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A CLASSIFICATION OF A CLASS OF MARTINGALE-LIKE SEQUENCES

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Abstract. A sequence (X_n) is said to be a game fairer with time if for every $\varepsilon > 0$ we have $\lim_n \sup_{m \geq n} P\Big(||E_n(X_m) - X_n|| > \varepsilon\Big) = 0$. It is known that every L^1-bounded Banach space-valued game fairer with time has a unique Riesz-Talagrand decomposition: $X_n = M_n + P_n$, where (M_n) is a uniformly integrable martingale and (P_n) converges to zero in probability. The aim of this note is to give a classification of a class of martingale-like sequences considerably more general than games fairer with time for which the above Riesz-Talagrand decomposition still holds.

1. Notations and definitions

Throuthout this note let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $(\mathcal{A}_n)$ an increasing sequence of complete sub-σ-fields of $\mathcal{A}$ and T the set of all bounded stopping times w.r.t. $(\mathcal{A}_n)$. Given a separable Banach space E we denote by $L^1(E)$ the Banach space of all (equivalence classes of) $\mathcal{A}$-measurable and Bochner integrable functions $X : \Omega \to E$ with the L^1-norm: $E(||X||) = \int_\Omega ||X|| dP < \infty$. Unless otherwise stated we shall consider only sequences (X_n) in $L^1(E)$ which are assumed to be adapted to $(\mathcal{A}_n)$, i.e. each X_n is $\mathcal{A}_n$-measurable. Let (X_n) be such a sequence and $\tau \in T$. We define the function X_τ and subset $\mathcal{A}_\tau$ of $\mathcal{A}$ by

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega) \quad \text{and} \quad \mathcal{A}_\tau = \big\{A \in \mathcal{A},\ A \cap \{\tau = n\} \in \mathcal{A}_n\big\}.$$

Then it is known (cf. [8]) that every $\mathcal{A}_\tau$ is a complete sub-σ-field of $\mathcal{A}$ and each X_τ is $\mathcal{A}_\tau$-measurable. For other related notions of martingale-like sequences, the reader is referred to the recent monograph of Edgar and Sucheston [2]. Here we recall only the following.

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Definition 1.1. A sequence (X_n) is said to be

(a) *a martingale* if $X_n(m) = X_n$ for all $m \geq n$, where given σ, $\tau \in T$ with $\sigma \leq \tau$, $X_\sigma(\tau)$ denotes the $\mathcal{A}_\sigma$-conditional expectation of X_τ.

(b) *a pramart* if for every $\varepsilon > 0$ there exists p such that for all σ, $\tau \in T$ with $p \leq \sigma \leq \tau$ we have

$$P\big(||X_\sigma(\tau) - X_\sigma|| > \varepsilon\big) < \varepsilon.$$

(c) *a martingale in the limit* if

$$\lim_n \sup_{m \geq n} ||X_n(m) - X_n|| = 0, \quad \text{a.s.}$$

(d) *a mil* if for every $\varepsilon > 0$ there exists p such that for all $n \geq p$ we have

$$P\left(\sup_{p \leq q \leq n} ||X_q(n) - X_q|| > \varepsilon\right) < \varepsilon.$$

(e) *a game which becomes fairer with time* if for every $\varepsilon > 0$ there exists p such that for all $n \geq p$ we have

$$\sup_{p \leq q \leq n} P\big(||X_q(n) - X_q|| > \varepsilon\big) < \varepsilon.$$

Martingales in the limit were first introduced by Mucci (1976), pramarts by Millet and Sucheston (1980) and mils by Talagrand (1985). But games which become fairer with time were earlier introduced and considered by Blake (1970). Clearly, by definition every above-mentioned class of martingale-like sequences is contained in the next one. Moreover, it is also known that any converse inclusion fails even in the real-valued case. The first result on games fairer with time is due to Blake [1] who proved that every real-valued game fairer with time, uniformly a.s. bounded by an integrable function converges in L^1. Three years later, this result was (independently) extended by Subramanian [9] and Mucci [6] to the uniformly integrable case. Recently, applying the structure results of Talagrand [10] we have proved in [3] that every L^1-bounded E-valued game fairer with time (X_n) has a unique Riesz-Talagrand decomposition: $X_n = M_n + P_n$, where (M_n) is a uniformly integrable martingale and the potential (P_n) goes to zero in probability. As a continuation of [3,4] we shall consider in this note a classification of the following class of martingale-like sequences.

