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A COMPLEXITY CHARACTERISTIC OF PETRI NET LANGUAGES

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ABSTRACT. A new complexity characteristic of Petri net languages is introduced. Some necessary conditions for Petri net languages are established and a series of simple languages not acceptable by Petri nets are given.

1. Introduction

The notion of Petri net was first introduced by C. A. Petri in his model of parallel and distributed computing systems. In the past few years, the theory and application of Petri nets were investigated extensively by many authors, (see, for example, [11, 12, 13, 14, 15]).

In this paper we are concerned with a new complexity characteristic of Petri net languages. Analogous complexity characteristics were earlier considered, for instance, by Myhill-Nerode for the languages of finite automata [1], by S. N. Cole for iterative array of finite automata languages [2], by Phan Dinh Dieu and the author of the present paper for probabilistic finite automaton and probabilistic automaton with a time-variant structure languages [3, 4, 5].

The definitions of Petri nets and of languages acceptable by them are recalled in Section 2. In Section 3, a new complexity characteristic of Petri net languages is introduced. An estimation of this characteristic allows us to formulate some necessary conditions for Petri net languages from which a series of rather simple languages not acceptable by Petri nets is derived. The proof of the above mentioned necessary conditions are given in Section 4. Finally, in Section 5, some extensions are considered.

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2. Notations and definitions

We first recall some necessary notions and definitions. For a finite alphabet Σ, Σ^* (Σ^r) denotes the set of all words (resp. of all words of length r) in the alphabet Σ. Let Λ denote the empty word. For any word $\omega \in \Sigma^*$, $l(\omega)$ denotes the length of ω. Every subset $L \subseteq \Sigma^*$ is called a language over the alphabet Σ. Let N be the set of all non-negative integers and $N^+ = N \backslash \{0\}$.

Definition 1. A free-labeled Petri net $\mathcal{N}$ is given by a list

$$\mathcal{N} = (P, T, I, O, \mu_0, M_f),$$

where
$P = \{p_1, ..., p_n\}$ is a finite set of *places*;
$T = \{t_1, ..., t_m\}$ is a finite set of *transitions*, $P \cap T = \emptyset$;
$I : P \times T \to N$ is the *input function*;
$O : T \times P \to N$ is the *output function*;
$\mu_0 : P \to N$ is the *initial marking*;
$M_f = \{\mu_{f_1}, ..., \mu_{f_k}\}$ is a finite set of *final marking*.

Definition 2. A marking μ (global configuration) of a Petri net $\mathcal{N}$ is a function $\mu : P \to N$ from the set of places to N.

The marking μ can also be defined as a n-vector $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_i = \mu(p_i)$ and $|P| = n$.

Definition 3. A transition $t \in T$ is said to be *firable at the marking* μ iff

$$\forall p \in P : \mu(p) \geq I(p, t).$$

Let t be firable at μ and if t fires, then the Petri net $\mathcal{N}$ shall change its state from marking μ to a new marking μ' which is defined as follows:

$$\forall p \in P : \mu'(p) = \mu(p) - I(p, t) + O(t, p).$$

We set $\delta(\mu, t) = \mu'$ and the function δ is said to be the *function of changing state of the net*.

A *firing sequence* can be defined as a sequence of transitions such that the firing of each its prefix will be led to a marking at which the following transition will be firable. By $\mathcal{F}_{\mathcal{N}}$ we denote the set of all firing sequences of the net $\mathcal{N}$.

We now extend the function δ for a firing sequence by induction as follows.

Let $t \in T^*, t_j \in T, \mu$ be a marking, at which tt_j is a firing sequence, then

$$\begin{cases} \delta(\mu, \Lambda) & = \mu \\ \delta(\mu, tt_j) & = \delta(\delta(\mu, t), t_j). \end{cases}$$

Definition 4. *The language acceptable by free-labeled Petri net* $\mathcal{N}$ is the set:

$$L(\mathcal{N}) = \{t \in T^* \big| (t \in \mathcal{F}_{\mathcal{N}}) \wedge (\delta(\mu_0, t) \in M_f)\},$$

The set of all free-labeled Petri net languages is denoted by $\mathcal{L}^f$.

