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UNIQUENESS OF GLOBAL SEMICLASSICAL SOLUTIONS FOR SOME SYSTEMS OF FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We consider the Cauchy problem for weakly coupled systems of first-order nonlinear partial differential equations and prove some uniqueness results of global semiclassical solutions. Our method is based on the theory of multivalued functions and of differential inclusions.

1. Introduction

The global existence and uniqueness of generalized solutions for convex Hamilton-Jacobi equations were well studied by several methods by the variational method [CH], by the method of envelopes [H], [AK], by the vanishing viscosity method [F], [K], etc. The global theory for nonconvex Hamilton-Jacobi equations has been recently considered by M. G. Crandall, L. C. Evans, P. L. Lions, H. Ishii ([CEL], [CL], [I]). They have introduced the notion of "viscosity solutions" to define generalized solutions and characterized their properties. By these contributions, the global existence and uniqueness of generalized solutions have been established almost completely.

Since the equations are of first order, the generalized solutions being just continuous (as regular as possible) should contain singularities. So what kinds of phenomena may appear when we extend the classical (local) solutions? Furthermore, an estimate for the solutions is sometimes needed. In these procedures, one must go back to the Haar lemma (see [T], [VS1, VS2]). In [VS1, VS2], a Carathéodory global extension of the classical Haar lemma and its applications to the stability questions concerning (global) solutions of the Cauchy problem for nonlinear partial differential equations were established. In particular, an answer to an open problem of Kruzkhov was therein given by the study of the widest class between

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the class of continuously differentiable functions and the class of Lipschitz continuous functions in which the Cauchy problem for a single first-order partial differential equation has a unique global solution. The existence of such solutions (called *global semiclassical*) was investigated in [VHG].

The aim of the present paper is to give a generalization of [VS1, VS2] to the case of weakly coupled systems of first-order partial differential equations. In Section 2, we give the definition of global semiclassical solutions of the Cauchy problem for such systems and formulate the uniqueness theorems. Section 3 is devoted to the proof of our uniqueness results. Finally, we give an example that distinguishes our uniqueness theorems.

2. Uniqueness of global semiclassical solutions

Let T be a positive number, $\Omega_{\mathrm{T}} := (0,\mathrm{T})\times\mathbb{R}^n$, $\nabla_x := \Big(\dfrac{\partial}{\partial x_1},\dots,\dfrac{\partial}{\partial x_n}\Big)$, $n\geq 1$; $||\cdot||_n$ and $\langle\cdot,\cdot\rangle$ be the norm and scalar product in $\mathbb{R}^n$, respectively.

We consider the Cauchy problem for the following system of first-order nonlinear partial differential equations:

$$\frac{\partial u_j}{\partial t}+\mathrm{H}_j(t,x,u,\nabla_x u_j)=0,\quad (t,x)\in\Omega_{\mathrm{T}},\ j=1,\dots,m, \tag{1}$$

$$u_j(0,x)=u_j^0(x),\quad j=1,\dots,m, \tag{2}$$

where H_j is a function of $(t,x,p,q_j)\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$ for each $j=1,\dots,m$. The vectors $\mathbf{p}=(p^1,\dots,p^m)$ and $\mathbf{q}_j=(q_j^1,\dots,q_j^n)$ are corresponding to $\mathbf{u}=(u_1,\dots,u_m)$ and $\nabla_x u_j=\Big(\dfrac{\partial u_j}{\partial x_1},\dots,\dfrac{\partial u_j}{\partial x_n}\Big)$, respectively. A system of the form (1) is called a weakly coupled system.

