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COEFFICIENT MULTIPLIERS FOR SOME CLASSES OF DIRICHLET SERIES IN SEVERAL COMPLEX VARIABLES

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Abstract. This paper deals with coefficient multipliers for some classes of Dirichlet series with complex frequencies, including those that define entire functions in $\mathbf{C}^n$.

1. Introduction

A holomorphic function, including entire one, can be identified with the sequence of its Taylor coefficients. One way for getting information about the Taylor coefficients of such functions is to describe the multipliers of the space of sequences of Taylor coefficients into various other sequence spaces. In this direction there have been many articles that deal with multipliers of spaces of Bloch functions, Lipschitz functions, of Hardy spaces, Bergman spaces, etc. (see, e.g., [1, 2, 7, 14, 15, 16]).

We recall that for two sequence spaces A and B the symbol (A, B) denotes the sequence space of multipliers from A to B,

$$(A, B) = \{u = (u_k); (u_k a_k) \in B,\ \forall (a_k) \in A\}.$$

By definition, a sequence space A is said to be normal [6] (or solid [1]) if whenever A contains (a_k) it also contains (b_k) with $|b_k| \le |a_k|$ for $k = 1, 2, \dots$ Equivalently, A is normal if $\ell^\infty \subset (A, A)$. Furthermore, for a sequence space A there always exists a largest normal subspace, denoted by $s(A)$, that is contained within it, and a smallest normal superspace, denoted by $S(A)$, that contains it. More precisely, $s(A) = (\ell^\infty, A)$ and $S(A)$ is the intersection of all the normal spaces that contain A [1].

Various concepts of duality for sequence spaces are given in [3, 4, 6]. Let D be a fixed sequence space. Then the *D-dual* of a sequence space A,

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denoted by A^D, is defined to be (A, D), the multipliers from A to D. The Köthe dual is obtained when $D = \ell^1$, and will be denoted by A^α (it is also denoted by A^K). The Abel dual is obtained when D is the space of Abel-summable sequences, that is, the space of sequences $\big(d_k\big)$ for which

$$\lim_{r\to 1}\sum_{k=1}^{\infty} d_k r^k$$

exists. We denote the Abel dual of A by A^γ (it is also denoted by A^a). Note that when $d_k \geq 0$, then the existence of this limit is equivalent to $\sum d_k < +\infty$. It is clear that $A^\alpha \subset A^\gamma$. The inverse inclusion is true if the space A is normal [1]. The spaces A^α and A^γ were studied in several papers (see, e.g., [1, 2, 7]).

What about Dirichlet series with complex frequencies? This question stems from the fact that each entire function in $\mathbf{C}^n$ as well as each holomorphic function in a convex domain of $\mathbf{C}^n$ can always be represented in the form of Dirichlet series with complex frequencies (see, e.g., [5, 8] and the references therein). This fact has many important applications in the theory of functional equations.

In the present paper we study coefficient multipliers for some classes of Dirichlet series with complex frequencies, including those that define entire functions in $\mathbf{C}^n$. Section 2 deals with preliminaries on sequence spaces closely related to Dirichlet series with a given sequence $\big(\lambda^k\big)$ of complex frequencies. Namely, we are concerned with the space E_0 that generates entire Dirichlet series in $\mathbf{C}^n$ and the space E_1 that seems to be the maximal among those we are interested in. We establish some dualities of the spaces E_0 and E_1. It turns out that the spaces E_0 and E_1, under rather general conditions on the sequence of frequencies, are the Köthe duals of each other. These results are obtained in the spirit of [9] for the case of holomorphic Dirichlet series. In Section 3 we consider the generalized Köthe duals of the spaces E_j $(j = 0, 1)$, i.e., the multipliers between spaces E_j and ℓ^p $(0 < p \leq \infty)$. Finally, in Section 4 we study conditions for a given sequence to be a multiplier for spaces E_0 and E_1 as well as between them.