3. A complexity characteristic of free-labeled Petri net languages

Let $L \subseteq \Sigma^*$. We define $E_r \pmod L$ to be an equivalence relation in Σ^r given by

$$x_1 E_r x_2 \pmod L \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L$$

for all $x_1, x_2 \in \Sigma^r$.

It is easy to show that the relation $E_r \pmod L$ is reflexive, symmetric and transitive. Therefore, it is an equivalence relation in Σ^r.

We define the function $H_L(r)$ to be the number of equivalence classes determined by the relation $E_r \pmod L$ in Σ^r, i.e.

$$H_L(r) = \operatorname{Rank} E_r (\operatorname{mod} L)$$

Remark that the function $H_L(r)$ is not given by an algorithm in general. Nevertheless, we call $H_L(r)$ a function. The function $H_L(r)$ is considered to be a complexity characteristic of the language L over Σ^r and in the sequel, we shall use it for formulating some necessary conditions for Petri net languages.

First we consider the case of $\mathcal{L}^f$ (class of free-labeled languages), and then in the Section 5 we shall consider the case of labeled Petri net languages.

Theorem 1. *Let $L \in \mathcal{L}^f$ be accepted by a free-labeled Petri net with m transitions. Then, there exists a polynomial P_m of degree m such that for any integer $r \geq 1$, we have*

$$H_L(r) \leq P_m(r).$$

The proof of Theorem 1 will be presented in Section 4.

By using the above necessary condition, we can give many simple examples of languages which do not belong to the class of free-labeled Petri net languages.

Example 1. Let Σ be an alphabet consisting of $k \geq 2$ letters and

$$L_1 = \{xx \big| x \in \Sigma^*\}.$$

We shall prove that $L_1 \notin \mathcal{L}^f$. First we will show that if $x_1, x_2 \in \Sigma^r$, $x_1 \neq x_2$, then $x_1 \overline{E}_r x_2 \,(\mathrm{mod}\, L_1)$. In fact, assume to the contrary that there exists $x_1, x_2 \in \Sigma^r$, $x_1 \neq x_2$, but $x_1 E_r x_2 \,(\mathrm{mod}\, L_1)$. According to the definition of $E_r \,(\mathrm{mod}\, L_1)$ we have

$$\forall \omega \in \Sigma^* : x_1\omega \in L_1 \leftrightarrow x_2\omega \in L_1$$

Now if we choose $\omega = x_1$, then

$$x_1 x_1 \in L_1 \quad \text{but} \quad x_2 x_1 \notin L_1$$

This contracdicts the hypothesis $x_1 E_r x_2 \,(\mathrm{mod}\, L_1)$, so if $x_1 \neq x_2$ then $x_1 \overline{E}_r x_2 \,(\mathrm{mod}\, L_1)$. In the other word, $H_{L_1}(r) = \mathrm{Rank}\, E_r(\mathrm{mod}\, L_1)$ is equal at least the cardinality of the set Σ^r, i.e. $H_{L_1}(r) \geq k^r$. For any $m \in N^+$, if we take r enough large, we shall have

$$H_{L_1}(r) \geq k^r \geq P_m(r) \quad (k \geq 2).$$

According to the Theorem 1, the language $L_1 \notin \mathcal{L}^f$.

Example 2. Let Σ be an alphabet consisting of $k \geq 2$ letters and

$$L_2 = \{xx^R \big| x \in \Sigma^*\},$$

where x^R is the inverse image of x. If $x = x_1 x_2 \dots x_j$, then $x^R = x_j x_{j-1} \dots x_1$.

Example 3. Let $\Sigma = \{0, 1, a\}$ and

$$L_3 = \{\omega a^k \big| \omega \in \{0,1\}^*, k = B(\omega)\},$$

where $B(\omega)$ is the integer represented by ω as a binary number. By an argument analogous to that in Example 1, it is easy to show that $L_2 \notin \mathcal{L}^f$ and $L_3 \notin \mathcal{L}^f$.

