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AN OPERATIONAL METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS WITH THE CAPUTO DERIVATIVES

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Abstract. In the present paper, we first develop the operational calculus of Mikusiński's type for the Caputo fractional differential operator. This calculus is used to obtain exact solutions of an initial value problem for linear fractional differential equations with constant coefficients and fractional derivatives in Caputo's sense. The initial conditions are given in terms of the field variable and its derivatives of integer order. The obtained solutions are expressed through Mittag-Leffler type functions. Special cases and integral representations of solutions are presented.

1. Introduction

Fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and in engineering (see, for example, [1]–[5], [9], [10], [16], [17]). In its turn, mathematical aspects of fractional differential equations and methods of their solution were discussed by many authors: the iteration method in [24], the series method in [1], the Fourier transform technique in [3], [12], special methods for fractional differential equations of rational order or for equations of special type in [2], [13], [15], [18], [19], [21], the Laplace transform technique in [9], [10], [16], [17], [21], [22], the operational calculus method in [11], [15]. Let us note that in mathematical treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative of order μ ($\mu \geq 0$) is normally used:

$$(1)\quad (D^{\mu} f)(x) := \left(\frac{d}{dx}\right)^{m} (J^{m-\mu} f)(x),\ m-1 < \mu \leq m,\ m \in \mathbf{N},\ x > 0,$$

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where

$$(2)\qquad (J^{\mu} f)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)dt, \quad \mu > 0,\ x > 0,$$

$$(J^0 f)(x) := f(x), \quad x > 0$$

is the Riemann-Liouville fractional integral of order μ. In this interpretation, the fractional derivative is left-inverse (and not right-inverse) to the corresponding fractional integral, which is the natural generalization of the Cauchy formula for the n-fold primitive of a function $f(t)$. As to the initial value problem for fractional differential equations with the fractional derivatives in the Riemann-Liouville sense, there are some troubles with the initial conditions, see [11], [15], [21], [22], [23]. Namely, these initial conditions should be given as the (bounded) initial values of the fractional integral $J^{m-\mu}$ and of its integer derivatives of order $k = 1, 2, \ldots, m-1$. On the other hand, in modeling of real processes the initial conditions are normally expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order. In order to meet this physical requirement, an alternative definition of fractional derivative was introduced by Caputo [4] and adopted by Caputo and Mainardi [5] in the framework of the theory of Linear Viscoelasticity:

$$(3)\quad (D_*^{\mu} f)(x) := (J^{m-\mu} f^{(m)})(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt,$$

$$m-1 < \mu \le m,\ m \in \mathbf{N},\ x > 0.$$

In a series of papers (see [9], [16], [17] and the references there) the Caputo fractional derivative was considered and some of the simplest linear fractional differential equations with constant coefficients and fractional derivatives in the Caputo sense were solved by using the Laplace transform technique.

In the present paper we use the operational calculus method to solve an initial value problem for a general linear fractional differential equation with constant coefficients and with the Caputo fractional derivatives. We first develop a modification of the Mikusiński operational calculus for the Caputo fractional derivative, a modification that is more suitable for application to the solution of differential and integral equations of fractional order. Firstly, we consider the space C_{α} of functions which can have an integrable singularity in the neighborhood of zero, instead of the space C$[0, \infty)$ of functions continuous on the half-axis used in Mikusiński's operational calculus. Since the kernel of the Riemann-Liouville integral operator J^μ belongs to the space C_α if $\mu - 1 > \alpha$ and $\alpha \geq -1$, this function space seems to be more suitable for the development of the operational calculus for the Caputo derivative. Secondly, the abovementioned property allows us to consider the Riemann-Liouville integral operator J^μ as the convolution product with the power function $x^{\mu-1}/\Gamma(\mu)$ in the space C_α if $\mu - 1 > \alpha$, and this power function plays in our considerations the same role, as the function $\{1\}$ in Mikusiński's operational calculus. As an application of this calculus we obtain solutions of linear fractional differential equations with constant coefficients and Caputo derivatives.

2. Some properties of the caputo fractional derivative

Since both the Riemann-Liouville and the Caputo derivatives ((1) and (3), respectively) are defined through the Riemann-Liouville fractional integral (2) and this operator plays a very important role in the development of the corresponding operational calculus, there are some coinciding elements in the operational calculi for both fractional derivatives. The operational calculus for the Riemann-Liouville fractional derivative was already developed in [8], [11], [15] and we shall cite some theorems from there without proofs.

We begin by defining the function space C_α, $\alpha \in \mathbf{R}$, which was used by Dimovski [6] in his development of the operational calculus for the hyper-Bessel differential operator.

Definition 2.1. A real or complex-valued function $f(x)$, $x > 0$, is said to be in the space C_α, $\alpha \in \mathbf{R}$, if there exists a real number p, $p > \alpha$, such that

$$f(x) = x^p f_1(x)$$

with a function $f_1(x)$ in $C[0, \infty)$.

Clearly, C_α is a vector space and the set of spaces C_α is ordered by inclusion according to

$$C_\alpha \subset C_\beta \Leftrightarrow \alpha \geq \beta. \tag{4}$$

Theorem 2.1. *The Riemann-Liouville fractional integral operator J^μ, $\mu \geq 0$, is a linear map of the space C_α, $\alpha \geq -1$, into itself, that is,*

$$J^\mu : C_\alpha \to C_{\mu+\alpha} \subset C_\alpha.$$

Remark 2.1. In the case $f \in C_\alpha$ for a value $\alpha \geq -1$ and for $\mu \geq 1$ we have $J^\mu f \in C_0 \subset C[0, \infty)$.

It is important to note, that the operator J^μ, $\mu > 0$, has the following convolutional representation in the space C_α, $\alpha \geq -1$:

$$(5) \qquad (J^\mu f)(x) = (h_\mu \circ f)(x), \quad h_\mu(x) := x^{\mu-1}/\Gamma(\mu),\ f \in C_\alpha,$$

where

$$(g \circ f)(x) = \int_0^x g(x-t)f(t)dt, \quad x > 0$$

is the Laplace convolution. For the Laplace convolution itself the inclusion

$$(6) \qquad g \circ f \in C_{\alpha_1+\alpha_2+1} \subseteq C_{-1},\ f \in C_{\alpha_1},\ g \in C_{\alpha_2},\ \alpha_1, \alpha_2 \geq -1$$

holds true. The representation (5) and the commutativity of the Laplace convolution (see [6], [20]) lead to the following property of the Riemann-Liouville fractional integral:

$$(J^\delta J^\eta f)(x) = (J^\eta J^\delta f)(x), \quad f \in C_\alpha,\ \alpha \geq -1,\ \delta \geq 0,\ \eta \geq 0.$$

