm ρ_M on E_b' induced by M. Since $A \subset \ell^1(I)\widehat{\otimes}_\pi E_b'$ is bounded, it implies that for every $z = \left(t_{ij}\right)_{j\geq 1, i\in I} \subset A$ we have

$$\sum_{j\geq 1, i\in I} |t_{ij}|\rho_M(e_j^*) \leq C.$$

However, this is impossible by choosing $\{z^k\} \subset A$. Hence $H_b(\ell^1(I)\widehat{\otimes}_\pi E_b')$ is a subspace of $\lim\limits_k \operatorname{proj} H_b(F(k))$.

(iv) Put

$$\mathbf{M} = \left\{\sigma : I \times \mathbf{N} \longrightarrow \mathbf{N} : \sigma(i,j) \neq 0 \text{ only for finitely many } (i,j) \in I \times \mathbf{N}\right\}.$$

For $\sigma \in \mathbf{M}$ and $z = (t_{ij})$, $t_{ij} \in \mathbf{C}$, $i \in I$, $j \in \mathbf{N}$ put

$$\sigma^\sigma = \prod_{i,j} \sigma_{(i,j)}^{\sigma(i,j)}, \quad \sigma! = \prod_{i,j} \sigma(i,j)!,$$
$$|\sigma| = \sum_{i,j} \sigma(i,j), \quad z^\sigma = \prod_{i,j} t_{i,j}^{\sigma(i,j)},$$

where the usual convention $0! = 1$ and 0^0 is defined to be 1. By a modification of Ryan [11] it follows that the topology of $\lim\limits_k \operatorname{proj} H_b(F(k))$ can be defined by the system of semi-norms $\left\{|||\cdot|||_{(r,k)}\right\}_{r>0, k\geq 1}$ given by

$$|||f|||_{(r,k)} = \sup\left\{\frac{|a_{\sigma(f)}|\sigma^\sigma r^{|\sigma|} b_{\cdot,k}^\sigma}{|\sigma|^{|\sigma|}} \ :\ \sigma \in \mathbf{M}\right\} \tag{3}$$

where

$$a_\sigma(f) = \Big(\frac{1}{2\pi i}\Big)^n \int\limits_{|\lambda_{ij}|=1} \frac{f\Big(\sum\limits_{\sigma(i,j)\neq 0} \lambda_{ij} d_i \otimes e_j^*\Big)}{\prod\limits_{i,j} \lambda_{i,j}^{\sigma(i,j)+1}}\, d\lambda,$$

$$b_{i,j,k} = \frac{1}{\|e_j^*\|_k^*}\ ,\ n = \#\{(i,j), \sigma(i,j)\neq 0\},$$

$$d\lambda = \prod_{i,j} d\lambda_{i,j},\ \{d_i\}_{i\in I} \text{ is the canonical basis of } \ell^1(I).$$

(v) Since $\big(E'', \beta(E'', E')\big)$ has property $(\underline{DN})$ we can choose $p \geq 1$ such that

$$\forall q\ \exists k, C, d > 0 :\ \|\cdot\|_q^{1+d} \leq C\|\cdot\|_k\ \|\cdot\|_p^d \text{ on } E''. \tag{4}$$

Let $\{e_j\}_{j\geq 1}$ be the coefficient functional sequence associated to a basis $\{e_j^*\}$. Since $\{e_j^*\}_{j\geq 1}$ is an absolute basis, it follows that $\{e_j\}_{j\geq 1} \subset E''$ and

$$\|e_j\|_k = \frac{1}{\|e_j^*\|_k^*} = b_{i,j,k}.$$

Now applying (4) for $\{e_j\}_{j\geq 1}$ we get

$$b_{i,j,q}^{1+d} \leq C b_{i,j,k}\cdot b_{i,j,p}^d \quad \text{for every } i, j. \tag{5}$$

From (3), (5) we have

$$|||f|||_{(r,q)}^{1+d} = \sup\Big\{\frac{|a_\sigma(f)|\sigma^\sigma r^{|\sigma|} b_{\cdot,q}^\sigma}{|\sigma|^{|\sigma|}}\ :\ \sigma\in\mathbf{M}\Big\}^{1+d}$$

$$\leq \sup\Big\{\frac{|a_\sigma(f)|}{|\sigma|^{|\sigma|}}\sigma^\sigma r^{|\sigma|(1+d)} C^{|\sigma|} b_{\cdot,k}^\sigma : \sigma\in\mathbf{M}\Big\} \sup\Big\{\frac{|a_\sigma(f)|}{|\sigma|^{|\sigma|}}\sigma^\sigma b_{\cdot,p}^\sigma : \sigma\in\mathbf{M}\Big\}^d$$

$$= |||f|||_{(Cr^{1+d},k)} |||f|||_{(1,p)}^d$$

for $f \in \lim\operatorname{proj} H_b(F(k))$. Consequently, $\lim\operatorname{proj} H_b(F(k))$ has property $(\underline{DN})$. Theorem B is proved. □