Denote by $\mathrm{Lip}(\Omega_{\mathrm{T}})$ the set of all locally Lipschitz continuous functions u defined on Ω_{T}, i.e. all functions u with the property that for any compact $\mathrm{K}\subset\Omega_{\mathrm{T}}$ there exists a number $\mathrm{L}=\mathrm{L}(\mathrm{K})\geq 0$ such that

$$|u(t_1,x_1)-u(t_2,x_2)|\leq \mathrm{L}\big(|t_1-t_2|+\|x_1-x_2\|_n\big),\quad \forall(t_1,x_1),\ (t_2,x_2)\in\mathrm{K}.$$

Furthermore, we use the notation

$$\mathrm{Lip}([0,\mathrm{T})\times\mathbb{R}^n):=\mathrm{Lip}(\Omega_{\mathrm{T}})\cap \mathrm{C}([0,\mathrm{T})\times\mathbb{R}^n).$$

As in [VS1], [VS2] let $\mathrm{V}(\Omega_{\mathrm{T}})$ be the following subclass of $\mathrm{Lip}([0,\mathrm{T})\times\mathbb{R}^n)$:

$$\mathrm{V}(\Omega_{\mathrm{T}}):=\Big\{u: u\in\mathrm{Lip}\big([0,\mathrm{T})\times\mathbb{R}^n\big); u \text{ is differentiable for all } x\in\mathbb{R}^n \text{ and for almost all } t\in(0,\mathrm{T})\Big\}.$$

It is obvious that

$$\mathrm{C}([0,\mathrm{T})\times\mathbb{R}^n)\cap\mathrm{C}^1\big(\Omega_\mathrm{T}\big)\subset\mathrm{V}\big(\Omega_\mathrm{T}\big)\subset\mathrm{Lip}\big([0,\mathrm{T})\times\mathbb{R}^n\big).$$

Finally, set

$$\mathrm{V}_m\big(\Omega_\mathrm{T}\big):=\underbrace{\mathrm{V}\big(\Omega_\mathrm{T}\big)\times\cdots\times\mathrm{V}\big(\Omega_\mathrm{T}\big)}_{m\text{ times}}.$$

Definition. A vector function $u\in\mathrm{V}_m\big(\Omega_\mathrm{T}\big)$ is called a global semiclassical solution of (1), (2) if u satisfies the condition (2) for all $x\in\mathbb{R}^n$ and u satisfies the system (1) for all $x\in\mathbb{R}^n$ and for almost all $t\in(0,\mathrm{T})$.

We now formulate some uniqueness results for global semiclassical solutions of the problem (1), (2).

Theorem 1. *Suppose that $H_j(t,x,p,q_j)$, $j=1,\dots,m$, satisfy the following condition: There exist nonnegative functions $k_j(.)\in L^1(0,T)$ and nonnegative functions $h_j(.)$ locally bounded in $\mathbb{R}^n$ such that*

$$\begin{aligned}(3)\qquad &\big|H_j(t,x,p,q_j)-H_j(t,x,p',q_j')\big|\\ &\le k_j(t)\big[(1+\|x\|_n)\|q_j-q_j'\|_n+h_j(x)\|p-p'\|_m\big],\end{aligned}$$

for all $x\in\mathbb{R}^n$ and for almost all $t\in(0,T)$; $p,p'\in\mathbb{R}^m$; $q_j,q_j'\in\mathbb{R}^n$. If $u=(u_1,\dots,u_m)$ and $v=(v_1,\dots,v_m)$ are global semiclassical solutions of the problem (1), (2), then $u\equiv v$ in Ω_T (i.e., $u_j\equiv v_j$ in Ω_T for each $j=1,\dots,m$).

Remark 1. The conditions (3) are satisfied if there exist the functions $k_j:(0,\mathrm{T})\longrightarrow(0,+\infty]$, Lebesgue integrable on $[0,\mathrm{T}]$, such that the functions $Q_j:(0,\mathrm{T})\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}$, given by

$$\mathrm{Q}_j(t,x,p,q_j):=\frac{\mathrm{H}_j(t,x,p,q_j)}{k_j(t)(1+\|x\|_n)},$$

are globally Lipschitz continuous in the variable $q_j\in\mathbb{R}^n$ uniformly with respect to $(t,x,p)\in(0,\mathrm{T})\times\mathbb{R}^n\times\mathbb{R}^m$ and globally Lipschitz continuous in the variable $p\in\mathbb{R}^n$ uniformly with respect to $(t,x,q_j)\in(0,T)\times\mathrm{X}\times\mathbb{R}^n$ for all compact $\mathrm{X}\subset\mathbb{R}^n$.