Next, using the associativity of the Laplace convolution and the Euler integral of the first kind for the evaluation of $(h_\delta \circ h_\eta)(x)$, we obtain

$$(7) \qquad (J^\delta J^\eta f)(x) = (J^{\delta+\eta} f)(x), \quad f \in C_\alpha,\ \alpha \geq -1,\ \delta \geq 0,\ \eta \geq 0,$$

that is also well known. In particular, it follows from (7) that

$$(8) \qquad (\underbrace{J^\mu \dots J^\mu}_{n} f)(x) = (J^{n\mu} f)(x), \quad f \in C_\alpha,\ \alpha \geq -1,\ \mu \geq 0,\ n \in \mathbf{N}.$$

It is obvious, that the Caputo derivative (3) is not defined on the whole space C_α. Let us introduce a subspace of C_α, which is suitable for dealing with the Caputo derivative.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space

$$C_\alpha^m,\ m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},\ \text{iff } f^{(m)} \in C_\alpha.$$

Remark 2.2. The space C_α^m doesn't coincide with the space $C_\alpha^{(m)} = \{f :$ there exist $p > \alpha,\ \tilde{f} \in C^m[0, \infty)$ such that $f(x) = x^p \tilde{f}(x)\}$, considered in

[14]. For example, if $f(x) = \cos(x)/\sqrt{x}$, $x > 0$, then $f \in C_{-1}^{(1)}$, $f \notin C_{-1}^{1}$ and for the function $f(x) \equiv 1$, $x > 0$, we have $f \notin C_0^{(1)}$, $f \in C_0^1$.

We state five of the properties of the space C_α^m, which will be used in the further discussions.

1) C_α^m is a vector space.

2) $C_\alpha^0 \equiv C_\alpha$.

3) If $f \in C_\alpha^m$ for a value $\alpha \geq -1$ and an index $m \geq 1$, then $f^{(k)}(0+) := \lim\limits_{x \to 0+} f^{(k)}(x) < +\infty$, $0 \leq k \leq m-1$, and the function

$$\tilde{f}(x) = \begin{cases} f(x), & x > 0, \\ f(0+), & x = 0 \end{cases}$$

is in $C^{m-1}[0, \infty)$.

Proof. $f \in C_\alpha^m$ means $f^{(m)} := \phi \in C_\alpha$. Let us fix $x = X > 0$. Since $f^{(m)} = \phi \in C[\xi, X]$, $0 < \xi < X$, we arrive at

$$\int_\xi^X f^{(m)}(t)\,dt = f^{(m-1)}(X) - f^{(m-1)}(\xi),$$

where the left- and right-hand sides as functions of ξ are continuous on $(0, X]$. Furthermore, since

$$\lim_{\xi \to 0+} \int_\xi^X f^{(m)}(t)\,dt = \int_0^X \phi(t)\,dt < +\infty,$$

we get

$$f^{(m-1)}(0+) := \lim_{\xi \to 0+} f^{(m-1)}(\xi) = f^{(m-1)}(X) - \int_0^X \phi(t)\,dt < +\infty,$$

hence

$$f^{(m-1)}(X) = \int_0^X \phi(t)\,dt + f^{(m-1)}(0+).$$

By liberating $x = X$ and putting $f^{(m-1)}(0) := f^{(m-1)}(0+)$ we recognize $f^{(m-1)}$ as continuous on $[0, \infty)$ and obtain the representation

$$f(x) = (J^m \phi)(x) + \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{x^k}{k!}, \quad x \geq 0, \tag{9}$$

$$\phi(t) = f^{(m)}(t), \quad f^{(k)}(0+) = \lim_{x \to 0+} f^{(k)}(x) < +\infty, \quad 0 \leq k \leq m-1.$$

4) If $f \in C_\alpha^m$ for a value $\alpha \geq -1$, then $f \in C^m(0, \infty) \cap C^{m-1}[0, \infty)$.

Indeed, since $f^{(m)} \in C_\alpha$, we have $f^{(m)} \in C(0, \infty)$. The inclusion $f \in C^{(m-1)}[0, \infty)$ follows from property 3).

5) For real $\alpha \geq -1$ and index $m \geq 1$ the following equivalence holds:

$$f \in C_\alpha^m \Leftrightarrow f(x) = (J^m \phi)(x) + \sum_{k=0}^{m-1} c_k \frac{x^k}{k!}, \quad x \geq 0, \ \phi \in C_\alpha.$$

The first statement ($\Rightarrow$) is already proved (see (9)), the second one ($\Leftarrow$) is checked directly. Let us note that in this case we have $f^{(k)}(0) = c_k$, $k = 0, 1, \ldots, m-1$, $f^{(m)} = \phi$.

On the basis of the properties 1)-5) of the function space C_α^m we prove now some theorems, important for the development of the corresponding operational calculus.

Theorem 2.2. *Let $f \in C_{-1}^m$, $m \in \mathbf{N}_0$. Then the Caputo fractional derivative $D_*^\mu f$, $0 \leq \mu \leq m$, is well defined and the inclusion*

$$D_*^\mu f \in \begin{cases} C_{-1}, & m-1 < \mu \leq m, \\ C^{k-1}[0, \infty) \subset C_{-1}, & m-k-1 < \mu \leq m-k, \ k = 1, \ldots, m-1 \end{cases}$$

holds true.

Proof. In the case $m - 1 < \mu \leq m$ the inclusion under consideration follows from the definition of the Caputo derivative D_*^μ and Theorem 2.1. The property 4) of the space C_α^m, $m \geq 1$, and the corresponding mapping properties of the Riemann-Liouville fractional integral (see [24]) give us the inclusion $D_*^\mu f \in C^{k-1}[0, \infty)$ for $m-k-1 < \mu \leq m-k$, $k = 1, \ldots, m-1$. The inclusion $C^{k-1}[0, \infty) \subset C[0, \infty) \subset C_{-1}$ follows from (4).

Theorem 2.3. *Let $f \in C_{-1}^m$, $m \in \mathbf{N}$ and $m-1 < \mu \leq m$. Then the Riemann-Liouville and the Caputo fractional derivatives are connected by the relation:*

$$(D^\mu f)(x) = (D_*^\mu f)(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0+)}{\Gamma(1+k-\mu)} x^{k-\mu}, \quad x > 0. \tag{10}$$

Proof. Making use of the representation (9), we get

$$\begin{aligned}
(D^\mu f)(x) &= \left(\frac{d}{dx}\right)^m (J^{m-\mu} f)(x) \\
&= \left(\frac{d}{dx}\right)^m \left(J^{m-\mu}\left[(J^m f^{(m)})(t) + \sum_{k=0}^{m-1} f^{(k)}(0+)\frac{t^k}{k!}\right]\right)(x) \\
&= (J^{m-\mu} f^{(m)})(x) + \left(\frac{d}{dx}\right)^m \left(J^{m-\mu}\left[\sum_{k=0}^{m-1} f^{(k)}(0+)\frac{t^k}{k!}\right]\right)(x) \\
&= (D_*^\mu f)(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0+)}{\Gamma(1+k-\mu)} x^{k-\mu}, \quad x > 0.
\end{aligned}$$

