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ON A REPRESENTATION OF CONVEX VECTOR FUNCTIONS AND THE MAXIMAL CYCLIC MONOTONICITY OF THEIR SUBDIFFERENTIAL

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Abstract. In this paper, we give a formula expressing a convex vector function as the supremum of a certain collection of affine functions and prove the maximal cyclic monotonicity of its subdifferential.

1. Introduction

The problem of characterizing the convexity of functions in terms of the monotonicity of their subdifferential operators is a very natural problem in nonsmooth analysis and has been studied intensively by many authors. Some new results are presented in [1], [3], where lower semicontinuous convex functions have been characterized via their Clarke subdifferential [1] or upper and lower Dini derivatives [3]. To the set-valued case, some efforts were also made to obtain necessary and sufficient conditions for a set-valued map to be convex (see, for example, [7], [8]).

In this paper, we consider convex vector functions from a subset of R^n to R^m, where R^m is ordered by a convex, closed and pointed cone. As in the scalar case, we shall represent convex vector functions as the supremum of a certain collection of affine functions concerning their subdifferentials. By this we prove that the subdifferential of a convex vector function defined on a relatively open set is a maximal cyclically monotone operator.

The paper is organized as follows. The next section contains some preliminaries which are needed in the sequel. Section 3 is devoted to the representation of convex vector functions. The last section is about the maximal cyclic monotonicity of subdifferential.

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2. Preliminaries

Let $C \subseteq R^m$ be a convex cone. We define a partial order $\succeq_C$ on R^m as follows:

$$x \succeq_C y \Leftrightarrow x - y \in C, \quad x, y \in R^m.$$

Sometimes we write $\succeq$ instead of $\succeq_C$ if it is clear which cone is under consideration.

Let S be a nonempty subset of R^m. An element $b \in R^m$ is said to be an upper bound of S with respect to C if $b \succeq x$ for every $x \in S$. An element $a \in R^m$ is said to be a supremum of S with respect to C if a is an upper bound of S and $a \preceq b$ for every upper bound b of S.

The cone C is called pointed if $C \cap (-C) = \{0\}$. It is clear that if C is pointed and S has a supremum then that element is unique. In this case, we denote by $\mathrm{Sup}S$ the supremum of S.

Denote by C' the positive polar cone of C, i.e.

$$C' := \{\xi \in L(R^m, R) |\ \xi(c) \geq 0, \forall c \in C\},$$

where $L(R^m, R)$ denotes the space of linear functionals on R^m.

From now on, the cone C is assumed convex, closed and pointed. We note that by [2, Chapter 1, Proposition 1.10], $\mathrm{int}C' \neq \emptyset$. The following result will be needed in the sequel.

Lemma 2.1. *Let S be a nonempty subset of R^m and let $a \in R^m$. If $\xi(a) = \sup \xi(S)$, for every $\xi \in \mathrm{int}C'$, then*

$$a = \sup S.$$

Proof. Let $x \in S$ be arbitrary. For every $\xi \in \mathrm{int}C'$, one has

$$\xi(a) \geq \xi(x). \tag{1}$$

Obviously, (1) also holds for every $\xi \in C'$. Since C is closed and convex, we get $a \succeq x$. Hence, a is an upper bound of S. Now let b be any upper bound of S. It is easy to see that $\xi(b) \geq \sup \xi(S) = \xi(a)$, for every $\xi \in \mathrm{int}C'$. Hence, $b \succeq a$. Thus, $a = \sup S$. The proof is complete. □

Now, let f be a vector function from a nonempty subset $D \subseteq R^n$ to R^m. We say that f is convex (or more precisely, C-convex) if for every $x, y \in D$, $\lambda \in [0, 1]$, one has

$$\lambda f(x) + (1 - \lambda) f(y) \succeq f(\lambda x + (1 - \lambda) y).$$

The subdifferential of f at $x \in D$ is defined as the set

$$\partial f(x) := \{A \in L(R^n, R^m) |\ f(y) - f(x) \succeq A(y - x), \forall y \in D\},$$

where $L(R^n, R^m)$ denotes the space of linear mappings from R^n to R^m. The graph of the subdifferential (as a set-valued mapping) is denoted by graph ∂f.

Finally, we recall the following result.

Lemma 2.2 ([4], Theorem 4.6). *Let f be a convex function from a nonempty subset $D \subseteq R^n$ to R^m and let $x \in D, \xi \in C'$. If one of the following conditions holds*

(i) $\text{int}D \neq \emptyset$, $x \in \text{int}D$

(ii) $\text{int}D = \emptyset$, $x \in \text{ri}D$, $\xi \neq 0$,

then $\partial(\xi \circ f)(x) = \xi \partial f(x)$.