Theorem 2. *Suppose that $H_j(t,x,p,q_j)$, $j=1,\dots,m$, satisfy the following condition: There exist functions $k_j(.)=k_{jK}(.)$ and $h_j(.)=h_{jK}(.)$ as* *in Theorem 1 for every compact $K \subset \mathbb{R}^n$ such that* (3) *holds for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,T)$; $p, p' \in \mathbb{R}^m$, $q_j, q_j' \in K$. If $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ are global semiclassical solutions of the problem* (1), (2) *with*

$$\operatorname*{ess.\,sup}_{(t,x)\in\Omega_T} \max\big\{\|\nabla_x u_j\|_n, \|\nabla_x v_j\|_n\big\} < \infty, \quad j = 1, \ldots, m,$$

then $u \equiv v$ in Ω_T.

The proof of Theorems 1, 2 will be based on the following theorem.

Theorem 3. *Let $u \in V_m(\Omega_T)$. If there exist nonnegative functions $k_j(.) \in L^1(0,T)$ and nonnegative functions $h_j(.)$ locally bounded in $\mathbb{R}^n$ such that*

$$(4) \qquad \Big|\frac{\partial u_j}{\partial t}(t,x)\Big| \le k_j(t)\Big[(1 + \|x\|_n)\|\nabla_x u_j(t,x)\|_n + h_j(x)\|u(t,x)\|_m\Big],$$

$j = 1, \ldots, m$, for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,T)$, then

$$(5) \qquad \max_{j=\overline{1,m}} \big|u_j(t,x)\big| \le \exp\big\{M(x)f(t)\big\} \cdot \sup_{\|y\|_n \le N(t,x)} \max_{j=\overline{1,m}} |u_j(0,y)|,$$

where

$$N(t,x) := (1 + \|x\|_n)\exp(f(t)) - 1,$$

$$M(x) := \sup\big\{|h(y)| : \|y\|_n \le N(T,x)\big\},$$

$$(6) \qquad k(t) := \max_{j=\overline{1,m}} k_j(t), \quad h(x) := \max_{j=\overline{1,m}} h_j(x), \quad f(t) := \int_0^t mk(\tau)d\tau.$$

Corollary 1. *Let $u \in V_m(\Omega_T)$ and $u(0,x) \equiv 0$, $x \in \mathbb{R}^n$. If conditions* (4) *are satisfied for almost all $t \in (0,T)$ and for all $x \in \mathbb{R}^n$, then $u(t,x) \equiv 0$ in Ω_T.*

From Theorem 3 we also get the following result which describes a criterion of continuous dependence on initial values for semiclassical solutions of the problem (1), (2).

Theorem 4. *Suppose that $H_j(t,x,p,q_j)$, $j = 1, \ldots, m$, satisfy the conditions* (3) *in Theorem 1. If $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ are* *global semiclassical solutions of the Cauchy problem for the equation* (1) *with the initial conditions*

$$u_j(0,x) = \varphi_j(x), \quad v_j(0,x) = \psi_j(x),$$

$$x \in \mathbb{R}^n, \quad \varphi_j(.), \psi_j(.) \in C(\mathbb{R}^n), \ j = 1, \dots, m,$$

then we have the estimate

$$\begin{aligned} &\max_{j=\overline{1,m}} \left|u_j(t,x) - v_j(t,x)\right| \\ &\le \exp\left\{M(x) \cdot f(t)\right\} \cdot \sup_{\|y\|_n \le N(t,x)} \max_{j=\overline{1,m}} |\varphi_j(y) - \psi_j(y)|, \end{aligned}$$

where $M(x)$*,* $f(t)$*,* $N(t,x)$ *are defined as in Theorem 3.*

3. Proof of theorems 1, 2, 3 and 4

First, we prove Theorem 3. For an arbitrary $(t_0, x_0) \in \Omega_{\mathrm{T}}$, we have to show that (5) holds at $(t,x) = (t_0, x_0)$.