Note that Dirichlet series with real frequencies on the complex plane have been treated in our recent works [12, 13].

2. Preliminaries on sequence spaces E_0 and E_1

We use the following basic notations: $\mathcal{O}(\mathbf{C}^n)$ denotes the space of entire functions in $\mathbf{C}^n$, with the compact-open topology, i.e., the topology of uniform convergence on compact subsets of $\mathbf{C}^n$. If $z, \zeta \in \mathbf{C}^n$ then $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$; $\langle z, \zeta\rangle = z_1\zeta_1 + \cdots + z_n\zeta_n$.

Let $\{\lambda^k\}$, $\lambda^k = (\lambda_1^k, \ldots, \lambda_n^k)$, $k = 1, 2, \ldots$, be a sequence of complex vectors in $\mathbf{C}^n$. Consider a multiple Dirichlet series with complex frequencies

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z\rangle}, \quad z \in \mathbf{C}^n. \tag{2.1}$$

We make a characterization of the coefficients of the series (2.1) for when it converges absolutely in whole $\mathbf{C}^n$ (which is important and necessary for further study).

Theorem 2.1. *If the Dirichlet series* (2.1) *converges absolutely for all* $z \in \mathbf{C}^n$ *and* $|\lambda^k| \to \infty$ *as* $k \to \infty$*, then*

$$\limsup_{k\to\infty} \frac{\log |c_k|}{|\lambda^k|} = -\infty. \tag{2.2}$$

Conversely, if the coefficients of (2.1) *satisfy condition* (2.2) *and if*

$$\limsup_{k\to\infty} \frac{\log k}{|\lambda^k|} < +\infty, \tag{2.3}$$

then the series (2.1) *converges absolutely for all* $z \in \mathbf{C}^n$.

The following elementary result is used often in the sequel.

Lemma 2.2. *Condition* (2.3) *is equivalent to*

$$\exists\, \rho > 0 : \sum_{k=1}^{\infty} e^{-\rho|\lambda^k|} < +\infty. \tag{2.4}$$

Proof of Theorem 2.1. Necessity. Let the Dirichlet series (2.1) converges absolutely for all $z \in \mathbf{C}^n$ and $|\lambda^k| \to \infty$ as $k \to \infty$. Assume that (2.2) is false, i.e.,

$$\limsup_{k\to\infty} \frac{\log |c_k|}{|\lambda^k|} > -\infty.$$

Then we can find a number $M > 0$ and an increasing sequence of positive integers (k_j) such that

$$\frac{\log |c_{k_j}|}{|\lambda^{k_j}|} > -M, \quad \forall\, j \geq 1,$$

or equivalently,

$$|c_{k_j}| > e^{-M|\lambda^{k_j}|}. \tag{2.5}$$

On the other hand, there exists a subsequence of $\big(k_j\big)$, which we can, for the sake of simplicity and without loss of generality, denote by the notation $\big(k_j\big)$ itself, such that

$$\arg\lambda_s^{k_j} \to \varphi_s \ \text{ as } \ j \to \infty, \quad (s = 1, 2, \dots, n).$$

Taking into account the fact that $\big(\arg\lambda_s^{k_j} - \varphi_s\big) \to 0$ as $j \to \infty$, for a vector $z_0 = (re^{-i\varphi_1}, \dots, re^{-i\varphi_n}) \in \mathbf{C}^n,\ r > 2M$, there is N such that $\forall\ j \geq N$,

$$\Big|e^{\langle\lambda^{k_j}, z_0\rangle}\Big| = e^{\mathrm{Re}\langle\lambda^{k_j}, z_0\rangle} = e^{r\sum\limits_{s=1}^{n}|\lambda_s^{k_j}|\cos(\arg\lambda_s^{k_j} - \varphi_s)} \geq e^{\frac{1}{2}r\sum\limits_{s=1}^{n}|\lambda_s^{k_j}|}. \tag{2.6}$$