3. The property $(\underline{DN})$ and pluripolar sets

In this section we establish the relation between the property $(\underline{DN})$ on a Frechet space and the existence of pluripolar sets on its strongly dual space E'_b. This result has been shown earlier by Dineen-Meise-Vogt [3] in the case E is nuclear. Here we have

Theorem 3.1. *Let E be a Frechet-Montel space such that E'_b has an absolute basis. Then the following are equivalent*

(i) *E has property $(\underline{DN})$,*

(ii) *$H(E'_b)$ has property $(\underline{DN})$,*

(iii) *E'_b contains a non-pluripolar compact set.*

Proof. (i) $\Leftrightarrow$ (ii) follows from the fact that E is a subspace of $H(E'_b)$ and by the Theorem B. (iii) $\Rightarrow$ (i) follows from [3], where as (ii) $\Rightarrow$ (iii) is an immediate consequence of the following result.

Proposition 3.2. *Let E be a Frechet-Montel space having the approximation property. If $H(E'_b)$ has property $(\underline{DN})$, then E'_b contains a non-pluripolar compact set.*

Proof. Since $H(E'_b)$ has property $(\underline{DN})$, there exists a compact set B in E'_b satisfying property $(\underline{DN})$ on $H(E'_b)$ such that

$$\forall q\ \exists k, C, d > 0 : \big\| \cdot \big\|_q^{1+d} \leq C \big\| \cdot \big\|_k \ \big\| \cdot \big\|_B^d,$$

where $\{\|\cdot\|_q\}_{q\geq 1}$ is the fundamental system of semi-norms on $H(E'_b)$ given by

$$\|\sigma\|_q = \sup\Big\{|\sigma(z)| : z \in U_q^0\Big\}, \quad \sigma \in H(E'_b),$$

and $\{U_q\}_{q\geq 1}$ is a neighbourhood basis of $0 \in E$, U_q^0 is a polar of U_q.

We shall prove that B is not pluripolar. If B is pluripolar, we can find a plurisubharmonic function φ on E'_b such that

$$\varphi \not\equiv -\infty \quad \text{and} \quad \varphi\big|_B = -\infty.$$

Consider the Hartogs domain $\Omega_\varphi \subset E'_b \times \mathbf{C}$ defined by

$$\Omega_\varphi = \Big\{(z, \lambda) \in E'_b \times \mathbf{C} : |\lambda| < e^{-\varphi(z)}\Big\}.$$

Note that Ω_φ is pseudoconvex in $E'_b \times \mathbf{C}$. Since E'_b and hence $E'_b \times \mathbf{C}$ has the approximation property, there exists $f \in H(\Omega_\varphi)$ such that Ω_φ is the domain of existence of f [12]. Write the Hartogs expansion of f at $(0,0) \in \Omega_\varphi$ as

$$f(z,\lambda) = \sum_{n=0}^{\infty} h_n(z)\lambda^n \quad \text{for } (z,\lambda) \in \Omega_\varphi,$$

where

$$h_n(z) = \frac{1}{2\pi i} \int\limits_{|\lambda| = \frac{1}{2}e^{-\varphi(z)}} \frac{f(z,\lambda)}{\lambda^{n+1}} d\lambda, \quad n \geq 0.$$

Since φ is upper semi-continuous, h_n is holomorphic on E'_b for each $n \geq 0$. On the other hand, since $\varphi\big|_B = -\infty$ it follows that the series $\sum_{n=0}^{\infty} h_n(z)\lambda^n$ converges to f uniformly on $K \times r\overline{\Delta}$ for all $r > 0$, where $\overline{\Delta} = \big\{z : |z| \leq 1\big\}$ and K is an arbitrary compact set in B. Hence

$$\lim_{n\to\infty} \sup \frac{1}{n} \log \big\|h_n\big\|_B = -\infty.$$

Let $q \geq 1$. Choose k, C, $d > 0$ such that

$$\big\|h_n\big\|_q^{1+d} \leq C\big\|h_n\big\|_k \big\|h_n\big\|_B^d, \quad \forall n \geq 1.$$

These inequalities imply that

$$\begin{aligned}\lim_n \sup \frac{1+d}{n} \log \big\|h_n\big\|_q &\leq \log C + \lim_n \sup \frac{1}{n} \log\big\|h_n\big\|_k + \lim_n \sup \frac{d}{n} \log\big\|h_n\big\|_B \\ &= -\infty.\end{aligned}$$

Hence, the series $\sum_{n\geq 0} h_n(z)\lambda^n$ converges uniformly on every compact set in $E'_b \times \mathbf{C}$. Since Ω_φ is the domain of existence of f, we infer that $E'_b \times \mathbf{C} \subset \Omega_\varphi$. This is impossible, because $\varphi \not\equiv -\infty$. □

Here arises the question whether the implication (i) ⇒ (iii) of Theorem 3.1 holds if we do not assume that E'_b has an absolute basis. Concerning this question we have the following