Definition 1.2. A sequence (X_n) is called a *quasi-game fairer with time,* briefly a *quasi-game* (cf. [4]) if for every $\varepsilon > 0$ there exists p such that for every $m \geq p$ there is $p_m \geq m$ such that for all $n \geq p_m$ we have

$$\sup_{p \leq q \leq m} P\big(||X_q(n) - X_q|| > \varepsilon\big) < \varepsilon. \tag{2.1}$$

It is clear that by definition, every game fairer with time is a quasi-game. We shall see in the next section that the class of quasi-games considerably generalises that of games fairer with time. Moreover, one can classify it into an increasing generalized family of subclasses for which the above Riesz-Talagrand decomposition still holds.

2. Main results.

The first result we begin with is the following characterization of quasi-games, where F denotes the set of all functions f from N to N.

Lemma 2.1. *A sequence (X_n) is a quasi-game if and only if there exists an $f \in F$ such that (X_n) is a game of size f, i.e. for every $\varepsilon > 0$ there exists p such that for every $m \geq p$ and $n \geq m + f(m)$ we have*

$$\sup_{p \leq q \leq m} P\big(||X_q(n) - X_q|| > \varepsilon\big) < \varepsilon. \tag{2.2}$$

Proof. Suppose first that a sequence (X_n) is a game of size f for some $f \in F$. Then by taking each $p_m = m + f(m)$, it follows that the sequence (p_m) does not depend on the choice of ε and (2.2) implies (2.1) automatically. This proves the sufficiency of the statement. The interest of the result consists in showing the necessity of the condition. For this purpose, let (X_n) be a quasi-game. Then by the same definition, one can construct a strictly increasing sequence $\big(p(n)\big)$ such that for every k there exists an increasing sequence $\big(p_n(k)\big)$ such that for every $m \geq p(k)$ and $n \geq p_m(k)$ we have

$$\sup_{p(k) \leq q \leq m} P\big(||X_q(n) - X_q|| > 1/k\big) < 1/k. \tag{2.3}$$

Now define the function $f \in F$ as follows. For $m < p(1)$ set $f(m) = 1$ and for every $m \geq p(1)$, i.e., $p(q) \leq m < p(q+1)$ for some $q \geq 1$, set

$$f(m) = \max\{p_m(s),\ s \leq q\} - m.$$

We claim that the sequence (X_n) must be a game of size f. Indeed, let $\varepsilon > 0$ be given. Then $1/k < \varepsilon$ for some k. Thus if $m \geq p(k)$ and $n \geq m + f(m)$ we have

$$p(k + j - 1) \leq m < p(k + j)$$

for some $j \geq 1$, hence it is evident that

$$n \geq m + f(m) = \max\{p_m(s),\ s \leq k + j - 1\} \geq p(k).$$

This with (2.3) implies that

$$\sup_{p(k) \leq q \leq m} P\big(||X_q(n) - X_q|| > \varepsilon\big) < \varepsilon.$$

So we obtain (2.1) by taking $p = p(k)$ and $p_m = p_m(k)$. The proof is then completed.

Now let define the following partial order $(<^*)$ on F as follows: $f <^* g$ iff card$\{g \leq f\}$ is finite. Then we get the following classification.

Theorem 2.2. *When f runs over $(F, <^*)$, the set of all quasi-games is classified into an inreasing family of games of size f. Moreover, if $(\Omega, \mathcal{A}, P)$ is a nonatomic probability space then there exists a stochastic basis $(\mathcal{A}_n)$ such that for any f, $g \in F$ with $f <^* g$, the class of real-valued games of size f is strictly contained in that of games of size g.*

Proof. The first statement follows from Lemma 2.1. Without any loss of generality we prove (for simplicity) the second statement only for the case where $(\Omega, \mathcal{A}, P)$ is the Lebesque probability space on $[0, 1)$ and f, $g \in F$ with $f(m) < g(m)$, $m \in N$. To do this, for every $n \in N$ let $a_n = \prod_{j \leq n} 2^j$, Q_n the partition of $[0, 1)$ in a_n intervals of equal length, $\mathcal{A}_n$ the σ-field generated by Q_n and

$$a(g) = \min\{m + g(m),\ m \in N\}.$$

Further, let define the sequence (X_n) of real-valued functions as follows. For $n < a(g)$ set $X_n = 0$, for $n \geq a(g)$ set

$$b_n = \max\{m, m + g(m) \leq n\}, \quad c_n = \max\{m \in N,\ m + f(m) \leq n\}$$

and put

$$X_n = a_n/a_{b_n+1} \quad \text{or} \quad X_n = -a_n/a_{b_n+1}$$

on the first interval of Q_n which is contained in the $(2p-1)$-th or $(2p)$-th interval in Q_{b_n+1}, resp., for all $1 \leq p \leq a_{b_n+1}/2$. It is easily checked that defined in such a way, the sequence (X_n) has the following typical properties:

(a) For every $m \geq a(g)$, $n \geq m + g(m)$ and $q = b_n + 1$ we have

$$X_q(n) = 1 \quad \text{or} \quad = -1$$

on the $(2p-1)$-th or $(2p)$-th interval of Q_{b_n+1}, resp., for all $1 \leq p \leq a_{b_n+1}/2$. Hence $X_q(n) = 0$, for all $q < b_n + 1$.
On the other hand, since $g(m) \geq 2$, $m \in N$, for all $n \geq a(g)$ we have

$$b_n + 2 \leq n - g(b_n) + 2 \leq n$$

and

$$P\big(\{X_n \neq 0\}\big) = a_{b_n+1}/a_n = \prod_{j=b_n+2}^{n} 2^{-j} \leq 2^{-n}.$$

This implies
(b) (X_n) converges to zero, a.s.

Now let $0 < \varepsilon < 1$, $p \geq a(g)$, $m \geq p$ and $n \geq m + g(m)$ be given. Then $n \geq a(g)$ and by the definition of b_n we have

$$m \leq b_n < b_n + 1.$$

This with (a) yields that for all $p \leq q \leq m$ we have

$$P\big(||X_q(n) - X_q|| > \varepsilon\big) = P\big(||X_q|| > \varepsilon\big).$$

Therefore by (b), (X_n) must be a game of size g. To show that (X_n) is not a game of size f, it should be noted that each $b_n \leq c_n$ and (b_n) increases to infinity when n runs to infinity. Let $\left(c_n^1\right)$ denote the strict increasing subsequence of (c_n) such that $\left(c_n^1\right)$ contains all elements of (c_n). Then by setting each $n_k = c_k^1 + f\left(c_k^1\right)$, the subsequence (n_k) also stricly increases. Moreover, by definition we have $c_n = c_k^1$ for all $n_k \leq n < n_{k+1}$, $k \in N$. Finally, given $p \in N\big(a(g)\big)$ let k_1 denote the first index k such that $p \leq c_k^1$. Then for all $k \geq k_1$ we have

$$n_k = c_k^1 + f\left(c_k^1\right) < c_k^1 + g\left(c_k^1\right).$$

Suppose first that $b_{n_k} = c_k^1 = c_{n_k}$. Then by definition we have

$$n_k \geq b_{n_k} + g\left(b_{n_k}\right) = c_k^1 + g\left(c_k^1\right) > c_k^1 + f\left(c_k^1\right) = n_k.$$

It is imposible. Thus $b_{n_k} < c_k^1$, hence $b_{n_k} + 1 \leq c_{n_k}$. Consequently, by taking

$$q = b_{n_k} + 1,\ m = c_{n_k},\ n = m + f(m),$$

we have

$$P\big(||X_q(n) - X_q|| > \varepsilon\big) \geq 1 - P\big(||X_q|| > \varepsilon\big).$$

This with (b) implies that (X_n) fails to be a game of size f, proving the theorem.

In the simplest cases, given $k \in N$, take $f(m) = k$ and $g_1(m) = k + 1$ or $g_2(m) = m$, $m \in N$. Then it is easily checked that

$$\begin{aligned} &f <^* g_i,\ i = 1, 2;\ a(g_1) = 3,\ a(g_2) = 2;\\ &b_n^1 = \max\{m, m + g_1(m) \leq n\} = n - (k + 1) \end{aligned}$$

and

$$b_n^2 = \max\{m, m + g_2(m) \leq n\} = [n/2], \quad n \in N,$$

where $[a]$ denotes the integer part of the real number a. Thus, if in the above definition of X_n we replace b_n by b_n^1 or b_n^2, resp., then the construction in the proof of the theorem gives the following.