From (4) and (6) we have

$$\left|\frac{\partial u_j}{\partial t}(t,x)\right| \le k(t)\Big[(1 + \|x\|_n)\|\nabla_x u_j(t,x)\|_n + h(x)\|u(t,x)\|_m\Big], \tag{4'}$$

$j = 1, \dots, m$, for all $x \in \mathbb{R}^n$ and for almost all $t \in (0, \mathrm{T})$. Let

$$\overline{\mathrm{B}}_r = \overline{\mathrm{B}}_r^n := \{y \in \mathbb{R}^n : \|y\|_n \le r\}, \quad r > 0.$$

We consider the multivalued function $\mathrm{F} : \Omega_{\mathrm{T}} \rightsquigarrow \mathbb{R}^n$ given by

$$\mathrm{F}(t,x) := \overline{\mathrm{B}}_{mk(t)(1+\|x\|_n)}, \quad (t,x) \in \Omega_{\mathrm{T}},$$

and the differential inclusion

$$\frac{dx(t)}{dt} \in \mathrm{F}(t, x(t)). \tag{7}$$

For every $(t_0, x_0) \in \Omega_{\mathrm{T}}$, we will denote by $\Sigma_{\mathrm{I}}(t_0, x_0)$ the set of all absolutely continuous functions $x(.)\colon \mathrm{I} := [0, t_0] \longrightarrow \mathbb{R}^n$ which satisfy almost everywhere in I the differential inclusion (7) subject to the constraint $x(t_0) = x_0$. From Theorem VI -13 of [CV] it follows that $\Sigma_{\mathrm{I}}(t_0, x_0)$ is a nonempty compact set in $\mathrm{C}(\mathrm{I}, \mathbb{R}^n)$. Therefore, we obtain the following

Lemma 1. *For every $t \in I = [0, t_0]$, the set*

$$Z(t, t_0, x_0) := \big\{x(t) : x(.) \in \Sigma_I(t_0, x_0)\big\}$$

is a nonempty compact set in $\mathbb{R}^n$.

To prove Theorem 3 we also need the following lemma whose proof is similar to that of [VS1, Lemma 2].

Lemma 2 ([VS1], Lemma 2). *The inclusion*

$$Z(t, t_0, x_0) \subset B_{(1+\|x_0\|_n)\exp\{\int_t^{t_0} mk(\tau)d\tau\}-1}$$

is satisfied for each $t \in I$.

We now define a function $\varphi : \mathrm{I} \longrightarrow \mathbb{R}$ as

$$\varphi(t) := \max\big\{\mathcal{L}(t, x) : x \in \mathrm{Z}(t, t_0, x_0)\big\},$$

where $t \in \mathrm{I}$ and $\mathcal{L}(t, x) := \max\limits_{j=\overline{1,m}} |u_j(t, x)|$. Of course, $\varphi(\cdot) \in \mathrm{C(I)}$ (see [VS1]).

Lemma 3. *For an arbitrary number $\theta \in (0, t_0)$, $\varphi(.)$ is absolutely continuous on $[\theta, t_0]$.*

Proof. By Lemma 1, it follows that the set

$$\Gamma(t_0, x_0) := \big\{(\tau, y) : \tau \in \mathrm{I}, y \in \mathrm{Z}(\tau, t_0, x_0)\big\}$$

is a compact set in $\mathbb{R}^{n+1}$. Since $u = (u_1, \dots, u_m) \in \mathrm{V}_m(\Omega_\mathrm{T})$, there exists $\mathrm{L} \geq 0$ such that

$$\big|u_j(t_1, x_1) - u_j(t_2, x_2)\big| \leq \mathrm{L}\big(|t_1 - t_2| + \|x_1 - x_2\|_n\big),$$

$\forall\ (t_1, x_1), (t_2, x_2) \in \mathrm{J} := \big([\theta, t_0] \times \mathbb{R}^n\big) \bigcap \Gamma(t_0, x_0), j = 1, \dots, m.$

Take $t_1, t_2 \in [\theta, t_0]$ and assume that $\varphi(t_1) \geq \varphi(t_2)$. By the definition of the function φ and Lemma 1, we may assume without loss of generality that $\varphi(t_1) = |u_1(x_1, x(t_1))|$ for some $x(.) \in \Sigma_I(t_0, x_0)$. As