In finding the last expression we have made use of the rules for fractional integration and differentiation of the power function $p_\nu(x) = x^\nu$, $\nu > -1$, namely,

$$(J^\mu p_\nu)(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\mu)} x^{\nu+\mu}, \quad \mu \geq 0,\ x > 0, \tag{11}$$

$$(D^\mu p_\nu)(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\mu)} x^{\nu-\mu}, \quad \mu \geq 0,\ x > 0. \tag{12}$$

Corollary 2.1. *It follows from the representation* (10) *that the Riemann-Liouville fractional derivative* $(D^\mu f)(x)$ *is not, in the general case, in the space* C_{-1}, *if* $f \in C_{-1}^m$. *There are only three exceptional cases:*

1) *If* $\mu = m \in \mathbf{N}$, *then*

$$D^\mu f \equiv D_*^\mu f \equiv f^{(m)} \in C_{-1}.$$

2) *If* $f^{(k)}(0+) = 0$, $k = 0, \ldots, m-1$, *then*

$$D^\mu f \equiv D_*^\mu f \in C_{-1}.$$

3) *If* $0 < \mu < 1$, *then* $D^\mu f \in C_{-1}$ *because of*

$$(D^\mu f)(x) = (D_*^\mu f)(x) + \frac{f(0)}{\Gamma(1-\mu)} x^{-\mu}.$$

Theorem 2.4. *Let* $m-1 < \mu \le m$, $m \in \mathbf{N}$, $\alpha \ge -1$ *and* $f \in C_\alpha^m$. *Then*

$$(J^\mu D_*^\mu f)(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{x^k}{k!}, \quad x \ge 0. \tag{13}$$

Proof. Using the relations (3), (7), and (9) we get

$$\begin{aligned}(J^\mu D_*^\mu f)(x) &= (J^\mu J^{m-\mu} f^{(m)})(x) = (J^m f^{(m)})(x) \\ &= f(x) - \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{x^k}{k!}, \quad x \ge 0.\end{aligned}$$

Theorem 2.5. *Let* $f \in C_{-1}^m$, $m \in \mathbf{N}_0$, $f(0) = \cdots = f^{(m-1)}(0) = 0$ *and* $g \in C_{-1}^1$. *Then the Laplace convolution*

$$h(x) = \int_0^x f(t) g(x-t)\, dt$$

is in the space C_{-1}^{m+1} *and* $h(0) = \cdots = h^{(m)}(0) = 0$.

Proof. Let $m = 0$. The property 4) of the space C_{-1}^1 gives us the inclusion $g \in C[0,\infty)$. In this case $h(x) = x^\varepsilon h_1(x)$ for an $\varepsilon > 0$, $h_1 \in C[0,\infty)$, i.e., $h \in C[0,\infty)$ and $h(0) = 0$. We have also

$$h'(x) = \int_0^x f(t) g'(x-t)\, dt + f(x) g(0), \quad x > 0.$$

Since $g' \in C_{-1}$ and $f \in C_{-1}$ it follows from the inclusion (6) that $\int_0^x f(t) g'(x-t)\, dt \in C_{-1}$ and, consequently, $h' \in C_{-1}$, which proves our theorem in the case $m = 0$. For $m = 1$, it follows from the property 4) of the space C_{-1}^1, that $f \in C[0,\infty)$. Using the condition $f(0) = 0$ and the representation obtained above for h' we arrive at $h'(0) = 0$ and

$$h''(x) = \int_0^x f'(x-t)g'(t)\,dt + f'(x)g(0), \quad x > 0.$$

Since $g' \in C_{-1}$ and $f' \in C_{-1}$, we get the inclusion $h'' \in C_{-1}$. Repeating the same arguments m times, we arrive at $h(0) = \cdots = h^{(m)}(0) = 0$ and the representation

$$h^{(m+1)}(x) = \int_0^x f^{(m)}(x-t)g'(t)\,dt + f^{(m)}(x)g(0), \quad x > 0,$$

which gives us the inclusion $h \in C_{-1}^{m+1}$.

3. Operational calculus for the caputo derivative

For the sake of simplicity we shall consider in our further discussions the case of the space C_{-1}, which turns out to be the most interesting one for applications. As in the case of Mikusiński's calculus, we have

Theorem 3.1. *The space C_{-1} with the operations of the Laplace convolution $\circ$ and ordinary addition becomes a commutative ring $(C_{-1}, \circ, +)$ without divisors of zero.*

This ring can be extended to the quotient field $\mathcal{M}$ by following the lines of Mikusiński [20]:

$$\mathcal{M} := C_{-1} \times (C_{-1} \setminus \{0\}) / \sim,$$

where the equivalence relation $(\sim)$ is defined, as usual, by

$$(f, g) \sim (f_1, g_1) \Leftrightarrow (f \circ g_1)(x) = (g \circ f_1)(x).$$

Thus, we can consider the elements of the field $\mathcal{M}$ as convolution quotients f/g and define the operations in $\mathcal{M}$ as follows:

$$\frac{f}{g} + \frac{f_1}{g_1} := \frac{f \circ g_1 + g \circ f_1}{g \circ g_1}$$

and

$$\frac{f}{g} \cdot \frac{f_1}{g_1} := \frac{f \circ f_1}{g \circ g_1}.$$

The proof of the fact that the set $\mathcal{M}$ is a commutative field with respect to the operations "+" and "·" is based on Theorem 3.1.

It is easily seen that the ring C_{-1} can be embedded in the field $\mathcal{M}$ by the map $(\mu > 0)$:

$$f \mapsto \frac{h_\mu \circ f}{h_\mu},$$

with, by (5), $h_\mu(x) = x^{\mu-1}/\Gamma(\mu)$.

Defining the operation of multiplication with a scalar λ from the field **R** (or **C**) by the relation

$$\lambda \frac{f}{g} := \frac{\lambda f}{g}, \quad \frac{f}{g} \in \mathcal{M},$$

and remembering the fact, that the set C_{-1} is a vector space, we check that the set $\mathcal{M}$ is a vector space too. Since the constant function $f(x) \equiv \lambda,\ x > 0$, is in the space C_{-1}, we should distinguish the operation of multiplication with a scalar in the vector space $\mathcal{M}$ and the operation of multiplication with a constant function in the field $\mathcal{M}$. In this last case we shall write

$$\{\lambda\} \cdot \frac{f}{g} = \frac{\lambda h_{\mu+1}}{h_\mu} \cdot \frac{f}{g} = \{1\} \cdot \frac{\lambda f}{g}. \tag{14}$$