Therefore, since $|z| \leq \sum\limits_{s=1}^{n}|z_s|\ \ \forall\ z \in \mathbf{C}^n$, from (2.5)-(2.6) it follows that

$$\begin{aligned}\Big|c_{k_j}e^{\langle\lambda^{k_j}, z_0\rangle}\Big| &\geq e^{-M|\lambda^{k_j}| + \frac{1}{2}r\sum\limits_{s=1}^{\infty}|\lambda_s^{k_j}|} \\ &\geq e^{-M\sum\limits_{s=1}^{n}|\lambda_s^{k_j}| + \sum\limits_{s=1}^{n}r|\lambda_s^{k_j}|/2} \\ &= e^{(r/2-M)\sum\limits_{s=1}^{n}|\lambda_s^{k_j}|} \geq 1,\end{aligned}$$

which shows that the series (2.1) does not converges absolutely at the point z_0, a contradiction.

Sufficiency. Suppose that condition (2.2) holds. Then for $\varepsilon > 0$ there exists N such that $\forall\ k \geq N$,

$$|c_k| \leq \varepsilon^{|\lambda^k|}.$$

Take an arbitrary vector $z \in \mathbf{C}^n$ and let $|z| = R$. We have

$$\sum_{k=1}^{\infty}\Big|c_k e^{\langle\lambda^k, z\rangle}\Big| \leq \sum_{k=1}^{\infty}|c_k|e^{\Re\langle\lambda^k, z\rangle} \leq \sum_{k=1}^{\infty}|c_k|e^{R|\lambda^k|} \leq \sum_{k=1}^{\infty}(\varepsilon e^R)^{|\lambda^k|}.$$

By Lemma 2.2, choosing $\varepsilon \in (0, e^{-R-\rho})$, where ρ is taken as in (2.4), completes the proof of the theorem.

Remark 2.3. From the proof of the sufficiency part of Theorem 2.1 it follows that if conditions (2.2)-(2.3) hold, then the series (2.1) converges absolutely for the topology of the space $\mathcal{O}(\mathbf{C}^n)$. Theorem 2.1 also shows that with the sequence of coefficients satisfying condition (2.2) and the sequence of frequencies satisfying condition (2.3), the series (2.1) represents an entire function in $\mathbf{C}^n$.

In connection with Theorem 2.1, given a sequence $\Lambda := \left(\lambda^k\right)_{k=1}^{\infty}$ of complex vectors in $\mathbf{C}^n$, we can associate to it the following two sequence spaces

$$E_1 = \left\{c = \left(c_k\right);\ \exists M\ \forall k\ |c_k| \le e^{M|\lambda^k|}\right\},$$

$$E_0 = \left\{c = \left(c_k\right);\ |c_k|^{1/|\lambda^k|} \to 0,\ k \to \infty\right\}.$$

We can define these spaces in a uniform way by requiring

$$\limsup_{k\to\infty} \frac{\log|c_k|}{|\lambda^k|} \begin{cases} < +\infty, \\ = -\infty. \end{cases}$$

The space E_0 is a proper subspace of E_1, for the element $\left(c_k\right)$ with $c_k = e^{M|\lambda^k|}$, $M \in \mathbf{R}$, belongs to E_1 but does not belong to E_0. These spaces were introduced in [10, 11]. We refer the readers to these papers for various properties of E_0 and E_1.

It should also be noted that in [10, 11] the spaces E_0 and E_1 were studied under the condition which is much stronger than condition (2.4), namely

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|} = 0. \tag{2.7}$$

The class of entire function of the form (2.1), where $\left(\lambda^k\right)$ satisfies condition (2.3) and $\left(c_k\right)$ satisfies condition (2.2), is denoted by $E(\Lambda, \mathbf{C}^n)$.