$x(t_2) \in \mathrm{Z}(t_2, t_0, x_0)$ we have

$$\begin{aligned} 0 &\le \varphi(t_1) - \varphi(t_2) = \big|u_1(t_1, x(t_1))\big| - \varphi(t_2) \\ &\le \big|u_1(t_1, x(t_1))\big| - \big|u_1(t_2, x(t_2))\big| \le \big|u_1(t_1, x(t_1)) - u_1(t_2, x(t_2))\big| \\ &\le \mathrm{L}\big[|t_1 - t_2| + \|x(t_1) - x(t_2)\|_n\big] = \mathrm{L}\big[|t_1 - t_2| + \big\| \int\limits_{[t_1,t_2]} \frac{dx}{dt}(t)dt \big\|_n\big] \\ &\le \mathrm{L}\Big[|t_1 - t_2| + \int\limits_{[t_1,t_2]} \|\frac{dx}{dt}(t)\|_n dt\Big] \\ &\le \mathrm{L}\Big[|t_1 - t_2| + \int\limits_{[t_1,t_2]} \big\{mk(t)(1 + \|x(t)\|_n)\big\}dt\Big]. \end{aligned}$$

By Lemma 2, this inequality becomes

$$|\varphi(t_1) - \varphi(t_2)| \le$$

$$\mathrm{L}\Big[|t_1 - t_2| + m(1 + \|x_0\|_n)\exp\Big\{\int\limits_{\theta}^{t_0} mk(t)dt\Big\} \int\limits_{[t_1,t_2]} k(t)dt\Big],\ \forall\, t_1, t_2 \in [\theta, t_0].$$

Therefore, the absolute continuity of the Lebesgue integral implies that of φ on I. $\square$

Going back to the proof of Theorem 3 we see that the inequality (5) holds at $(t, x) = (t_0, x_0)$ if we show that

$$\varphi(t) \le \varphi(0)\exp\{M(x_0).f(t)\}, \quad \forall t \in [0, t_0]. \tag{8}$$

For arbitrary $\mu > 0$, let

$$\psi(t) = \psi_\mu(t) := (\varphi(0) + \mu)\exp\{(M(x_0) + \mu)(f(t) + \mu t)\},\ t \in [0, t_0].$$

To get (8) we only have to prove that

$$\varphi(t) < \psi(t), \quad \forall t \in [0, t_0],$$

or equivalently,

$$\omega(t) := \psi(t) - \varphi(t) > 0, \quad \forall t \in [0, t_0].$$

It is clear that $\omega(0) = \mu > 0$. It is sufficient to show that $\omega(t) \geq \omega(0)$ $\forall\, t \in [0, t_0]$.

Assume this is false. Then there exists $t' \in (0, t_0]$ such that $\omega(t') < \omega(0)$. Since $k(.) \in L^1(0, \mathrm{T}\ )$, f is an absolutely continuous function on $[0, \mathrm{T}]$. This together with the fact that $u \in \mathrm{V}_m(\Omega_{\mathrm{T}})$ imply that there exists a set $\mathrm{G} \subset (0, \mathrm{T})$, $mes(\mathrm{G}) = 0$, such that $f(.)$ is differentiable at points of $(0, \mathrm{T}) \setminus \mathrm{G}$ with $f'(t) = mk(t)$, u is differentiable and satisfies (4') at points of $\Omega_{\mathrm{T}} \setminus (\mathrm{G} \times \mathbb{R}^n)$. By the absolute continuity of ω on $[\theta, t_0]$, which is due to Lemma 3, we have $mes\{\omega(\mathrm{G} \cap [\theta, t_0])\} = 0$ for all $\theta \in (0, t_0)$. Hence $mes\{\omega(\mathrm{G} \cap [0, t_0]\} = 0$. Therefore, since the interval

$$\mathrm{K} := \big(\max\{0, \omega(t')\}, \omega(0)\big)$$

is nonempty, it follows that there exists a number $\lambda \in \mathrm{K} \setminus \omega(\mathrm{G} \cap [0, t_0])$. Because $\omega(.) \in \mathrm{C}(\mathrm{I})$, we have

$$\lambda \in \omega([0, t']) \setminus \omega(\mathrm{G} \cap [0, t_0]).$$

Let $t_* = \inf\{t \in [0, t'] : \omega(t) = \lambda\}$. It is clear that $t_* \in (0, t') \setminus G$, $\omega(t_*) = \lambda$ and $\omega(t) > \lambda, \forall t \in [0, t_*)$. Take $x_*(.) \in \Sigma_{\mathrm{I}}(t_0, x_0)$ such that $\varphi(t_*) = \mathcal{L}(t_*, x_*(t_*))$. We can suppose that