It is easy to check that the element $I = \frac{h_\mu}{h_\mu}$ of the field $\mathcal{M}$ is the unity of this field with respect to the operation of multiplication. Let us prove that this element of the field $\mathcal{M}$ is not reduced to a function lying in the ring C_{-1} and, consequently, it can be regarded as a generalized function. Indeed, let the unity $I = \frac{h_\mu}{h_\mu}$ be reduced to some function $f \in C_{-1}$. Then we have

$$\frac{h_\mu}{h_\mu} \sim \frac{f \circ h_\mu}{h_\mu} \Leftrightarrow h_\mu \circ h_\mu = (f \circ h_\mu) \circ h_\mu. \tag{15}$$

Making use of the representations (5) and (8) we rewrite the last relation in the form

$$\frac{x^{2\mu-1}}{\Gamma(2\mu)} = (J^{2\mu} f)(x), \tag{16}$$

where $(J^{2\mu} f)(x)$ is the Riemann-Liouville fractional integral (2). It is well known (see [15] for the case of the space C_{-1}), that the Riemann-Liouville fractional derivative (1) is left-inverse to the Riemann-Liouville fractional integral. Applying the operator $D^{2\mu}$ to the equality (16) and using the formula (12) for evaluation of the right-hand side, we obtain

$$0 \equiv f(x), \ x > 0. \tag{17}$$

Relations (16) and (17) lead to the contradictory formula

$$\frac{x^{2\mu-1}}{\Gamma(2\mu)} \equiv 0, \quad x > 0,$$

which shows, that the relation (15) is contradictory for any $f \in C_{-1}$ and, consequently, the unity element I of the field $\mathcal{M}$ is not reduced to function in the ring C_{-1}. Later we shall consider some other elements of the field $\mathcal{M}$ possessing this property, in particular, the element which will play an important role in the applications of operational calculus and is given by

Definition 3.1. The algebraic inverse of the Riemann-Liouville fractional integral operator J^μ is said to be the element S_μ of the field $\mathcal{M}$, which is reciprocal to the element h_μ in the field $\mathcal{M}$, that is,

$$S_\mu = \frac{I}{h_\mu} \equiv \frac{h_\mu}{h_\mu \circ h_\mu} \equiv \frac{h_\mu}{h_{2\mu}}, \tag{18}$$

where (and in what follows) $I = \dfrac{h_\mu}{h_\mu}$ denotes the identity element of the field $\mathcal{M}$ with respect to the operation of multiplication.

As we have already seen, the Riemann-Liouville fractional integral J^μ can be represented as a multiplication (convolution) in the ring C_{-1} (with the function h_μ, see (5)). Since the ring C_{-1} is embedded into the field $\mathcal{M}$ of convolution quotients, this fact can be rewritten as follows:

$$(J^\mu f)(x) = \frac{I}{S_\mu} \cdot f. \tag{19}$$

As to the Caputo fractional derivative, there exists no convolution representation in the ring C_{-1} for it, but it is reduced to the operator of multiplication in the field $\mathcal{M}$.

Theorem 3.2. *Let $f \in C_{-1}^m$, $m-1 < \mu \le m$, $m \in \mathbf{N}$. Then the following relation holds true in the field $\mathcal{M}$ of convolution quotients:*

$$D_*^\mu f = S_\mu \cdot f - S_\mu \cdot f_\mu, \; f_\mu(x) := \sum_{k=0}^{m-1} f^{(k)}(0+)\frac{x^k}{k!}. \tag{20}$$

Proof. Theorem 2.4 gives us the representation

$$(J^\mu D_*^\mu f)(x) = f(x) - f_\mu(x), \quad f_\mu(x) = \sum_{k=0}^{m-1} f^{(k)}(0+)\frac{x^k}{k!}, \quad x \ge 0.$$

Upon multiplying both sides of this relation by S_μ, if we apply the relation (19), we shall obtain the assertion (20).

We already know (see (8)), that for $\mu > 0$, $n \in \mathbf{N}$

$$h_\mu^n(x) := \underbrace{h_\mu \circ \ldots \circ h_\mu}_{n} = h_{n\mu}(x).$$

Let us extend this relation to an arbitrary positive real power exponent:

$$h_\mu^\lambda(x) := h_{\lambda\mu}(x), \quad \lambda > 0. \tag{21}$$

We have for any $\lambda > 0$ the inclusion $h_\mu^\lambda \in C_{-1}$, and the following relations can be easily checked ($\alpha > 0$, $\beta > 0$):

$$h_\mu^\alpha \circ h_\mu^\beta = h_{\alpha\mu} \circ h_{\beta\mu} = h_{(\alpha+\beta)\mu} = h_\mu^{\alpha+\beta}, \tag{22}$$

$$h_{\mu_1}^\alpha = h_{\mu_2}^\beta \;\Leftrightarrow\; \mu_1\alpha = \mu_2\beta. \tag{23}$$

Then we define a power function of the element S_μ with an arbitrary real power exponent λ:

$$S_\mu^\lambda = \begin{cases} h_\mu^{-\lambda}, & \lambda < 0, \\ I, & \lambda = 0, \\ \frac{I}{h_\mu^\lambda}, & \lambda > 0. \end{cases} \tag{24}$$

Using this definition and the relations (22) and (23), we get ($\alpha, \beta \in \mathbf{R}$):

$$S_\mu^\alpha \cdot S_\mu^\beta = S_\mu^{\alpha+\beta}, \tag{25}$$

$$S_{\mu_1}^\alpha = S_{\mu_2}^\beta \;\Leftrightarrow\; \mu_1\alpha = \mu_2\beta. \tag{26}$$

For many applications it is important to know the functions of S_μ in $\mathcal{M}$ which can be represented by means of the elements of the ring C_{-1}. One useful class of such functions is given by the following theorem.

Theorem 3.3. *Let the power series*

$$\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} x_1^{i_1} \times \cdots \times x_n^{i_n},\; x = (x_1,\dots,x_n) \in \mathbf{R}^n,\; a_{i_1,\dots,i_n} \in \mathbf{R},$$

be convergent at a point $x_0 = (x_{10},\dots,x_{n0})$ *with all* $x_{k0} \neq 0,\; k = 1,\dots,n,$ *and* $\beta > 0,\; \alpha_i > 0,\; i = 1,\dots,n.$
Then the function of S_μ

$$S_\mu^{-\beta} \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} (S_\mu^{-\alpha_1})^{i_1} \times \cdots \times (S_\mu^{-\alpha_n})^{i_n}$$

$$= \sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} h_{(\beta+\alpha_1 i_1+\cdots+\alpha_n i_n)\mu}(x),$$

where $h_\mu(x)$ *is given by* (5)*, defines an element of the ring* C_{-1}*.*

For the proof of this theorem we refer to [11]. We give here some operational relations, which will be used in the further discussions. For more operational relations we refer to [8], [11], and [15].

$$\frac{I}{S_\mu - \rho} = x^{\mu-1} E_{\mu,\mu}(\rho x^\mu), \tag{27}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\; |z| < \infty.$$