Example 4. Let Σ be an alphabet consisting $k \geq 2$ letters and

$$L_{4,l} = \{\tau_1\tau_2\cdots\tau_n\tau_0 \big| \forall i : \tau_i \in \Sigma^*, l(\tau_i) = l = \text{const}; \exists\tau_i = \tau_0\}.$$

We shall prove that for l enough large, $L_{4,l} \notin \mathcal{L}^f$. In fact, each subset $W = \{P_1, P_2, \cdots, P_q\} \subseteq \Sigma^l$ is associated with a word

$$\alpha(W) = P_1P_2\cdots P_q \underbrace{P_q\cdots P_q}_{k^l - q times} \in \Sigma^r,\ r = l.k^l.$$

It is easy to verify that

$$\alpha(W)\omega \in L_{4,l} \leftrightarrow \omega \in W.$$

Thus, if $W_1 \neq W_2$ then $\alpha(W_1)\overline{E}_r\alpha(W_2)\,(\operatorname{mod} L_{4,l})$. Therefore, $H_{L_{4,l}}(r) \geq 2^{k^l} = C^r$ with $C = 2^{\frac{1}{l}}$ because $r = l.k^l$. So, r is enough large when l is enough large and we have

$$H_{L_{4,l}} \geq C^r \geq P_m(r).$$

According to Theorem 1, the language $L_{4,l} \notin \mathcal{L}^f$.

Finally, we extend the range of application for Theorem 1.

Let $\Sigma^{\leq r} = \bigcup\limits_{i=1}^{r} \Sigma^i$, and $L \subseteq \Sigma^*$. We define the equivalence relation $E_{\leq r}\,(\operatorname{mod} L)$ by

$$x_1E_{\leq r}x_2\,(\operatorname{mod} L) \Leftrightarrow \forall\omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L$$

for all $x_1, x_2 \in \Sigma^{\leq r}$ and $G_L(r) = \operatorname{Rank} E_{\leq r}\,(\operatorname{mod} L)$ in $\Sigma^{\leq r}$.

Theorem 1 now will be extended as follows.

Theorem 2. *Let $L \in \mathcal{L}^f$ be accepted by a free-labeled Petri net with m transitions. Then, there exists a polynomial P_{m+1} of degree $m+1$ such that for any integer $r \geq 1$, we have*

$$G_L(r) \leq P_{m+1}(r).$$

The proof of Theorem 2 will be presented at the end of Section 4.

Example 5. Let Σ be an alphabet consisting $k \geq 2$ letters and $c \notin \Sigma$. We define

$$L_{5,l} = \{\tau_1 c \tau_2 c \cdots c \tau_n c \tau_0 / \forall i : \tau_i \in \Sigma^*, l(\tau_i) \leq l = \text{const}; \exists \tau_i = \tau_0\}.$$

Using Theorem 2 we shall prove that $L_{5,l} \notin \mathcal{L}^f$ for l enought large. As in Example 4, each subset $W = \{P_1, P_2, \cdots, P_q\} \subseteq \Sigma^{\leq l}$ is associated with a word

$$\alpha(W) = P_1 c P_2 c \cdots c P_q c \in \Sigma^{\leq r},$$

where $r = (|\Sigma^{\leq l}|)(l+1) = \big(\dfrac{k(k^l - 1)}{k-1}\big)(l+1)$ with $k = |\Sigma|$. It is easy to see that

$$\alpha(W)\omega \in L_{5,l} \leftrightarrow \omega \in W$$

Therefore

$$G_{L_{5,l}}(r) \geq 2^{|\Sigma^{\leq l}|} = 2^{\frac{1}{l+1}(|\Sigma^{\leq l}|)(l+1)} = C^r$$

with $C = 2^{\frac{1}{l+1}}$. Thus, if l is enough large then so does r and we have

$$G_{L_{5,l}} \geq C^r \geq P_{m+1}(r).$$

According to Theorem 2, $L_{5,l} \notin \mathcal{L}^f$.