$$\varphi(t_*) = |u_1(t_*, x^*)| = s u_1(t_*, x^*),$$

where $s :=$ sign$u_1(t_*, x^*)$ and $x^* := x_*(t_*)$. Choose $l \in \mathbb{R}^n$ with $\|l\|_n = 1$ and

$$\big\langle s\nabla_x u_1(t_*, x^*), l\big\rangle = -\|\nabla_x u_1(t_*, x^*)\|_n.$$

Let $y(p)$ be a continuously differentiable solution on $\mathbb{R}$ for the system of ordinary equations: $\dfrac{dy(p)}{dp} = (1 + \|y(p)\|_n)l$, which satisfies the condition $y(f(t_*)) = x^*$. Since f is absolutely continuous on $[0, \mathrm{T}]$, so is the function $x(t) = y(f(t))$. Furthermore, $x(t_*) = x^*$ and

$$\frac{dx}{dt} = mk(t)(1 + \|x(t)\|_n) \cdot l.$$

Consider the function $\overline{x}$ given by:

$$\overline{x}(t) := \begin{cases} x(t) & \text{if } 0 \leq t \leq t_*, \\ x_*(t) & \text{if } t_* \leq t \leq t_0. \end{cases}$$

It is easy to see that $\overline{x} \in \Sigma_{\mathrm{I}}(t_0, x_0)$. Hence, by the definition of φ, for every $t \in [0, t_*)$, we have

$$su_1(t, x(t)) \leq |u_1(t, x(t))| \leq \varphi(t) = \psi(t) - \omega(t) < \psi(t) - \lambda.$$

This means that $n(t) := \psi(t) - su_1(t, x(t)) - \lambda > 0,\ \forall\, t \in [0, t_*)$. Moreover,

$$n(t_*) = \psi(t_*) - su_1(t_*, x(t_*)) - \lambda = \omega(t_*) - \lambda = 0.$$

Thus t_* is a point where the function $n(.)$ attains its minimum on $[0, t_*]$. Since $t_* \in (0, \mathrm{T}) \setminus \mathrm{G}$, we see that u_1 is differentiable at (t_*, x^*), $x(.)$ is differentiable at t_* and so is ψ. Therefore $\dfrac{dn(t)}{dt}\Big|_{t=t_*} \leq 0$, that is to say

$$(\mathrm{M}(x_0) + \mu)(mk(t_*) + \mu)\psi(t_*) \leq s\frac{\partial u_1}{\partial t}(t_*, x^*) + \big\langle s.\nabla_x u_1(t_*, x^*), \frac{dx}{dt}(t_*)\big\rangle.$$

Consequently,

$$\begin{aligned}\Big|\frac{\partial u_1}{\partial t}(t_*, x^*)\Big| \geq mk(t_*)(1 + \|x^*\|_n)\|\nabla_x u_1(t_*, x^*)\|_n \\ + (\mathrm{M}(x_0) + \mu)\big(mk(t_*) + \mu\big)\psi(t_*).\end{aligned}$$

Because $\mu > 0$ and $k(t_*) \geq 0$, the last inequality implies that

$$\begin{aligned}\Big|\frac{\partial u_1}{\partial t}(t_*, x^*)\Big| > k(t_*)(1 + \|x^*\|_n)\|\nabla_x u_1(t_*, x^*)\|_n \\ + \sup\{h(y) : \|y\|_n \leq \mathrm{N}(T, x_0)\}.mk(t_*)(|u_1(t_*, x^*)| + \lambda).\end{aligned}$$

Since $|u_1(t_*, x^*)| = \max\limits_{j=\overline{1,m}} |u_j(t_*, x^*)|$,

$$m|u_1(t_*, x^*)| \geq \|u(t_*, x^*)\|_m.$$

On the other hand, as $x^* = x_*(t_*) \in Z(t_*, t_0, x_0)$, Lemma 2 gives

$$\|x^*\|_n \leq (1 + \|x_0\|_n)\exp\Big\{\int_t^{t_0} mk(\tau)d\tau\Big\} - 1 \leq \mathrm{N}(T, x_0).$$