Proposition 3.3. *Let E be a Frechet space having property (DN). Then E'_b contains a non-pluripolar bounded set.*

Proof. By Vogt [14] E is isomorphic to a subspace of $B\widehat{\otimes}_\pi s$, where B is a Banach space. Let $R : \big(B\widehat{\otimes}_\pi s\big)' \cong B'\widehat{\otimes}_\pi s' \to E'_b$ be the restriction map. Since every Banach space is a quotient space of $\ell^1(I)$ for some index set I, we may assume without loss of generality that $B' \cong \ell^1(I)$. On the other hand, if $B\widehat{\otimes}_\pi s$ has property (DN), so does $H_b\big(B'\widehat{\otimes}s'\big) = H_b\big(\ell^1(I)\widehat{\otimes}_\pi s'\big)$ and from the definition of property (DN) it is easy to check that s has property (DN). Hence we may assume that $A = \mathrm{conv}(U \otimes U_p^0) \subset \ell^1(I)\widehat{\otimes}_\pi s'$ such that the semi-norm on $H_b(\ell^1(I)\widehat{\otimes}_\pi s')$ induced by A is the (DN)-norm for $H_b(\ell^1(I)\widehat{\otimes}_\pi s')$, where U is the unit ball of $\ell^1(I)$ and U_p is a neighbourhood of $0 \in s$ induces the (DN)-norm for s.

Put $B = R(A)$. If B is pluripolar in E'_b, there exists a plurisubharmonic function φ on E'_b such that $\varphi \not\equiv -\infty$ and $\varphi\big|_B = -\infty$. Put

$$\Omega = \Big\{(\omega,\lambda) \in \big(\ell^1(I)\widehat{\otimes}_\pi s\big) \times \mathbf{C} : |\lambda| < e^{-\varphi R(\omega)}\Big\}.$$

It follows that Ω is pseudoconvex in $\big(\ell^1(I)\widehat{\otimes}_\pi s\big) \times \mathbf{C}$ and $A \times \mathbf{C} \subset \Omega$.

For each countable subset J of I let $\Omega_J = \Omega \cap \big(\ell^1(J)\widehat{\otimes}_\pi s'\big) \times \mathbf{C}$. Then Ω_J is the domain of existence of a holomorphic function f_J. Write

$$f_J(\omega,\lambda) = \sum_{n\geq 0} h_{J,n}(\omega)\lambda^n \quad \text{for } (\omega,\lambda) \in \Omega_J,$$

where

$$h_{J,n}(\omega) = \frac{1}{2\pi i} \int\limits_{|t| = \frac{1}{2}e^{-\varphi R(\omega)}} \frac{f(w,t)}{t^{n+1}} dt.$$

Since φ is upper-continuous, it follows that $h_{J,n}$ are holomorphic on $\ell^1(J)\widehat{\otimes}_\pi s'$. Put $A_J = A \cap (\ell^1(J)\widehat{\otimes}_\pi s')$. Since $A_J \times \mathbf{C} \subset \Omega_J$, the series $\sum\limits_{n\geq 0} h_{J,n}(\omega)\lambda^n$ converges uniformly to f_J on $K \otimes r\overline{\Delta}$ for $r > 0$, where $\overline{\Delta} = \{z \in \mathbf{C} : |z| \leq 1\}$ and K is a compact set in A_J. Thus,

$$\limsup_n \frac{1}{n} \log \big\|h_{J,n}\big\|_{A_J} = -\infty.$$

Let $q \geq 1$. Choose $k, C > 0$ such that

$$\big\|h_{J,n}\big\|_q^2 \leq C \big\|h_{J,n}\big\|_k \big\|h_{J,n}\big\|_{A_J}$$

This inequality yields

$$\limsup_n \frac{2}{n}\log\left\|h_{J,n}\right\|_q \leq \log C + \limsup_n \log\left\|h_{J,n}\right\|_k + \limsup_n \log\left\|h_{J,n}\right\|_{A_J}$$
$$= -\infty$$

where

$$\left\|h_{J,n}\right\|_q = \sup\Big\{|h_{J,n}(\omega)| : \omega \in \operatorname{conv}\big(U_J \otimes U_q^0\big)\Big\}$$

$U_J = U \cap \ell^1(J)$ and similarly for $\left\|h_{J,n}\right\|_k$. Hence the series $\sum_{n\geq 0} h_{J,n}(\omega)\lambda^n$ converges uniformly on every compact set in $\big(\ell^1(J)\widehat{\otimes}_\pi s'\big) \times \mathbf{C}$. On the other hand, since Ω_J is the domain of existence of f_J, it implies that $\big(\ell^1(J)\widehat{\otimes}_\pi s'\big) \times \mathbf{C} \subset \Omega_J$. This shows $\varphi R = -\infty$ on $\ell^1(J)\widehat{\otimes}_\pi s'$. Since J is an arbitrary countable set, $\varphi \equiv -\infty$. This is impossible. Hence B is not pluripolar in E_b'.

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