Example 2.3.
(a) For every $k \in N$, there exists a real-valued game of size $k + 1$ which is not a game of size k, hence it is not a game fairer with time either.
(b) There exists a real-valued game of size g for some $g \in F$ such that it is not a game of size k for any $k \in N$.

However, it is easily checked that the same proof of Lemma 2.2 [3] can be applied to establish the following property of quasi-games.

Property 2.4. *Every quasi-game contains a subsequence which is a mil.*

Further, by mixing the "upcroasing" method due to J. L. Doob's early contribution and the recent stopping time technique we are able to prove the following new result.

Theorem 2.5. *Let (X_n) be a quasi-game with*

$$\lim_n \inf E\big(||X_n||\big) < \infty. \tag{2.4}$$

Then (X_n) converges to zero in probability if (and only if) it contains a subsequence, take (X_{n_k}) which converges to zero in probability.

Proof. By Lemma 2.1 there exists some $f \in F$ such that (X_n) is a game of size f. Assume to the contrary that (X_n) does not converge to zero in probability. Then there exists $a > 0$ such that

$$\sup_{q \geq n} P\big(||X_q|| > 5a/4\big) > a, \quad n \in N. \tag{2.5}$$

We claim first that for every $0 < \varepsilon < a/4$ and $m_1 \in N$ there exists $m_2 > m_1$ such that for every $A \in \mathcal{A}_{m_1}$ with $P(A) < a/4$ and $n \geq m_2 + f(m_2)$ there exists $B \in \mathcal{A}_{m_2}$ with $B \cap A = \emptyset$ and $P(B) < \varepsilon$ such that

$$\int_B ||X_n|| dP \geq a^2/4. \tag{2.6}$$

To prove the claim let $0 < \varepsilon < a/4$ and $m_1 \in N$ be given. Then by the definition of quasi-games of size f, one can find some $p \geq m_1$ such that for all $m \geq p$ and $n \geq m + f(m)$ we have

$$P\big(||X_p|| > 5a/4\big) > a \quad \text{and} \sup_{p \leq q \leq m} P\big(||X_q(n) - X_q|| > a/4\big) < \varepsilon/2. \tag{2.7}$$

Thus, there exists a finite sequence $(y_i, i \leq s)$ of the closed unit ball of E^* such that $P(C^1) > 7a/8$, where

$$C_i^1 = \{(y_i, X_p) > 5a/4\} \setminus \bigcup_{j<i} \{(y_i, X_p) > 5a/4\} \text{ and } C^1 = \bigcup_{i \leq s} C_i^1.$$

On the other hand, by the property of the subsequence (n_k), there is some $m_2 \in \{n_k\}$ with $p \leq m_2$ such that $P(D) < \varepsilon/2$, where

$$D = \{||X_{m_2}|| > a/4\}.$$

Now let $A \in \mathcal{A}_{m_1}$ with $P(A) < a/4$ and $n \geq m_2 + f(m_2)$. Define

$$G = \{||X_p(n) - X_p|| > a/4\} \text{ and } H = \{||X_{m_2}(n) - X_{m_2}|| > a/4\}.$$

Then by (2.7) we have $P(G) < \varepsilon/2$ and $P(H) < \varepsilon/2$. It follows that if we set $C_j^2 = C_j^1 \setminus (A \cup G)$, $j \leq s$ and $C^2 = \bigcup_{j \leq s} C_j^2$ we get

$$P(C^2) > 7a/8 - 3a/8 = a/2.$$

Finally, define

$$D^1 = D \cup H,\ D_j^2 = C_j^2 \setminus D^1,\ B_j = C_j^2 \cap D^1,\ j \leq s \text{ and } B = \bigcup_{j \leq s} B_j.$$

Clearly, $B \in \mathcal{A}_{m_2}$, $B \cap A = \emptyset$ and $P(B) \leq P(D^1) < \varepsilon$. Thus, to prove the claim it remains to show that B also satisfies (2.5). To see this let $j \leq s$ be given. Then we have

$$\int\limits_{C_j^2} (y_j, X_n)dP = \int\limits_{C_j^2} \big(y_j, X_p(n)\big)dP \geq aP(C_j^2)$$

because $C_j^2 \in \mathcal{A}_p$ and on C_j^2 one has $\big(y_j, X_p(n)\big) \geq (y_j, X_p) - a/4 \geq a$. Similarly, because $D_j^2 \in \mathcal{A}_{m_2}$ and on D_j^2 we have

$$\big(y_j, X_{m_2}(n)\big) \leq (y_j, X_{m_2}) + a/4 \leq a/2.$$

Then

$$\int\limits_{D_j^2} (y_j, X_n)dP = \int\limits_{D_j^2} \big(y_j, X_{m_2}(n)\big)dP \leq a/2P\left(C_j^2\right).$$