Note that $E(\Lambda, \mathbf{C}^n) \subset \mathcal{O}(\mathbf{C}^n)$ and the equality holds if and only if $\left(e^{\langle\lambda^k, z\rangle}\right)_{k=1}^{\infty}$ forms an absolutely representing system in the space $\mathcal{O}(\mathbf{C}^n)$ (see, e.g., [5, 8]).

In the rest of this section we study some properties such as normality, perfectness of the spaces E_j $(j = 0, 1)$ as well as a description of the Köthe dual of these spaces. Here we follow the terminology of [6].

First note that whenever E_j contains (c_k) it also contains (d_k) with $|d_k| \le |c_k|$ for $k = 1, 2, \dots$. So this space is normal.

Denote by E_j^α the Köthe dual of the space E_j, i.e.,

$$E_j^\alpha = \left\{ (u_k);\ \sum_{k=1}^{\infty} c_k u_k \text{ converges absolutely for all } (c_k) \in E_j \right\}.$$

Also we consider the following set

$$E_j^\beta = \left\{ (u_k);\ \sum_{k=1}^{\infty} c_k u_k \text{ converges for all } (c_k) \in E_j \right\}.$$

We make a characterization of the Köthe dual for the spaces E_j ($j = 0, 1$).

Proposition 2.4. (i) *If $(d_k) \in E_0^\beta$, then*

$$\limsup_{k\to\infty} |d_k|^{1/|\lambda^k|} < +\infty, \text{ i.e., } (d_k) \in E_1. \tag{2.8}$$

Conversely, if the sequence (d_k) satisfies condition (2.8) *and, in addition, the sequence (λ^k) satisfies condition* (2.3)*, then $(d_k) \in E_0^\alpha$. In other words, we have*

$$E_0^\beta \subset E_1 \subset E_0^\alpha.$$

(ii) *If $(d_k) \in E_1^\beta$, then*

$$\lim_{k\to\infty} |d_k|^{1/|\lambda^k|} = 0, \text{ i.e., } (d_k) \in E_0. \tag{2.9}$$

Conversely, if the sequence (d_k) satisfies condition (2.9) *and, in addition, the sequence (λ^k) satisfies condition* (2.3)*, then $(d_k) \in E_1^\alpha$. In other words, we have*

$$E_1^\beta \subset E_0 \subset E_1^\alpha.$$

Proof. We shall prove (i). For (ii) it is analogous.

Necessity. Let $(d_k) \in E_0^\beta$. Suppose that (2.8) is not true, i.e.,

$$\limsup_{k\to\infty} |d_k|^{1/|\lambda^k|} = +\infty.$$

Then there exists an increasing sequence $\big(k_j\big)_{j=1}^{\infty}$ of positive integers such that

$$\lim_{j\to\infty} |d_{k_j}|^{1/|\lambda^{k_j}|} = +\infty.$$

Define a sequence $\big(c_k\big)$ as follows

$$c_k = \begin{cases} 1/|d_k|, & \text{if } k = k_j,\ j = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\lim_{k\to\infty} |c_k|^{1/|\lambda^k|} = \lim_{j\to\infty} 1\big/|d_{k_j}|^{1/|\lambda^{k_j}|} = 0,$$

which means that $\big(c_k\big)$ is in E_0. However, the series $\sum\limits_{k=1}^{\infty} c_k d_k$ does not converge, a contradiction.

Sufficiency. Assume that (2.8) holds. Then there exists a constant C such that

$$|d_k| \le C^{|\lambda^k|},\ \forall\ k \ge 1.$$

Take an arbitrary element $c = \big(c_k\big) \in E_0$. For $\varepsilon \in (0, e^{-\rho}/C)$, where ρ is from (2.4), there exists N such that $\forall\ k > N$

$$|c_k| < \varepsilon^{|\lambda^k|}.$$

Hence

$$\sum_{k=1}^{\infty} |c_k d_k| \le \sum_{k=1}^{\infty} \big(\varepsilon C\big)^{|\lambda^k|} \le \sum_{k=1}^{\infty} e^{-\rho|\lambda^k|} < +\infty,$$

due to (2.4). This completes the proof.