The relation (27) can formally be obtained as a geometric series:

$$\frac{I}{S_\mu - \rho} = \frac{I}{\dfrac{I}{h_\mu} - \rho} = \frac{h_\mu}{I - \rho h_\mu} = \sum_{k=0}^{\infty} \rho^k h_\mu^{k+1}$$
$$= \sum_{k=0}^{\infty} \frac{\rho^k x^{(k+1)\mu-1}}{\Gamma(\mu k + \mu)} = x^{\mu-1} E_{\mu,\mu}(\rho x^\mu).$$

The m-fold convolution of the right-hand side of the relation (27) gives us the operational relation:

$$\frac{I}{(S_\mu - \rho)^m} = x^{\mu m - 1} E^m_{\mu, m\mu}(\rho x^\mu),\ m \in \mathbf{N}, \tag{28}$$

where

$$E^m_{\alpha,\beta}(z) := \sum_{i=0}^{\infty} \frac{(m)_i z^i}{i!\Gamma(\alpha i + \beta)},\ \alpha, \beta > 0,\ |z| < \infty,\ (m)_i = \prod_{k=0}^{i-1}(m+k).$$

Let $\beta > 0$, $\alpha_i > 0$, $i = 1, \ldots, n$. We then have the operational relation

$$\frac{S_\mu^{-\beta}}{I - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{-\alpha_i}} = x^{\beta\mu-1} E_{(\alpha_1\mu, \ldots, \alpha_n\mu), \beta\mu}(\lambda_1 x^{\alpha_1\mu}, \ldots, \lambda_n x^{\alpha_n\mu}) \tag{29}$$

with the multivariate Mittag-Leffler function

$$E_{(a_1, \ldots, a_n), b}(z_1, \ldots, z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\ge 0, \ldots, l_n \ge 0}} (k; l_1, \ldots, l_n) \frac{\prod\limits_{i=1}^{n} z_i^{l_i}}{\Gamma\big(b + \sum\limits_{i=1}^{n} a_i l_i\big)}$$

and the multinomial coefficients

$$(k; l_1, \ldots, l_n) := \frac{k!}{l_1! \times \cdot \times l_n!}\ .$$

For the general properties of the Mittag-Leffler functions we refer to [7].

4. Fractional differential equations

We first apply the developed operational calculus to some simple fractional differential equations, which have already been considered by using the Laplace transform technique (see [9] and references there). We begin with the initial value problem ($\mu > 0$)

$$(30)\qquad \begin{cases} (D_*^{\mu} y)(x) - \lambda y(x) = g(x), \\ y^{(k)}(0) = c_k \in \mathbf{R},\ k = 0, \ldots, m-1,\ m-1 < \mu \le m,\ \lambda \in \mathbf{R}. \end{cases}$$

The function g is assumed to lie in C_{-1} if $\mu \in \mathbf{N}$, in C_{-1}^1 if $\mu \notin \mathbf{N}$, and the unknown function $y(x)$ is to be determined in the space C_{-1}^m.

Making use of the relation (20), the initial value problem (30) can be reduced to the following algebraic equation in the convolution field $\mathcal{M}$:

$$S_\mu \cdot y - \lambda y = S_\mu \cdot y_\mu + g,\ y_\mu(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!},\ m-1 < \mu \le m,$$

whose unique solution in the field $\mathcal{M}$ has the form

$$y = y_g + y_h = \frac{I}{S_\mu - \lambda} \cdot g + \frac{S_\mu}{S_\mu - \lambda} \cdot y_\mu.$$

It turns out that the right-hand part of this relation can be interpreted as a function lying in the space C_{-1}^m, i.e., as a classical solution of the initial value problem (30). We shall prove this fact later for more general equations, here we shall only demonstrate formulas for this solution.

It follows from the operational relation (27) and the embedding of the ring C_{-1} into the field $\mathcal{M}$, that the first term of this relation, y_g (solution of the inhomogeneous fractional differential equation (30) with zero initial conditions), is represented in the form

$$(31)\qquad y_g(x) = \int_0^x t^{\mu-1} E_{\mu,\mu}(\lambda t^\mu) g(x-t)\, dt.$$

As to the second term, y_h, it is a solution of the homogeneous fractional differential equation (30) ($g(x)$ replaced by 0) with the given initial conditions, and we have

$$(32)\qquad y_h(x) = \sum_{k=0}^{m-1} c_k u_k(x),\ u_k(x) = \frac{S_\mu}{S_\mu - \lambda} \cdot \{\frac{x^k}{k!}\}.$$

Making use of the relation

$$\frac{x^k}{k!} = h_{k+1}(x) = h^{\mu}_{(k+1)/\mu}(x) = \frac{I}{S_\mu^{(k+1)/\mu}}, \tag{33}$$

the formula (25), and the operational relation (29), we get the representation of the functions $u_k(x)$, $k = 0, \ldots, m-1$, in terms of the generalized Mittag-Leffler function:

$$u_k(x) = \frac{S_\mu}{S_\mu - \lambda} \cdot \{\frac{x^k}{k!}\} = \frac{S_\mu^{-(k+1)/\mu}}{I - \lambda S_\mu^{-1}} = x^k E_{\mu,k+1}(\lambda x^\mu).$$

Furthermore, due to the representation (5) of the Riemann-Liouville fractional integral, we have

$$u_k(x) = (J^k u_0)(x),\ u_0(x) = E_{\mu,1}(\lambda x^\mu) := E_\mu(\lambda x^\mu),$$

where $E_\alpha(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ is the Mittag-Leffler function. Using the last representation we arrive at the relations

$$u_k^{(l)}(0) = \delta_{kl}, \quad k, l = 0, \ldots, m-1,$$

and therefore the m functions $u_k(x)$, $k = 0, \ldots, m-1$, represent the general solution of the homogeneous fractional differential equation (30). Summarizing the obtained results, we get the solution of the initial value problem (30) in the form

$$y(x) = \int\limits_0^x t^{\mu-1} E_{\mu,\mu}(\lambda t^\mu) g(x-t)\, dt + \sum_{k=0}^{m-1} c_k x^k E_{\mu,k+1}(\lambda x^\mu),$$

which can be rewritten in the case $\lambda \neq 0$ in terms of the Mittag-Leffler function:

$$y(x) = \frac{1}{\lambda} \int\limits_0^x \frac{d}{dt} \left(E_\mu(\lambda t^\mu)\right) g(x-t)\, dt + \sum_{k=0}^{m-1} c_k (J^k E_\mu(\lambda t^\mu))(x).$$

The next equation

$$\begin{cases} y'(x) - \lambda_1 (D_*^\mu y)(x) - \lambda_2 y(x) = g(x), \\ y(0) = c_0 \in \mathbf{R},\ 0 < \mu < 1,\ \lambda_1, \lambda_2 \in \mathbf{R} \end{cases} \tag{34}$$

with $\mu = 1/2$ and $\lambda_1 < 0$, $\lambda_2 < 0$ corresponds to the Basset problem, a classical problem in fluid dynamics (see [9], [17]). We treat the general problem (34). The function $g \in C_{-1}$ is given, and the unknown function $y(x)$ is to be determined in the space C^1_{-1}.