4. Proofs of theorems 1 and 2

Proof of Theorem 1. The proof is divided into 4 steps:

Step 1: Establishing an equation of changing state of a Petri net. Let $\mathcal{N} = (P, T, I, O, \mu_0, M_f)$, where $|P| = n; |T| = m$. We define the matrices I^- , O^+ , D as follows:

$$I^-[j,i] = (I(p_i, t_j))_{m \times n},$$

$$O^+[j,i] = (O(t_j, p_i))_{m \times n},$$

$$D = O^+ - I^-,$$

and set:

$$e[j] = (0, \cdots, 0, \underbrace{1}_{j-th}, 0, \cdots, 0)_{1 \times m}.$$

It is easy to verify the following properties:

- Transition t_j is firable at marking μ if

$$\mu \geq e[j]I^{-}.$$

- The result of firing for firable transition t_j at marking μ is

$$\mu' = \delta(\mu, t_j) = \mu - e[j]I^{-} + e[j]O^{+} = \mu + e[j]D.$$

- The result of firing for a firing sequence of transitions $x = t_{j_1} t_{j_2} \cdots t_{j_r}$ is

$$\mu' = \delta(\mu, x) = \mu + e[j_1]D + \cdots + e[j_r]D.$$

We set $e[j]D = \nu_j$, $j = 1, \cdots, m = |T|$, and f_j is the number of times the transition t_j occurs in x. We can now express the equation of changing state in the following form

$$\mu' = \delta(\mu, x) = \mu + \sum_{j=1}^{m} f_j \nu_j \tag{1}$$

$$\sum_{j=1}^{m} f_j = r \tag{2}$$

Step 2: Estimating the number of reachable states of a Perti net after a sequence of r transitions fired. Let S_r denote the set of all reachable states after firing r transitions of a Petri net. From the system of equations (1) - (2) it is clear that $|S_r|$ equals at most the number of non-negative integer solutions of the equation $\sum_{j=1}^{m} f_j = r$. Each distinct solution of the equation $\sum_{j=1}^{m} f_j = r$ is one-to-one associated with an unordered repetition sample of size r from m-element set $\{f_1, f_2, \ldots, f_m\}$. It follows that the number of distinct solutions of $\sum_{j=1}^{m} f_j = r$ equals the number of unordered repetion samples of size r from m-element set and equals

$$C^r_{m+r-1} = \frac{(m+r-1)!}{(m-1)!r!} = \frac{(m+r-1)\cdots(r+1)}{(m-1)!} \leq (m+r)^m.$$

Therefore $|S_r| \leq P_m(r)$.

Step 3: Extending the partial function δ to a total function over T^r. We assume that μ is fixed, $x \in T^r$, and we remark that the function $\delta(\mu, x)$ is only defined with firing sequence x of $\mathcal{N}$. We could extend δ to a totally defined $\tilde{\delta}$ by adding a new marking μ_ϵ defined as follows

- If x is a firing sequence of $\mathcal{N}$ at μ, then

$$\tilde{\delta}(\mu, x) = \delta(\mu, x).$$

- If x is not a firing sequence of $\mathcal{N}$ at μ, then

$$\tilde{\delta}(\mu, x) = \mu_\varepsilon.$$

- For all $x \in T^r$, $\tilde{\delta}(\mu_\varepsilon, x) = \mu_\varepsilon$.
- $\mu_\varepsilon \notin M_f$.

Set $\tilde{S}_r = S_r \cup \{\mu_\epsilon\}$. Then $\tilde{\delta}$ is a totally defined funtion from T^r into $\tilde{S}_r$ and $|\tilde{S}_r| = |S_r| + 1$. Hence $|\tilde{S}_r| \leq P_m(r)$.

Step 4: Proving the inequality $H_L(r) \leq |\tilde{S}_r|$. Assume to the contrary that $H_L(r) > |\tilde{S}_r|$, where $L = L(\mathcal{N})$. There exist $x_1, x_2 \in T^r$ such that $x_1 \overline{E}_r x_2 \pmod L$ and $\tilde{\delta}(\mu_0, x_1) = \tilde{\delta}(\mu_0, x_2)$. It follows from the last equation that both x_1, x_2 are (or are not) firing sequences and we could verify that

$$\forall \omega \in T^* : x_1\omega \in L \leftrightarrow x_2\omega \in L.$$

According to the definition, it follows that $x_1 E_r x_2 \pmod L$ which contradicts the hypothesis $x_1 \overline{E}_r x_2 \pmod L$. Therefore

$$H_L(r) \leq |\tilde{S}_r| \leq P_m(r).$$

So Theorem 1 is proved.