3. A representation of convex vector functions

From now on, we denote by $< .,. >$ the duality pairing between $L(R^n, R^m)$ and R^n. As in the scalar case, we shall show that a convex vector function can be represented as the supremum of a certain collection of affine functions.

Let $g : R^n \to R \cup \{+\infty\}$ be an extended-real-valued function. We say that g is proper if Dom$g$ is nonempty. Sometimes, we shall identify a proper function $g : R^n \to R \cup \{+\infty\}$ with the function $\bar{g} : \text{Dom}g \to R$ defined by

$$\bar{g}(x) = g(x), \ \ (\forall x \in \text{Dom}g).$$

Lemma 3.1. *Let $g : R^n \to R \cup \{+\infty\}$ be proper and convex. Then*

$$\partial g(x) = \partial(\text{cl}\, g)(x), \ \ (\forall x \in \text{ri}(\text{Dom}g)),$$

where cl g *denotes the closure of* g.

Proof. The inclusion $\partial(\text{cl}\, g)(x) \subseteq \partial g(x)$ is immediate from the definition of subdifferential and from the fact that $\text{cl}\, g \leq g$. For the converse inclusion, let $x^* \in \partial g(x)$, $y \in \text{Dom}(\text{cl}\, g)$ be arbitrary. For every $\lambda \in (0, 1)$, one has

$$g(\lambda y + (1 - \lambda)x) - g(x) \geq \lambda < x^*, y - x > .$$

Taking $\lambda \to 1$, we obtain

$$(\text{cl}\, g)(y) - (\text{cl}\, g)(x) \geq < x^*, y - x > .$$

Thus, $x^* \in \partial(\operatorname{cl} g)(x)$. The proof is complete. □

Lemma 3.2. *Let $g : R^n \to R \cup \{+\infty\}$ be a proper, convex function with the relatively open domain and let $x \in \operatorname{Dom} g, (x_0, x_0^*) \in \operatorname{graph} \partial g$. Then*

$$g(x) = g(x_0) + \sup\Big\{ \sum_{i=0}^{k-1} < x_i^*, x_{i+1} - x_i > \\ + < x_k^*, x - x_k > \mid (x_i, x_i^*) \in \operatorname{graph} \partial g,\ i = 1, \ldots, k,\ k \geq 1 \Big\}.$$

Proof. Consider the function $\operatorname{cl} g$. From the Rockafellar formula [6, Part 5, Theorem 24.8] one has

$$\operatorname{cl} g(x) = \operatorname{cl} g(x_0) + \sup\Big\{ \sum_{i=0}^{k-1} < x_i^*, x_{i+1} - x_i > \\ + < x_k^*, x - x_k > |(x_i, x_i^*) \in \operatorname{graph} \partial(\operatorname{cl} g),\ i = 1, \ldots, k,\ k \geq 1 \Big\}. \tag{2}$$

Since $\operatorname{Dom} g$ is relatively open and $x, x_0 \in \operatorname{Dom} g$ then by (2), one has

$$g(x) - g(x_0) = \sup\Big\{ \sum_{i=0}^{k-1} < x_i^*, x_{i+1} - x_i > \\ + < x_k^*, x - x_k > |(x_i, x_i^*) \in \operatorname{graph} \partial(\operatorname{cl} g),\ i = 1, \ldots, k,\ k \geq 1 \Big\}.$$

Taking into account the definition of subdifferential, one has

$$g(x) - g(x_0) \geq \sup\Big\{ \sum_{i=0}^{k-1} < x_i^*, x_{i+1} - x_i > \\ + < x_k^*, x - x_k > |(x_i, x_i^*) \in \operatorname{graph} \partial g,\ i = 1, \ldots, k,\ k \geq 1 \Big\}.$$

To complete the proof, we shall show that for every $\varepsilon > 0$, $(x_i, x_i^*) \in \operatorname{graph} \partial(\operatorname{cl} g)$, $(i = 1, \ldots, k)$, $k \geq 1$, there exist $(y_i, y_i^*) \in \operatorname{graph} \partial g$, $i = 1, \ldots, k$, such that

$$< x_0^*, y_1 - x_0 > + \sum_{i=1}^{k-1} < y_i^*, y_{i+1} - y_i > + < y_k^*, x - y_k > \\ \geq \sum_{i=0}^{k-1} < x_i^*, x_{i+1} - x_i > + < x_k^*, x - x_k > - \varepsilon. \tag{3}$$