With the help of the relation (20) the problem under consideration can be reduced to the algebraic equation in the field $\mathcal{M}$:

$$S_1 \cdot y - \lambda_1 S_\mu \cdot y - \lambda_2 y = g + S_1 \cdot y_1 - \lambda_1 S_\mu \cdot y_\mu, \ y_1(x) \equiv y_\mu(x) \equiv c_0.$$

Applying the relation (26) we represent a unique solution of this equation in the field $\mathcal{M}$ in the form

$$y = y_g + y_h = \frac{I}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot g + \frac{S_1 - \lambda_1 S_1^\mu}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot y_1, \ y_1(x) \equiv c_0.$$

Using now the relations (25) and (29) we arrive at the representation

$$\frac{I}{S_1 - \lambda_1 S_1^\mu - \lambda_2} = \frac{S_1^{-1}}{I - \lambda_1 S_1^{-(1-\mu)} - \lambda_2 S_1^{-1}} = E_{(1-\mu,1),1}(\lambda_1 x^{1-\mu}, \lambda_2 x)$$

with the multivariate Mittag-Leffler function. We also have, using the same technique and (14), (33), the relation

$$\begin{aligned} y_h(x) &= \frac{S_1 - \lambda_1 S_1^\mu}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot \{c_0\} = \left[I + \frac{\lambda_2}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \right] \cdot \frac{c_0 I}{S_1} \\ &= c_0 \left[\frac{I}{S_1} + \lambda_2 \frac{S_1^{-2}}{I - \lambda_1 S_1^{\mu-1} - \lambda_2 S_1^{-1}} \right] \\ &= c_0 \left[1 + \lambda_2 x E_{(1-\mu,1),2}(\lambda_1 x^{1-\mu}, \lambda_2 x) \right]. \end{aligned}$$

The unique solution of the initial value problem (34) has then the form

$$\begin{aligned} y(x) = &\int_0^x E_{(1-\mu,1),1}(\lambda_1 t^{1-\mu}, \lambda_2 t) g(x-t)\, dt \\ &+ c_0 \left[1 + \lambda_2 x E_{(1-\mu,1),2}(\lambda_1 x^{1-\mu}, \lambda_2 x) \right]. \end{aligned}$$

The inclusion $y \in C^1_{-1}$ will be shown in the next theorem for a more general situation.

We consider now the general linear differential equation with constant coefficients and the Caputo derivatives.

Theorem 4.1. *Let* $\mu > \mu_1 > \cdots > \mu_n \geq 0$, $m_i - 1 < \mu_i \leq m_i$, $m_i \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\lambda_i \in \mathbf{R}$, $i = 1, \ldots, n$. *The initial value problem*

$$\begin{cases} (D_*^{\mu} y)(x) - \sum_{i=1}^{n} \lambda_i (D_*^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k \in \mathbf{R}, \ k = 0, \ldots, m-1, \ m-1 < \mu \leq m, \end{cases} \tag{35}$$

where the function g *is assumed to lie in* C_{-1} *if* $\mu \in \mathbf{N}$, *in* C_{-1}^1 *if* $\mu \notin \mathbf{N}$, *and the unknown function* $y(x)$ *is to be determined in the space* C_{-1}^m, *has a solution, unique in the space* C_{-1}^m, *of the form*

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), \ x \geq 0. \tag{36}$$

Here

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\cdot),\mu}(t) g(x-t) dt \tag{37}$$

is a solution of the problem (35) *with zero initial conditions, and the system of functions*

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i x^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(x), \ k = 0, \ldots, m-1, \tag{38}$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl}$, $k, l = 0, \ldots, m-1$. *The function*

$$E_{(\cdot),\beta}(x) = E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\beta}(\lambda_1 x^{\mu-\mu_1}, \ldots, \lambda_n x^{\mu-\mu_n}) \tag{39}$$

is a particular case of the multivariate Mittag-Leffler function (29) *and the natural numbers* l_k, $k = 0, \ldots, m-1$, *are determined from the condition*

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases} \tag{40}$$

In the case $m_i \le k$, $i = 0, \dots, m-1$, *we set* $l_k := 0$, *and if* $m_i \ge k+1$, $i = 0, \dots, m-1$, *then* $l_k := n$.

Proof. Since $y \in C_{-1}^m$, the initial value problem (35) can be reduced to the following algebraic equation in the field $\mathcal{M}$ by using the formula (20):

$$S_\mu \cdot y - S_\mu \cdot y_\mu - \sum_{i=1}^{n} \lambda_i (S_{\mu_i} \cdot y - S_{\mu_i} \cdot y_{\mu_i}) = g, \tag{41}$$

$$y_\mu(x) = \sum_{k=0}^{m-1} c_k \frac{x^k}{k!}, \; y_{\mu_i}(x) = \sum_{k=0}^{m_i-1} c_k \frac{x^k}{k!}, \; i = 1, \dots, n.$$

Equation (41) has a unique solution in the field $\mathcal{M}$, which in view of the relation (26) is given by

$$y = y_g + y_h = \frac{I}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot g + \frac{S_\mu \cdot y_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu} \cdot y_{\mu_i}}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}}. \tag{42}$$

Using the operational relation (29), Theorems 3.1 and 3.3, we can interpret the element y_g of the field $\mathcal{M}$ as the function in the ring C_{-1}:

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\cdot),\mu}(t) g(x-t) dt,$$

where the function $E_{(\cdot),\mu}(x)$ is given by (39). In the case $\mu \notin \mathbf{N}$ the function $g(x)$ is in the space C_{-1}^1 and Theorem 2.5 gives us the inclusion $y_g \in C_{-1}^1$. Let us show that the function $y_g(x)$ is in the space C_{-1}^m. Multiplying the relation

$$y_g = \frac{I}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot g$$

by $(S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu})$ and then by $h_\mu = S_\mu^{-1}$, we readily get

$$y_g(x) = (J^\mu g)(x) + \sum_{i=1}^{n} \lambda_i (J^{\mu-\mu_i} y_g)(x). \tag{43}$$

It follows from this representation, that

$$(44)\qquad y_g(x) = (J^{\mu-\mu_1}\psi_1)(x), \quad \psi_1 \in \begin{cases} C_{-1}, & \mu \in \mathbf{N}, \\ C_{-1}^1, & \mu \notin \mathbf{N}. \end{cases}$$

