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THE REGULARITY OF SPACES OF GERMS OF FRECHET-VALUED BOUNDED HOLOMORPHIC FUNCTIONS

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Introduction

Several authors have studied the regularity of the space $\mathcal{H}(K)$ of germs of holomorphic functions on a compact set K in a locally convex space, for example, Chae [1] for K in a Banach space, Mujica [6] for K in a metric locally convex space, Dineen [2] and Soraggi [10] for K in certain locally convex spaces (non-necessarily metrizable). Recently, Vogt [14] gave a general characterization of the regularity of inductive limit of a sequence of Frechet spaces, in particular, of Köthe sequence spaces. In the present paper we shall find a necessary and sufficient condition for the space $\mathcal{H}_\infty(K,F)$ of germs of F-valued bounded holomorphic functions on a compact set K in E to be regular where E and F are two Frechet spaces. In the case where K is a compact set in $\mathbf{C}^n$, this problem was investigated in [11]. As an application we will study the regularity and completeness of the space $\mathcal{H}(K,F)$ of germs of F-valued holomorphic functions on K.

1. Preliminaries

We shall use standard notions from the theory of locally convex spaces as presented in the books of Pietsch [7] and Schaefer [9]. All locally convex spaces are assumed to be complex and Hausdorff.

1.1. Linear topological invariants

Let F be a Frechet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}$. For each subset $B\subset F$ define a generalized semi-norm $\|\cdot\|_B^*$: $F'\to[0,+\infty]$ on the dual space of F by

$$\|u\|_B^* = \sup\{|u(x)| : x\in B\}.$$

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Write $\|\cdot\|_k^*$ instead of $\|\cdot\|_{U_k}^*$, where $U_k = \{x \in F : \|x\|_k < 1\}$. We say that F has the property

(Ω) if $\forall p\ \exists q\ \forall k\ \exists d, C > 0\ \|\cdot\|_q^{*1+d} \leq C\|\cdot\|_k^*\|\cdot\|_p^{*d}$,

(DN) if $\exists p\ \forall q, d > 0\ \exists k, C > 0 : \|\cdot\|_q^{1+d} \leq C\|\cdot\|_k\|\cdot\|_p^d$,

($\overline{DN}$) if $\exists p\ \forall q\ \exists k\ \forall d > 0\ \exists C > 0 : \|\cdot\|_q^{1+d} \leq C\|\cdot\|_k\|\cdot\|_p^d$.

The above properties were introduced and investigated by Vogt in [12] [13] and [14]. In [13] Vogt has shown that $F \in (DN)$ (resp. $F \in \Omega$) if and only if F is isomorphic to a subspace (resp. quotient space) of $B\widehat{\otimes}_\pi s$ for a suitable Banach space, where s denotes the space of rapidly decreasing sequences.

1.2. The space of germs of holomorphic functions

Let E, F be locally convex spaces and D an open set in E. A function $f : D \to F$ is called holomorphic if f is continuous and $u \circ f$ is Gateaux holomorphic for all $u \in F'$. By $\mathcal{H}(D, F)$ we denote the space of F-valued holomorphic functions on D. We denote by τ_0 and τ_ω the compact open and Nachbin topologies on $\mathcal{H}(D, F)$. Let us recall that a semi-norm ρ on $\mathcal{H}(D, F)$ is said to be τ_ω-continuous if there exist a compact set $K \subset D$ and a continuous semi-norm α on F such that for every neighbourhood V of K, there exists $C_V > 0$:

$$\rho(f) \leq C_V \alpha_V(f) \text{ for } f \in \mathcal{H}(D, F),$$

where

$$\alpha_V(f) = \sup\Big\{\alpha(f(x)) : x \in V\Big\}.$$

Put for each compact set K in E,

$$\mathcal{H}_\infty(K, F) = \lim_{U \downarrow K} \operatorname{ind} \mathcal{H}_\infty(U, F)$$

and

$$\mathcal{H}(K, F) = \lim_{U \downarrow K} \operatorname{ind} \big[\mathcal{H}_\infty(U, F), \tau_w\big],$$

where

$$\mathcal{H}_\infty(U, F) = \Big\{f \in \mathcal{H}(U, F) : f(U) \text{ is bounded}\Big\}$$

equipped with the topology of uniform convergence on U. For details concerning holomorphic functions in locally convex spaces, we refer the reader to the book of Dineen [3].