Indeed, let $\varepsilon > 0$, $(x_i, x_i^*) \in \operatorname{graph} \partial(\operatorname{cl} g)$, $i = 1, 2, \ldots, k$ be given. First, we shall change the pair (x_k, x_k^*) by a pair $(y_k, y_k^*) \in \operatorname{graph} \partial g$ such that

$$\sum_{i=0}^{k-2} < x_i^*, x_{i+1} - x_i > + < x_{k-1}^*, y_k - x_{k-1} > + < y_k^*, x - y_k >$$

(4)

$$\geq \sum_{i=0}^{k-1} < x_i^*, x_{i+1} - x_i > + < x_k^*, x - x_k > - \frac{\varepsilon}{k} \cdot$$

The existence of a such a pair (y_k, y_k^*) is shown as follows. Since the function $< x_{k-1}^*, y - x_{k-1} >$ is continuous at x_k then there exists $\delta > 0$ such that $\|y - x_k\| < \delta$ implies

(5) $$< x_{k-1}^*, y - x_{k-1} > \geq < x_{k-1}^*, x_k - x_{k-1} > - \frac{\varepsilon}{2k} \cdot$$

Let y belong to the interval (x, x_k). Represent y as $y = (1-\lambda)x + \lambda x_k$ for some $\lambda \in (0, 1)$. Fix $x^* \in \partial g(x)$. By the monotonicity of subdifferential and by Lemma 3.1, for every $y^* \in \partial g(y)$, one has

$$< y^* - x_k^*, y - x_k > \geq 0,$$

$$< y^* - x^*, y - x > \geq 0,$$

or

$$< y^* - x_k^*, (1-\lambda)(x - x_k) > \ \geq 0,$$
$$< y^* - x^*, \lambda(x_k - x) > \ \geq 0.$$

Hence

$$< x^*, x - x_k > \geq < y^*, x - x_k > \geq < x_k^*, x - x_k > .$$

Thus, $< y^*, y - x_k >= (1-\lambda) < y^*, x - x_k > \to 0$ when $y \to x_k$, $y \in (x, x_k)$. Choose $y_k \in (x, x_k), y_k^* \in \partial g(y_k)$ such that

$$\|y_k - x_k\| < \delta, | < y_k^*, y_k - x_k > | < \frac{\varepsilon}{2k} \cdot$$

Since $< y_k^* - x_k^*, y_k - x_k > \geq 0$ and $y_k \in (x, x_k)$, $< y_k^* - x_k^*, x - x_k > \geq 0$. Hence

(6) $$< y_k^*, x - y_k > - < x_k^*, x - x_k > \geq - < y_k^*, y_k - x_k > > - \frac{\varepsilon}{2k} \cdot$$

It is clear that (5) and (6) imply (4).

Continuing this process for the pairs $(x_{k-1}, x^*_{k-1}), \ldots, (x_1, x^*_1)$, after k steps we find out $(y_i, y^*_i) \in \operatorname{graph} \partial g, i = 1, 2, \ldots, k$, satisfying (3). The Lemma is proved. □

Theorem 3.3. *Let f be a convex vector function from a nonempty relatively open subset $D \subseteq R^n$ to R^m and let $x \in D$, $(x_0, A_0) \in \operatorname{graph} \partial f$. Then*

$$f(x) = f(x_0) + \sup\Big\{ \sum_{i=0}^{k-1} < A_i, x_{i+1} - x_i > \\ + < A_k, x - x_k > \mid (x_i, A_i) \in \operatorname{graph} \partial f,\ i = 1, \ldots, k,\ k \geq 1 \Big\}.$$

Proof. Let $\xi \in \operatorname{int} C'$ be arbitrary. By Lemma 2.1 of [4], $\xi \circ f$ is convex and by Lemma 2.2 above, $\xi \circ A_0 \in \partial(\xi \circ f)(x_0)$. Then by Lemma 3.2, one has

$$(\xi \circ f)(x) = (\xi \circ f)(x_0) \\ + \sup\Big\{ < \xi \circ A_0, x_1 - x_0 > + \sum_{i=1}^{k-1} < x^*_i, x_{i+1} - x_i > \\ + < x^*_k, x - x_k > \mid (x_i, x^*_i) \in \operatorname{graph} \partial(\xi \circ f),\ i = 1, \ldots, k,\ k \geq 1 \Big\}.$$