Combining now the relations (44) and (43) and repeating the same arguments p times ($p = [\mu/(\mu-\mu_1)] + 1$) we arrive at the representation

$$(45)\qquad y_g(x) = (J^{\mu}\psi_p)(x), \quad \psi_p \in \begin{cases} C_{-1}, & \mu \in \mathbf{N}, \\ C_{-1}^1, & \mu \notin \mathbf{N}. \end{cases}$$

In the case $\mu = m \in \mathbf{N}$ it follows from (45) that $y_g^{(m)} = \psi_p \in C_{-1}$ and $y_g(0) = \cdots = y_g^{(m-1)}(0) = 0$. If $\mu \notin \mathbf{N}$, $m-1 < \mu < m$ then the function $h_\mu(x) = \dfrac{x^\mu}{\Gamma(\mu)} \in C_{-1}^{m-1}$ and we have $h_\mu(0) = \cdots = h_\mu^{(m-2)}(0) = 0$. Using now the representation (45) and Theorem 2.5 we get the inclusion $y_g \in C_{-1}^m$ and the relations $y_g(0) = \cdots = y_g^{(m-1)}(0) = 0$. The last considerations prove the part of Theorem 4.1 concerning the solution of the problem (35) with zero initial conditions.

Let us consider the element y_h of the representation (42). Using the expressions for the functions $y_\mu(x)$, $y_{\mu_i}(x)$, $i = 1,\ldots,n$, from (41), we have

$$y_h = \sum_{k=0}^{m-1} c_k u_k(x), \quad u_k = \frac{S_\mu - \sum\limits_{i=1}^{l_k} \lambda_i S_\mu^{\mu_i/\mu}}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot \left\{ \frac{x^k}{k!} \right\},$$

where the natural numbers l_k, $k = 0,\ldots,m-1$, are defined by (40). Applying the relation (33) for the function $\dfrac{x^k}{k!}$, some elementary transformations and then the operational relation (29), we find from the previous representation ($k = 0,\ldots,m-1$):

$$u_k = \frac{I}{S_\mu^{(k+1)/\mu}} \cdot \frac{S_\mu - \sum\limits_{i=1}^{l_k} \lambda_i S_\mu^{\mu_i/\mu}}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} =$$

$$
= \frac{I}{S_\mu^{(k+1)/\mu}} \cdot \left[I + \frac{\sum_{i=l_k+1}^{n} \lambda_i S_\mu^{\mu_i/\mu}}{S_\mu - \sum_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \right]
$$

$$
= S_\mu^{-(k+1)/\mu} + \sum_{i=l_k+1}^{n} \lambda_i \frac{S_\mu^{-(k+1+\mu-\mu_i)/\mu}}{I - \sum_{i=1}^{n} \lambda_i S_\mu^{-(\mu-\mu_i)/\mu}}
$$

$$
= \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i x^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(x),
$$

where the function $E_{(\cdot),\beta}(x)$ is given by (39). According to the definition of the numbers l_k, we have $m_i \le k$ for $i = l_k + 1, \ldots, n$. It follows then, that $k + \mu - \mu_i \ge \mu$, $i = l_k + 1, \ldots, n$. Using this inequality, we readily get the inclusion $u_k \in C_{-1}^m$, $k = 0, \ldots, m-1$ and the relations

$$
u_k^{(l)}(0) = \delta_{kl}, \quad k, l = 0, \ldots, m-1,
$$

and therefore the m functions $u_k(x)$, $k = 0, \ldots, m-1$, represent the general solution of the homogeneous fractional differential equation (35).

Remark 4.1. The results of Theorem 4.1 can be used in some cases for the initial value problem (35) with the Riemann-Liouville fractional derivatives instead of the Caputo fractional derivatives. In particular, as we have seen in Corollary 2.1, $(D^\mu y)(x) \equiv (D_*^\mu y)(x)$ if $\mu = m \in \mathbf{N}$ or $y^{(k)}(0) = 0$, $k = 0, \ldots, m-1$, $m - 1 < \mu \le m$. In the case $0 < \mu < 1$ we can use the relation

$$
(D^\mu y)(x) = (D_*^\mu y)(x) + \frac{y(0)}{\Gamma(1-\mu)} x^{-\mu}
$$

to reduce the initial value problem with the Riemann-Liouville fractional derivatives to the initial value problem of the type (35).

Remark 4.2. The developed theory can be rewritten without any difficulties for the case of the Caputo derivative of the complex order μ. In particular, Theorem 4.1 gives us then the solution of the initial value problem (35) for $\mu \in \mathbf{C}$ if $\Re(\mu) = m \in \mathbf{N}$, then $\mu = m$), $\Re(\mu) > \Re(\mu_1) > \cdots > \Re(\mu_n) \ge 0$ if $\Re(\mu_i) = m_i \in \mathbf{N}_0$, then $\mu_i = m_i$), $m_i - 1 < \Re(\mu_i) \le m_i$, $m_i \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mu_i, \lambda_i \in \mathbf{C}$, $i = 1, \ldots, n$, $y^{(k)}(0) = c_k \in \mathbf{C}$, $k = 0, \ldots, m-1$, $m - 1 < \Re(\mu) \le m$.

The solution (36)-(38) of the initial value problem (35) was obtained in terms of the Mittag-Leffler type function $E_{(\cdot),\beta}(x)$, which is given by its series representation

$$E_{(\cdot),\beta}(x) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} (k; l_1, \ldots, l_n) \frac{\prod_{i=1}^{n} (\lambda_i x^{\mu-\mu_i})^{l_i}}{\Gamma(\beta + \sum_{i=1}^{n} (\mu - \mu_i) l_i)} . \tag{46}$$

Let us find an integral representation of this function. We shall use the Hankel integral representation of the reciprocal Gamma-function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{Ha(0+)} e^{\zeta} \zeta^{-z} \, d\zeta, \ \ z \in \mathbf{C},$$

where $Ha(\varepsilon+)$ is the Hankel path, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|\zeta| = \zeta_0 > \varepsilon > 0$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis. Then we substitute this representation into (46) and get

$$E_{(\cdot),\beta}(x) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} (k; l_1, \ldots, l_n) \prod_{i=1}^{n} (\lambda_i x^{\mu-\mu_i})^{l_i} \times \frac{1}{2\pi i} \int_{Ha(0+)} e^{\zeta} \zeta^{-\beta - \sum_{i=1}^{n} (\mu-\mu_i) l_i} \, d\zeta.$$

Changing the order of integration and summation, using the multinomial formula with exponent -1 to get a closed form for a sum in integrand and substituting $\zeta = sx$, we finally get

$$E_{(\cdot),\beta}(x) = x^{1-\beta} \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{sx} s^{\mu-\beta} \, ds}{s^{\mu} - \sum_{i=1}^{n} \lambda_i s^{\mu_i}}, \tag{47}$$

$$\lambda = \max \left\{ 1, \left(\sum_{i=1}^{n} |\lambda_i| \right)^{1/(\mu-\mu_1)} \right\}.$$