Corollary 2.5. *If* (2.3) *holds, then* $\big(d_k\big) \in E_j^{\beta}$ *if and only if* $\big(d_k\big) \in E_j^{\alpha}$, *i.e.,* $E_j^{\alpha} = E_j^{\beta}$ $(j = 0, 1)$. *In this case, these sequence spaces can be defined as follows*

$$E_0^{\beta} = E_0^{\alpha} = E_1,$$

$$E_1^{\beta} = E_1^{\alpha} = E_0,$$

and therefore, the spaces E_0 *and* E_1 *are the Köthe duals each for other.*

It is clear that $E_j \subset E_j^{\alpha\alpha}$ $(j = 0, 1)$. A question arises when does the inverse inclusion hold? We can prove the following result.

Proposition 2.6. *Suppose that condition* (2.3) *holds. Then the sequence space E_j is perfect, i.e.,* $E_j^{\alpha\alpha} = E_j$ $(j = 0, 1)$.

Proof. We prove this statement for E_0. For E_1 it is analogous.

Assume that $(c_k) \notin E_0$. Then

$$\limsup_{k\to\infty} |c_k|^{1/|\lambda^k|} > 0.$$

Note that the value of the left-hand side can be finite as well as $+\infty$. In any case, there exist $M > 0$ and an increasing sequence $(k_j)_{j=1}^{\infty}$ of positive integers such that

$$|c_{k_j}|^{1/|\lambda^{k_j}|} > M, \quad \forall\, j \geq 1,$$

which is equivalent to

$$1/|c_{k_j}|^{1/|\lambda^{k_j}|} < 1/M.$$

Define a sequence (d_k) as follows

$$d_k = \begin{cases} 1/|c_k|, & \text{if } k = k_j,\ j = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{k\to\infty} |d_k|^{1/|\lambda^k|} \leq \limsup_{j\to\infty} |d_{k_j}|^{1/|\lambda^{k_j}|} \leq \frac{1}{M} < \infty,$$

which means, by Proposition 2.4, that $(d_k) \in E_0^{\alpha}$. However, the series $\sum_{k=1}^{\infty} c_k d_k$ does not converge. Hence, $(c_k) \notin E_0^{\alpha\alpha}$. This completes the proof.

Remark 2.7. A part of these results (which concerns the space E_0) was announced in [10] under the condition (2.7), i.e.,

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|} = 0.$$

Everywhere in what follows condition (2.3) for the sequence of frequencies (λ_n) is assumed to hold, i.e.,

$$\limsup_{k\to\infty} \frac{\log k}{\lambda^k} < +\infty,$$

which, by Lemma 2.2, is equivalent to

$$\exists\, \rho > 0 : \sum_{k=1}^{\infty} e^{-\rho|\lambda^k|} < +\infty.$$

3. Generalized Köthe duals for E_0 and E_1

In this section we study the generalized Köthe duals of the spaces E_0 and E_1. As for the space E_1 all proofs are similar to the case of E_0, hence we consider E_0 only.

As noted above, the Köthe dual of a sequence space is the sequence space of multipliers from this space to the space ℓ^1. A question arises what are about multipliers from E_0 to ℓ^p $(0 < p \leq \infty)$ and vice-versa?

First we note that $\big(u_k\big)$ is in E_0 if and only if $\big(u_k^{1/p}\big)$ is in E_0 (with any appropriate choice of the power), so the study for all $0 < p < \infty$ reduces to the case $p = 1$, which is the case of the Köthe duality already studied above. The same remark applies to the space E_0^α. Furthermore, $\ell^p \subset \ell^\infty$ for any $0 < p < \infty$. These facts allow us to obtain the following results.