By Lemma 2.2,

$$\Big\{ < \xi \circ A_0, x_1 - x_0 > + \sum_{i=1}^{k-1} < x^*_i, x_{i+1} - x_i > \\ + < x^*_k, x - x_k > \mid (x_i, x^*_i) \in \operatorname{graph} \partial(\xi \circ f),\ i = 1, \ldots, k,\ k \geq 1 \Big\} \\ = \xi \Big\{ \sum_{i=0}^{k-1} < A_i, x_{i+1} - x_i > \\ + < A_k, x - x_k > \mid (x_i, A_i) \in \operatorname{graph} \partial f, i = 1, \ldots, k, k \geq 1 \Big\}.$$

Hence,

$$\xi[f(x) - f(x_0)] = \sup \xi \Big\{ \sum_{i=0}^{k-1} < A_i, x_{i+1} - x_i > \\ + < A_k, x - x_k > \mid (x_i, A_i) \in \operatorname{graph} \partial f,\ i = 1, \ldots, k, k \geq 1 \Big\}.$$

Then by Lemma 2.1,

$$f(x) - f(x_0) = \sup\Big\{ \sum_{-=0}^{k-1} < A_i, x_{i+1} - x_i >$$
$$+ < A_k, x - x_k > |(x_i, A_i) \in \operatorname{graph} \partial f,\ i = 1, \dots, k,\ k \geq 1\Big\}.$$

The Theorem is proved. □

4. The maximal cyclic monotonicity of subdifferential

Let F be a set-valued map from a nonempty subset $D \subseteq R^n$ to $L(R^n, R^m)$. F is said to be cyclically monotone if one has

$$< A_0, x_1 - x_0 > + < A_1, x_2 - x_1 > + \cdots + < A_k, x_0 - x_k > \preceq 0,$$

for any set of pairs $(x_i, A_i) \in \operatorname{graph} F$, $i = 0, 1, \dots, k$ (k arbitrary). F is said to be maximal cyclically monotone if F is cyclically monotone and its graph is not properly contained in the graph of any other cyclically monotone map.

Theorem 4.1. *Let f be a convex vector function from a nonempty relatively open subset $D \subseteq R^n$ to R^m. Then ∂f is maximal cyclically montone.*

Proof. For any set of pairs $(x_i, A_i) \in \operatorname{graph} \partial f$, $i = 0, 1, \dots, k$ (k arbitrary), from the definition of subdifferential, one has

(7) $$f(x_{i+1}) - f(x_i) \succeq < A_i, x_{i+1} - x_i >,\ i = 0, 1, \dots, k-1.$$

(8) $$f(x_0) - f(x_k) \succeq < A_k, x_0 - x_k > .$$

By adding up inequalities (7), (8), we get

$$< A_0, x - 1 - x_0 > + < A_1, x_2 - x_1 > + \cdots + < A_k, x_0 - x_k > \preceq 0.$$

Hence, ∂f is cyclically monotone.

To complete the proof, let $x \in D$, $A \in L(R^n, R^m)$ and $A \notin \partial f(x)$. We shall show that there exist $k \geq 0$ and $(x_i, A_i) \in \operatorname{graph} \partial f$, $i = 0, 1, \dots, k$ such that

$$< A, x_0 - x > + < A_0, x_1 - x_0 >$$
$$+ < A_1, x_2 - x_1 > + \cdots + < A_k, x - x_k > \not\preceq 0.$$

Suppose to the contrary that for every $k \geq 0$, $(x_i, A_i) \in \operatorname{graph} \partial f$, $i = 0, 1, \cdots, k$, one has

$$< A, x_0 - x > + < A_0, x_1 - x_0 > \\ + < A_1, x_2 - x_1 > + \cdots + < A_k, x - x_k > \preceq 0.$$

Then

$$\sum_{i=0}^{k-1} < A_i, x_{i+1} - x_i > + < A_k, x - x_k > \preceq < A, x - x_0 > . \tag{9}$$

Fix $(x_0, A_0) \in \operatorname{graph} \partial f$. Then (9) implies that $< A, x - x_0 >$ is an upper bound of the set

$$\Big\{ \sum_{i=0}^{k-1} < A_i, x_{i+1} - x_i > \\ + < A_k, x - x_k > \mid (x_i, A_i) \in \operatorname{graph} \partial f,\ i = 1, \ldots, k, k \geq 1 \Big\}.$$

By Theorem 3.3, this implies

$$< A, x - x_0 > \succeq \ f(x) - f(x_0). \tag{10}$$

By Theorem 4.14 of [4], $\operatorname{Dom} \partial f = D$. From this and from the fact that (10) holds for every $(x_0, A_0) \in \operatorname{graph} \partial f$, we have $A \in \partial f(x)$. We arrive at a contradiction. The proof is complete. □

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