Finally we recall that an inductive limit $E = \liminf_{\alpha} E_\alpha$ is said to be regular if every bounded set in E is contained and bounded in some E_α.

2. The regularity of $\mathcal{H}_\infty(K, F)$ and the properties (DN), (Ω)

The aim of this section is to prove the following result.

2.1. Theorem.
(i) *The Frechet space F has the property (DN) if and only if $\mathcal{H}_\infty(K, F)$ is regular for every compact set K in every Frechet space $E \in (\Omega)$*
(ii) *The Frechet space E has the property (Ω) if and only if $\mathcal{H}_\infty(K, F)$ is regular for every compact set K in E and every Frechet space $F \in (DN)$.*

For the proof of the theorem we need some notations. Put

$$\mathbf{M} = \Big\{\alpha = (\alpha_j) \subset \mathbf{N} : \alpha_j = 0 \text{ for } j \text{ sufficiently large}\Big\}.$$

For each $\alpha \in \mathbf{M}$, $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots)$, and $z \in \mathbf{C}^{\mathbf{N}}$, put

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$
$$\alpha! = \alpha_1! \dots \alpha_n!,$$
$$z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

Note that (see [8])

$$\Big(\sum_{j \geq 1} z_j\Big)^n = \sum_{\alpha \in M_n} \frac{n!}{\alpha!} z^\alpha, \tag{1}$$

where

$$M_n = \big\{\alpha \in M : |\alpha| = n\big\}.$$

The following is a small modification of a theorem of Ryan [8].

2.2. Lemma. *Let $B(0, R) = \big\{z \in \ell^1 : \|z\| < R\big\}$, $R > 0$ and F a Banach space. Let $0 < R < 1$ and $f : B(0, R) \to F$ a bounded holomorphic function. Then there exists a unique family $\big\{a_\alpha\big\}_{\alpha \in \mathbf{M}} \subset F$ such that*

$$\sup\Big\{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}\Big(\frac{R}{2}\Big)^{|\alpha|}\|a_\alpha\| : \alpha \in \mathbf{M}\Big\} < \infty,$$

and

$$(2) \quad \sum_{\alpha\in\mathbf{M}_n} a_\alpha z^\alpha \text{ converges to } f \text{ in } \mathcal{H}_\infty(B(0,r),F) \text{ for all } 0<r<\frac{R}{2e}\cdot$$

2.3. Lemma. *Let $R : E \to G$ be a continuous linear surjective map between Frechet spaces and let $\mathcal{H}_\infty(O_E, F)$ be regular. Then so is $\mathcal{H}_\infty(O_G, F)$.*

Proof. Given A a bounded set in $\mathcal{H}_\infty(O_G, F)$. Then $\hat{R}(A)$ is bounded in $\mathcal{H}_\infty(O_E, F)$, where $\hat{R} : \mathcal{H}_\infty(O_G, F) \to \mathcal{H}_\infty(O_E, F)$ is the continuous linear induced by R. By the hypothesis we can find a balanced convex neighbourhood U of $0 \in E$ such that $\hat{R}(A)$ is contained and bounded in $\mathcal{H}_\infty(U, F)$. Put $V = R(U)$. The open mapping theorem yields that V is a neighbourhood of $0 \in G$. We check that A is contained and bounded in $\mathcal{H}_\infty(V, F)$. Given $g \in A$. Choose a balanced convex neighbourhood V_g of $0 \in G$ in V such that $g \in \mathcal{H}_\infty(V_g, F)$. Consider the Taylor expansions of $f = \hat{R}(g) \in \mathcal{H}_\infty(U, F)$ and g at $0 \in E$ and $0 \in G$ on U and V_g, respectively,