Proof of Theorem 2. In this case, the equation of state change (1)-(2) has the following form

$$\mu' = \delta(\mu, x) = \mu + \sum_{j=1}^{m} f_j \nu_j, \tag{1}$$

$$\sum_{j=1}^{m} f_j \leq r. \tag{3}$$

We can prove that the number of distinct solutions of the equation $\sum_{j=1}^{m} f_j \leq r$ equals the one of the equation $\sum_{j=1}^{m+1} f_j = r$ and equals

$$C^r_{(m+1)+r-1} \leq P_{m+1}(r).$$

We set $S_{\leq r} = \bigcup_{i=1}^{r} S_i$. It follows that

$$|S_{\leq r}| = P_{m+1}(r).$$

The rest of the proof is analogous to that of Theorem 1.

5. An extension to labeled petri net languages

We recall some definitions for labeled Petri net.

Definition 5. A labeled Petri net is a system

$$\mathcal{N} =< P, T, I, O, \sigma, \mu_0, M_f >,$$

where P, T, I, O, μ_0, M_f are the sames in the Definition 1.

Let $\sigma : T \to \Sigma$, where Σ is a finite alphabet (output alphabet). We also extend the function σ for a firing sequence as follows:

$$\text{if} \quad t = t_1 t_2 ... t_n \quad \text{then} \quad \sigma(t) = \sigma(t_1)\sigma(t_2)...\sigma(t_n).$$

Definition 6. The language acceptable by labeled Petri net $\mathcal{N}$ is a set

$$L(\mathcal{N}) = \{x \in \Sigma^* / \exists t \in T^* : (x = \sigma(t)) \wedge (t \in \mathcal{F}_{\mathcal{N}}) \wedge (\delta(\mu_0, t) \in M_f)\}.$$

The set of all labeled Petri net languages is denoted by $\mathcal{L}$. It is clear that $\mathcal{L}^f \subseteq \mathcal{L}$.

Now we generalize Theorems 1 and 2 for labeled Petri net languages.

Theorem 3. *If $L \in \mathcal{L}$ be accepted by a labeled Petri net with m transitions. Then, there exist polynomials P_m and P_{m+1} such that for any integer $r \geq 1$, we have*

$$H_L(r) \leq P_m(r),$$
$$G_L(r) \leq P_{m+1}(r).$$

Proof (Sketch). It consists of the following steps:

- Remark that σ is a non-erasing mapping. If $x = \sigma(t)$, then $l(x) = l(t)$. Let S_r denote the set of all reachable marking after firing r transitions and $S_{\leq r} = \bigcup_{i=1}^{r} S_i$, $\tilde{S}_r = S_r \cup \{\mu_\epsilon\}$. Let μ_ε be a new added marking, $\mu_\varepsilon \notin S_r$; $\tilde{S}_{\leq r} = S_{\leq r} \cup \{\mu_\varepsilon\}$.

- From $\sigma^{-1} : \Sigma^r \to T^r$ and $\tilde{\delta} : T^r \to \tilde{S}_r$ we define $\Delta : \Sigma^r \to \tilde{S}_r$ by $\Delta = \tilde{\delta}.\sigma^{-1}$. The mapping Δ may not be defined over all Σ^r. We could extend Δ to a totally defined function $\tilde{\Delta}$ as follows. For any $x \in \Sigma^r$, if $\sigma^{-1}(x)$ exists then

$$\tilde{\Delta}(\mu_0, x) = \tilde{\delta}(\mu_0, \sigma^{-1}(x)).$$

If $\sigma^{-1}(x)$ does not exist then

$$\tilde{\Delta}(\mu_0, x) = \mu_\varepsilon.$$

- From the inequalities $|\tilde{S}_r| \le P_m(r)$, $|\tilde{S}_{\le r}| \le P_{m+1}$ and arguing as in proof of Theorems 1 and 2 we can prove that

$$H_L(r) \le |\tilde{S}_r| \le P_m(r),$$

$$G_L(r) \le |\tilde{S}_{\le r}| \le P_{m+1}(r).$$

The proof is completed.

As an immediate consequence of Theorem 3 and of the estimate of the functions $H_{L_1}(r)$, $H_{L_2}(r)$, $H_{L_3}(r)$, $H_{L_{4,l}}(r)$ and $G_{L_{5,l}}(r)$ in Section 3, we can see that the languages L_1, L_2, L_3, $L_{4,l}$, $L_{5,l}$ do not belong to $\mathcal{L}$.

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