Corollary 4.1. *Applying the representation* (47) *to the formulas* (37) *and* (38), *we rewrite the solution* (36) *of the initial value problem* (35) *in the form*

$$y(x) = \int_0^x u_\delta(t)g(x-t) + \sum_{k=0}^{m-1} c_k u_k(x), \quad x \geq 0, \tag{48}$$

where

$$u_\delta(x) = \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{sx}\,ds}{s^\mu - \sum\limits_{i=1}^{n} \lambda_i s^{\mu_i}}, \tag{49}$$

$$\begin{aligned} u_k(x) &= \frac{x^k}{k!} + \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{sx} \sum\limits_{i=l_k+1}^{n} \lambda_i s^{\mu_i}}{s^\mu - \sum\limits_{i=1}^{n} \lambda_i s^{\mu_i}} \frac{ds}{s^{k+1}} \\ &= \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{sx}\left[s^\mu - \sum\limits_{i=1}^{l_k} \lambda_i s^{\mu_i}\right]}{s^\mu - \sum\limits_{i=1}^{n} \lambda_i s^{\mu_i}} \frac{ds}{s^{k+1}}, \; k = 0, \ldots, m-1. \end{aligned} \tag{50}$$

In particular, if $\mu_n = 0, \; \lambda_n \neq 0,$ *then* $l_0 = n-1$ *and we have*

$$u_0(x) = \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{sx}\left[s^\mu - \sum\limits_{i=1}^{n-1} \lambda_i s^{\mu_i}\right]}{s^\mu - \sum\limits_{i=1}^{n-1} \lambda_i s^{\mu_i} - \lambda_n} \frac{ds}{s} \; .$$

In this situation we have the relation

$$u_\delta(x) = \frac{1}{\lambda_n} u_0'(x).$$

If $\mu_n > 0$, *we get* $u_0(x) \equiv 1$.

Remark 4.3. The initial value problem (35) for the three cases: 1) $n = 1$, $\mu_1 = 0$, $\lambda_1 = -1$, 2) $n = 2$, $\mu = 1$, $\lambda_2 = -1$, $\mu_2 = 0$, and 3) $n = 2$,

$\mu = 2$, $\lambda_2 = -1$, $\mu_2 = 0$ was considered in [9] by using the Laplace transform method. In this research and survey paper the form (48) - (50) of the solution was obtained and used, by evaluating the contribution of poles of the integrand by the residue theorem and transforming the Hankel path $Ha(\lambda+)$ into the $Ha(0+)$, to represent it as a sum of oscillatory and monotone parts. In addition, asymptotic expansions, plots and interesting particular cases are given there. General results concerning the methods of evaluation of the poles of the integrand in the integral representations of the type (49), (50), asymptotic expansions of such representations as well as a lot of interesting particular cases can be found in the paper [12].

Finally, we consider some examples.

Example 1. Let the right-hand part of the fractional differential equation (35) be a power function:

$$g(x) = \frac{x^\alpha}{\Gamma(\alpha+1)},\ \alpha > -1,\ \text{if } \mu \in \mathbf{N},\ \alpha \geq 0,\ \text{if } \mu \notin \mathbf{N}.$$

Since

$$\frac{x^\alpha}{\Gamma(\alpha+1)} = h_{\alpha+1}(x) = h^{\mu}_{(\alpha+1)/\mu}(x) = S_\mu^{-(\alpha+1)/\mu},$$

we get by using (42), (25), (29), and (47) the following representations of the part $y_g(x)$ of the solution (36):

$$\begin{aligned} y_g &= \frac{I}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot g = \frac{I}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot \frac{I}{S_\mu^{(\alpha+1)/\mu}} \\ &= \frac{S_\mu^{-(\mu+\alpha+1)/\mu}}{I - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{-(\mu-\mu_i)/\mu}} = x^{\mu+\alpha} E_{(\cdot),\mu+\alpha+1}(x) \\ &= \frac{1}{2\pi i} \int\limits_{Ha(\lambda+)} \frac{e^{sx} s^{-\alpha-1}\, ds}{s^\mu - \sum\limits_{i=1}^{n} \lambda_i s^{\mu_i}} \,. \end{aligned}$$

Example 2. We consider now the equation (35) with $\mu_i = (n-i)\alpha$, $i = 1, \ldots, n$, $\mu = n\alpha$, $q - 1 < \mu \leq q$, $q \in \mathbf{N}$. Then the solution (36) can be represented in terms of the generalized Mittag-Leffler function $E^m_{\alpha,\beta}(x)$, see (28). Indeed, using the relation (26) and representing the corresponding rational function as a sum of partial fractions, we get

$$y_g = \frac{I}{S_\mu - \sum\limits_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot g = \frac{I}{S_\alpha^n - \sum\limits_{i=1}^{n} \lambda_i S_\alpha^{n-i}} \cdot g$$

$$= \left[\sum_{j=1}^{p} \sum_{m=1}^{n_j} \frac{c_{jm}}{(S_\alpha - \beta_j)^m} \right] \cdot g, \; n_1 + \cdots + n_p = n.$$

The operational relation (28) gives us the representation

$$y_g(x) = \int\limits_0^x u_\delta(t) g(x-t)\, dt$$

with

$$u_\delta(x) = \sum_{j=1}^{p} \sum_{m=1}^{n_j} c_{jm} x^{\alpha m - 1} E_{\alpha, m\alpha}^{m}(\beta_j x^\alpha).$$

We have also $(k = 0, \ldots, q-1)$

$$y_h(x) = \sum_{k=0}^{q-1} c_k u_k(x),$$

$$u_k = \frac{x^k}{k!} + \left\{ \frac{x^k}{k!} \right\} \cdot \frac{\sum\limits_{i=l_k+1}^{n} \lambda_i S_\alpha^{n-i}}{S_\alpha^n - \sum\limits_{i=1}^{n} \lambda_i S_\alpha^{n-i}} = \frac{x^k}{k!} + (J^{k+1} v_k)(x),$$

$$v_k(x) = \frac{\sum\limits_{i=l_k+1}^{n} \lambda_i S_\alpha^{n-i}}{S_\alpha^n - \sum\limits_{i=1}^{n} \lambda_i S_\alpha^{n-i}} = \sum_{j=1}^{p_k} \sum_{m=1}^{n_{jk}} \frac{c_{jmk}}{(S_\alpha - \beta_{jk})^m}$$

$$= \sum_{j=1}^{p_k} \sum_{m=1}^{n_{jk}} c_{jmk} x^{\alpha m - 1} E_{\alpha, m\alpha}^{m}(\beta_{jk} x^\alpha), \quad \sum_{j=1}^{p_k} n_{jk} = n.$$

In the case $\alpha \in \mathbf{Q}$ the generalized Mittag-Leffler function $E_{\alpha, m\alpha}^{m}(x)$ can be represented in terms of special functions of the hypergeometric type (see [9]).

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