Proposition 3.1. *A sequence $\big(u_k\big)$ is the multiplier from E_0 to ℓ^p $(0 < p \leq \infty)$ if and only if $\big(u_k\big)$ satisfies condition* (2.8). *In other words,*

$$\big(E_0, l^p\big) = E_0^\alpha = E_1, \quad \forall\, 0 < p \leq \infty.$$

Proof. We already have the inclusion $E_0^\alpha = \big(E_0, \ell^1\big) \subset \big(E_0, \ell^\infty\big)$. The inclusion $\big(E_0, \ell^\infty\big) \subset E_0^\alpha$ remains to be proved.

Let $\big(u_k\big) \in \big(E_0, \ell^\infty\big)$. Assume that $\big(u_k\big) \notin E_0^\alpha$, i.e.,

$$\limsup_{k\to\infty} |u_k|^{1/|\lambda^k|} = \infty.$$

Then there exists an increasing sequence $\big(k_j\big)$ of positive numbers such that

$$\lim_{j\to\infty} |u_{k_j}|^{1/|\lambda^{k_j}|} = \infty.$$

We define a sequence $\big(c_k\big)$ as follows:

$$c_k = \begin{cases} \lambda^{k_j} / \big|u_{k_j}\big|, & \text{if } k = k_j,\ j = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

In this case

$$\limsup_{k\to\infty} |c_k|^{1/|\lambda^k|} \le \limsup_{j\to\infty} |c_{k_j}|^{1/|\lambda^{k_j}|} = \limsup_{j\to\infty} \Big(\frac{1}{|u_{k_j}|^{1/|\lambda^{k_j}|}} \cdot |\lambda_{k_j}|^{1/|\lambda_{k_j}|} \Big) = 0,$$

which means that $\big(c_k\big) \in E_0$. However, it is clear that $\big(c_k u_k\big) \notin \ell^\infty$, and therefore, $\big(u_k\big) \notin \big(E_0, \ell^p\big)$, a contradiction.

Remark 3.2. The statement of Proposition 3.1 is valid when E_0 and E_1 are interchanged, i.e.,

$$\big(E_1, \ell^p\big) = E_1^\alpha = E_0, \ \forall\ 0 < p \le \infty.$$

The proof is analogous.

Proposition 3.3. *A sequence $\big(u_k\big)$ is the multiplier from ℓ^p $(0 < p \le \infty)$ to E_0 if and only if $\big(u_k\big)$ satisfies condition* (2.9). *In other words,*

$$\big(\ell^p, E_0\big) = E_0, \quad \forall\ 0 < p \le \infty.$$

Proof. Due to the inclusion $\big(\ell^\infty, E_0\big) \subset \big(\ell^1, E_0\big)$ it suffices to prove that $E_0 \subset \big(\ell^\infty, E_0\big)$ and $\big(\ell^1, E_0\big) \subset E_0$.

The first part follows already from the fact that E_0 is a normal space.

Next let $\big(u_k\big) \in \big(l^1, E_0\big)$. Assume that $\big(u_k\big) \notin E_0$, i.e.,

$$\limsup_{k\to\infty} |u_k|^{1/|\lambda^k|} > 0.$$

The value of the left-hand side can be finite as well as $+\infty$. In any case, there exist $Q > 0$ and an increasing sequence $\big(k_j\big)$ of positive numbers such that

$$|u_{k_j}|^{1/|\lambda^{k_j}|} \ge Q, \quad \forall j \ge 1,$$

which is equivalent to

$$\frac{1}{|u_{k_j}|} \le \Big(\frac{1}{Q}\Big)^{|\lambda^{k_j}|}.$$

We define a sequence $\big(\xi_k\big)$ as follows:

$$\xi_k = \begin{cases} \nu^{|\lambda^{k_j}|} / \big|u_{k_j}\big|, & \text{if } k = k_j,\ j = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