Because $\lambda > 0$, it follows that

(9)
$$\Big|\frac{\partial u_1}{\partial t}(t_*, x^*)\Big| > k(t_*)\big[(1 + \|x^*\|_n)\|\nabla_x u(t_*, x^*)\|_n + h(x^*)\|u(t_*, x^*)\|_m\big].$$

The inequality (9) contradicts (4') at $t = t_*$. This shows that there could not exist any $t' \in [0, t_0]$ with $\omega(t') < \omega(0)$, i.e., $\omega(t) \geq \omega(0)$ and consequently, $\varphi(t) < \psi(t)$ for all $t \in [0, t_0]$. Therefore (8) holds, and the proof of Theorem 3 is complete. $\square$

Corollary 1 is immediately deduced from the estimate (5) of Theorem 3.

Proof of Theorem 1. Set $\varphi = u - v$. It follows that $\varphi \in \mathrm{V}_m(\Omega_\mathrm{T})$ and that φ satisfies the following conditions:

$$\begin{aligned}\varphi(0, x) &= \big(\varphi_1(0, x), \dots, \varphi_m(0, x)\big) = 0,\\ \Big|\frac{\partial \varphi_j}{\partial t}(t, x)\Big| &= \big|\mathrm{H}_j(t, x, u, \nabla_x u_j) - \mathrm{H}_j(t, x, v, \nabla_x v_j)\big|,\\ &\leq k_j(t)\big[(1 + \|x\|_n)\|\nabla_x \varphi_j\|_n + h_j(x)\|\varphi\|_m\big],\end{aligned}$$

for all $x \in \mathbb{R}^n$ and for almost all $t \in (0, \mathrm{T})$. By Corollary 1, we conclude that $\varphi(t, x) = 0$ for all $(t, x) \in \Omega_\mathrm{T}$. In other words, $u \equiv v$ on Ω_T. $\square$

Proof of Theorem 2. By the definition of $\mathrm{V}(\Omega_\mathrm{T})$, it follows that for almost all $t \in (0, \mathrm{T})$ the functions $u_j(t, .)$, $v_j(t, .)$ are both differentiable and locally Lipschitz continuous on $\mathbb{R}^n$; hence (see [VS1])

$$\begin{aligned}\operatorname*{ess.\,sup}_{x \in \mathbb{R}^n}\Big|\frac{\partial u_j}{\partial x_i}(t, x)\Big| &= \sup_{x \in \mathbb{R}^n}\Big|\frac{\partial u_j}{\partial x_i}(t, x)\Big|,\\ \operatorname*{ess.\,sup}_{x \in \mathbb{R}^n}\Big|\frac{\partial v_j}{\partial x_i}(t, x)\Big| &= \sup_{x \in \mathbb{R}^n}\Big|\frac{\partial v_j}{\partial x_i}(t, x)\Big|,\end{aligned}$$

$i = 1, \dots, n$; $j = 1, \dots, m$. Taking essential supremum at both sides of the last two equalities with respect to the variable t on $(0, \mathrm{T})$ we have

$$\begin{aligned}\operatorname*{ess.\,sup}_{(t,x) \in \Omega_\mathrm{T}}\Big|\frac{\partial u_j}{\partial x_i}(t, x)\Big| &= \operatorname*{ess.\,sup}_{t \in (0,\mathrm{T})}\Big(\sup_{x \in \mathbb{R}^n}\Big|\frac{\partial u_j}{\partial x_i}(t, x)\Big|\Big),\\ \operatorname*{ess.\,sup}_{(t,x) \in \Omega_\mathrm{T}}\Big|\frac{\partial v_j}{\partial x_i}(t, x)\Big| &= \operatorname*{ess.\,sup}_{t \in (0,\mathrm{T})}\Big(\sup_{x \in \mathbb{R}^n}\Big|\frac{\partial v_j}{\partial x_i}(t, x)\Big|\Big),\end{aligned}$$