But note that $B_j \cap D_j^2 = \emptyset$ and $C_j^2 = B_j \cup D_j^2$. This yields

$$\begin{aligned}\int\limits_{B_j} ||X_n||dP &\geq \int_{B_j} (y_j, X_n)dP \\ &\geq \int\limits_{B_j} ||X_n||dP - \int\limits_{D_j^2} (y_j, X_n)dP \geq a/2P\left(C_j^2\right).\end{aligned}$$

Consequently, by summation over $j \leq s$ we get

$$\int\limits_{B} ||X_n||dP \geq a/2P\left(C^2\right) \geq a^2/4.$$

This proves (2.5) and the claim.

Now we are in a position to complete the proof as follows. First applying the claim to construct by induction a strictly increasing sequence (m_p) of N with the following property: whenever $A \in \mathcal{A}_{m_p}$ with $P(A) < a/4$ and $n \geq m_{p+1} + f\left(m_{p+1}\right)$, there exists a set $B \in \mathcal{A}_{m_{p+1}}$ with $B \cap A = \emptyset$ and $P(B) < a2^{-(p+1)}$ and $\int\limits_B ||X_n||dP \geq a^2/4$. Thus, given $p \in N$ and $n \geq \max\{m_i + f(m_i),\ i \leq p\}$ we can construct by finite induction, for $j \leq p$, disjoint sets B_j with $B_1 = \emptyset$, $P(B_j) < a2^{-(p+1)}$ and $\int\limits_{B_j} ||X_n||dP \geq a^2/4$. This implies

$$\int\limits_{B} ||X_n||dP \geq (p-1)a^2/4,$$

where $B = \bigcup_{j \le p} B_j$. Hence $\lim_n E(||X_n||) = \infty$, which contradicts the assumption (2.3), proving the theorem.

Corollary 2.6 ([4], Theorem 3.6). *Let (X_n) be a quasi-game satisfying* (2.3). *Then it admits a unique Riesz-Talagrand decomposition:*

$$X_n = M_n + P_n, \tag{2.8}$$

where (M_n) is a uniformly integrable martingale and (P_n) goes to zero in probability.

Proof. By (2.4) there exists a subsequence (n_k) of N such that the subsequence $(Y_k = X_{n_k})$ is still a quasi-game w.r.t. $(\mathcal{B}_k = \mathcal{A}_{n_k})$ such that

$$\lim_k E(||Y_k||) = \lim_n \inf E(||X_n||) < \infty.$$

Thus, by Property 2.4 one can find a further strictly increasing subsequence (k_p) of N such that the subsequence $\left(Z_p = Y_{k_p}\right)$ is an L^1-bounded mil w.r.t. $\left(\mathcal{C}_p = \mathcal{B}_{k_p}\right)$. Consequently, by the proof of Theorem 8 [10], (Z_p) admits a unique Riesz-Talagrand decomposition:

$$Z_p = L_p + W_p, \ p \in N$$

where (L_p) is a uniformly integrable martingale with

$$||L_p|| \le E_p \left(\lim_q \inf ||Y_{k_q}||\right), \ p \in N,$$

and the potential (W_p) converges to zero, a.s. Now define the sequences (M_n) and (P_m) as follows. For every $m \in N$ let

$$m(p) = \min\left\{q \in N, \ m \le n_{k_q}\right\}, \ M_m = L_m\big(m(p)\big),$$

and

$$P_m = X_m - M_m, m \in N.$$

Then it is clear that

$$X_n = M_n + P_n, \ n \in N,$$

where by the property of the martingale (L_p), (M_n) is also a uniformly integrable martingale. Thus, (P_n) is a quasi-game satisfying the condition:

$$\lim_n \inf E(||P_n||) \le 2 \lim_n \inf E(||X_n||) < \infty.$$

On the other hand, by the property of the sequence (W_p), the subsequence $\left(P_{n_{k_p}}\right)$ converges to zero, a.s., hence so does in probability when p runs to infinity. By Theorem 2.5 the potential (P_n) must converge itself to zero in probability as well. This proves (2.8) and the Corollary.

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