where $\nu \in (0, Qe^{-\rho})$ and ρ is taken from Lemma 2.2. Then we have

$$\sum_{k=1}^{\infty} |\xi_k| = \sum_{j=1}^{\infty} |\xi_{k_j}| = \sum_{j=1}^{\infty} \frac{\nu^{|\lambda^{k_j}|}}{|u_{k_j}|} \leq \sum_{j=1}^{\infty} \left(\frac{\nu}{Q}\right)^{|\lambda^{k_j}|} \leq \sum_{j=1}^{\infty} e^{-\rho|\lambda^{k_j}|} < \infty,$$

due to Lemma 2.2, which shows that $(\xi_k) \in l^1$. However,

$$\limsup_{k\to\infty} |\xi_k u_k|^{1/|\lambda^k|} = \limsup_{j\to\infty} |\xi_{k_j} u_{k_j}|^{1/|\lambda^{k_j}|} = \nu > 0,$$

which means that $(\xi_k u_k) \notin E_0$, a contradiction. Thus, $(l^1, E_0) \subset E_0$. This completes the proof of the proposition.

Remark 3.4. The statement of Proposition 3.3 is valid when E_0 and E_1 are interchanged, i.e.,

$$(\ell^p, E_1) = E_1, \ \forall\ 0 < p \leq \infty.$$

The proof is analogous.

4. Coefficient multipliers between E_0 and E_1

In this section we study conditions for a given sequence to be a coefficient multiplier for E_0, E_1 as well as between these spaces.

Proposition 4.1. *A sequence (u_k) is the multiplier for the space E_0 if and only if (u_k) satisfies condition* (2.8)*. In other words,*

$$(E_0, E_0) = E_1.$$

Proof. Let $(u_k) \in (E_0, E_0)$. Assume that $(u_k) \notin E_1$, i.e.,

$$\limsup_{k\to\infty} \frac{\log |u_k|}{|\lambda^k|} = +\infty,$$

which means that there exists an increasing sequence of positive integers (k_j) such that

$$\lim_{j\to\infty} \frac{\log |u_{k_j}|}{|\lambda^{k_j}|} = +\infty. \tag{4.1}$$

Then a sequence $\big(c_k\big)$ with

$$c_k = \begin{cases} 1/\left|u_{k_j}\right|, & \text{if } k = k_j,\ j = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

is in E_0, while $\big(c_k u_k\big)$ does not belongs to E_0, a contradiction. Thus $\big(E_0, E_0\big) \subset E_1$. Since the inverse inclusion is obvious, this completes the proof of the proposition.

Proposition 4.2. *A sequence $\big(u_k\big)$ is the multiplier for the space E_1 if and only if $\big(u_k\big)$ satisfies condition* (2.8). *In other words,*

$$\big(E_1, E_1\big) = E_1.$$

Proof. It is trivial that $E_1 \subset \big(E_1, E_1\big)$. Now let $\big(u_k\big) \in \big(E_1, E_1\big)$. Taking $\big(c_k\big) \in E_1$, where $c_k = 1,\ \forall k \geq 1$, we get that $\big(u_k\big) = \big(c_k u_k\big) \in E_1$.

Proposition 4.3. *A sequence $\big(u_k\big)$ is the multiplier from the space E_0 to the space E_1 if and only if $\big(u_k\big)$ satisfies condition* (2.8). *In other words,*

$$\big(E_0, E_1\big) = E_1.$$

Proof. Since $E_0 \subset E_1$ we have $\big(E_0, E_0\big) \subset \big(E_0, E_1\big)$ and therefore, by Proposition 4.1, $E_1 = (E_0, E_0) \subset \big(E_0, E_1\big)$. We now prove that $\big(E_0, E_1\big) \subset E_1$.