$i = 1, \dots, n$; $j = 1, \dots, m$. The above equalities and the hypotheses of Theorem 2 imply

$$r_j := \max\{\operatorname{ess.sup}_{t\in(0,\mathrm{T})}(\sup_{x\in\mathbb{R}^n} \|\nabla_x u_j(t,x)\|_n), \operatorname{ess.sup}_{t\in(0,\mathrm{T})}(\sup_{x\in\mathbb{R}^n} \|\nabla_x v_j(t,x)\|_n)\}$$
$$< \infty.$$

On the other hand, the hypotheses of Theorem 2 with $\mathrm{K} := \overline{\mathrm{B}}_r^n$, $r := \max_{1\le j\le n} r_j$ imply that there exist nonnegative functions $h_j = h_{j\mathrm{K}}$, locally bounded on $\mathbb{R}^n$, and functions $k_j = k_{j\mathrm{K}} \in L^1(0,\mathrm{T})$ such that for all $p_1, p_2 \in \mathbb{R}^m; q_2, q_2 \in$ K the inequalities (3) hold for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,\mathrm{T})$. Thus, if we put $\omega := u - v = (\omega_1, \dots, \omega_m)$, then $\omega(0,x) = 0$ and

$$\begin{aligned}
\Big|\frac{\partial\omega_j}{\partial t}(t,x)\Big| &= \Big|\mathrm{H}_j(t,x,u(t,x),\nabla_x u_j(t,x)) - \mathrm{H}_j(t,x,v_j(t,x),\nabla_x v_j(t,x))\Big| \\
&\le k_j(t)\Big[(1+\|x\|_n)\|\nabla_x u_j(t,x) - \nabla_x v_j(t,x)\|_n \\
&\qquad + h_j(x)\|u(t,x) - v(t,x)\|_m\Big] \\
&\le k_j(t)\Big[(1+\|x\|_n)\|\nabla_x \omega_j(t,x)\|_n + h_j(x)\|\omega(t,x)\|_m\Big],
\end{aligned}$$

for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,\mathrm{T})$. By Corollary 1, it follows that $\omega(t,x) \equiv 0$ on Ω_T. $\square$

The proof of Theorem 4 is immediately deduced from Theorem 3.

Finally, let us give a Cauchy problem which has no classical solution but admits a unique semiclassical solution.

Example. Take n :=1, T := 1, m := 2 and let $\mathrm{J} \subset [0,1]$ be the Cantor set, i.e. the set of all numbers of the form:

$$\mathrm{t} = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i},$$

where ε_i is either 0 or 2. We define the continuous function $\varphi(.)$ on $[0,1]$ by

$$\varphi(t) := \min\{|t-\xi| : \xi \in \mathrm{J}\}, \quad t \in [0,1].$$

It is clear that $\varphi(0) = 0$ and that the function $\varphi(.)$ is Lipschitz continuous of Lipschitz constant 1. Set $\psi(t) := \dfrac{d\varphi(t)}{dt}$.

We consider the following problem:

$$
(10) \qquad \begin{cases} \dfrac{\partial u_1}{\partial t}(t,x) + \psi(t)\sin\Big(\dfrac{\partial u_1}{\partial x}(t,x) - u_2(t,x)\Big) & = 0, \\ \dfrac{\partial u_2}{\partial t}(t,x) + \psi(t)\dfrac{\partial u_2}{\partial x}(t,x) & = 0, \\ \qquad\qquad u_1(0,x) & = \dfrac{\pi}{2}x, \\ \qquad\qquad u_2(0,x) & = \pi, \end{cases}
$$

where $\mathrm{H}_1(t,x,p,q_1) = \psi(t)\sin(q_1 - p_2)$ and $\mathrm{H}_2(t,x,p,q_2) = \psi(t)q_2$, $p = (p_1,p_2)$, $q = (q_1,q_2) \in \mathbb{R}^2$.

Functions H_i, $i = 1,2$, satisfy the condition of Theorem 1. Therefore, the problem (10) admits at most one global semiclassical solution. It is evident that $u_1(t,x) = \varphi(t) + \dfrac{\pi}{2}x, u_2(t,x) = \pi$ is the unique global semiclassical solution for (10). Let us note that there exists no classical solution even in the local sense for (10).

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