Let $\big(u_k\big) \in \big(E_0, E_1\big)$. Assume that $\big(u_k\big) \notin E_1$. By (4.1) we have

$$\lim_{j\to\infty} \frac{\log|u_{k_j}|}{|\lambda^{k_j}|} = +\infty.$$

For a sequence

$$c_k = \begin{cases} 1/\sqrt{\left|u_{k_j}\right|}, & \text{if } k = k_j,\ j = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

we see that $\big(c_k\big) \in E_0$. However,

$$\limsup_{k\to\infty} \frac{\log|c_k u_k|}{|\lambda^k|} = \limsup_{j\to\infty} \frac{\log|c_{k_j} u_{k_j}|}{|\lambda^{k_j}|} = \frac{1}{2} \lim_{j\to\infty} \frac{\log|u_{k_j}|}{|\lambda^{k_j}|} = +\infty,$$

which shows that $\big(c_k u_k\big) \notin E_1$, a contradiction.

Proposition 4.4. *A sequence* $\big(u_k\big)$ *is the multiplier from the space* E_1 *to the space* E_0 *if and only if* $\big(u_k\big)$ *satisfies condition* (2.9)*. In other words,*

$$\big(E_1, E_0\big) = E_0.$$

Proof. It is obvious that $E_0 \subset \big(E_1, E_0\big)$. For the inverse inclusion take $\big(u_k\big) \in (E_1, E_0)$. Then for a sequence $\big(c_k\big)$ with $c_k = 1, \ \forall\, k \geq 1$, which is in E_1 we obtain that $\big(u_k\big) = \big(c_k u_k\big) \in E_0$. The proposition is proved.

References

1. J. M. Anderson and A. L. Shields, *Coefficient multipliers of Bloch functions*, Trans. Amer. Math. Soc. **224** (1976), 255-265.
2. J. Burbea and S. Y. Li, *Weighted Hadamard products of holomorphic functions in the ball*, Pacific J. Math. **168** (1995), 237-270.
3. D. J. H. Garling, *The β-and γ-duality of sequence spaces*, Proc. Cambridge Philos. Soc. **63** (1967), 963-981.
4. C. N. Kellogg, *An extension of the Hausdorff-Young theorem*, Michigan Math. J. **18** (1971), 121-127.
5. Yu. F. Korobeinik, *Representing systems* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 325-355. English transl. in Math. USSR Izv. **12** (1978), 309-335.
6. G. Köthe, *Topological Vector Spaces* I, Springer-Verlag, Berlin - New York, 1969.
7. G. D. Levsina, *On coefficient multipliers of Lipshitz functions* (Russian), Mat. Zametki **52** (1992), 68-77.
8. Lê Hai Khôi, *Espaces conjugués, ensembles faiblement suffisants discrets et systèmes de représentation exponentielle*, Bull. Sci. Math. (2) **113** (1989), 309-347.
9. Lê Hai Khôi, *Holomorphic Dirichlet series in several variables*, Math. Scand. **77** (1995), 85-107.
10. Lê Hai Khôi, *Matrix transformations of entire Dirichlet series*, Vietnam J. Math. **24** (1996), 109-112.
11. Lê Hai Khôi, *Convolution operators on holomorphic Dirichlet series*, Tokyo J. Math. **20** (1997), 125-138.
12. Lê Hai Khôi, *Holomorphic Dirichlet series in the half plane*, Vietnam J. Math. **26** (1998), 259-271.
13. Lê Hai Khôi, *Multipliers for Dirichlet series in the complex plane*, SEA Bull. Math. **23** (1999), 33-42.
14. M. Mateljevic and M. Pavlovic, *Multipliers of H^p and BMOA*, Pacific J. Math. **146** (1990), 71-84.
15. J. M. Shi, *Duality and multipliers for mixed norm spaces in the ball* (II), Complex Variables **25** (1994), 131-157.

16. J. B. Xiao, *Coefficient multipliers of H^p spaces over bounded symmetric domains in* $\mathbf{C}^n$, Science in China (Series A) **38** (1995), 171-181.

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