$$f(x) = \sum_{n\geq 0} P_n f(x), \quad P_n f(x) = \frac{1}{2\pi i} \int\limits_{|\lambda|=\varepsilon_x>0} \frac{f(\lambda x)}{\lambda^{n+1}} d\lambda, \quad x \in E,$$

and

$$g(y) = \sum_{n\geq 0} P_n g(y), \quad P_n g(y) = \frac{1}{2\pi i} \int\limits_{|\lambda|=\varepsilon_y>0} \frac{g(\lambda y)}{\lambda^{n+1}} d\lambda, \quad y \in G.$$

Hence

$$\begin{aligned} P_n f(x) &= \frac{1}{2\pi i} \int\limits_{|\lambda|=\varepsilon_x} \frac{f(\lambda x)}{\lambda^{n+1}} d\lambda \\ &= \frac{1}{2\pi i} \int\limits_{|\lambda|=\varepsilon_x} \frac{gR(\lambda x)}{\lambda^{n+1}} d\lambda \\ &= P_n g(y) \end{aligned}$$

with $y = Rx$, where $\varepsilon_x > 0$ is sufficiently small such that $\lambda x \in R^{-1}(V_g) \cap U$ for $|\lambda| \leq \varepsilon_x$. Thus

$$\hat{R}(P_n g) = P_n f \quad \text{for } n \geq 0$$

and hence the series $\sum_{n\geq 0} P_n g(y)$ converges to $\hat{g} \in \mathcal{H}_\infty(V, F)$, for which $\hat{g} = g$ in $\mathcal{H}_\infty(O_G, F)$ and $\hat{R}(\hat{g}) = f$. This means that A is contained and bounded in $\mathcal{H}_\infty(V, F)$.

2.4. Lemma [11]. *Let F be a Frechet space with a continuous norm and let $\mathcal{H}_\infty(O_E, F)$ be regular, where O_E is the zero-element of a Frechet space E. Then $\mathcal{H}_\infty(K, F)$ is regular for all compact sets K in E.*

Proof of Theorem 2.1.

Necessity. Assume that $\mathcal{H}_\infty(K, F)$ is regular for all compact sets in an arbitrary Frechet space $E \in (\Omega)$. Then $\mathcal{H}_\infty(\overline{\Delta}, F)$ and $\Delta = \big\{z \in \mathbf{C} : |z| < 1\big\}$ is regular. By [11], F has (DN).

Assume that $\mathcal{H}_\infty(K, F)$ is regular for every compact set $K \subset E$ and every Frechet space $F \in (DN)$. In particular, $\mathcal{H}_\infty(O_E, s)$ is regular. By Vogt [14] we have $\forall \mu\ \exists k, n\ \forall K, m\ \exists N, S\ \forall \sigma \in \mathcal{H}_\infty(O_E, s)$,

$$\|\sigma\|_{k,m} \leq S\big(\|\sigma\|_{\mu,n} + \|\sigma\|_{K,N}\big). \tag{1}$$

Using (1) to $\sigma = ue_p$, $u \in E'$, $p \geq 1$ we get

$$\big\|u\big\|_k^* p^m \leq S\big(\big\|u\big\|_\mu^* p^n + \big\|u\big\|_K^* p^N\big).$$

The inequality yields

$$\big\|u\big\|_k^* \leq S\Big(\frac{1}{t}\big\|u\big\|_\mu^* + t^d\big\|u\big\|_K^*\Big)\ \forall u \in E',\ \forall t \geq 1,$$

where $t = p^{m-n}$ and $d = \dfrac{N-m}{m-n} > 0$. We can increase S so that the inequality holds for $t > 0$. This means $E \in (\Omega)$.

Sufficiency. Assume that E and F has the property (Ω) and (DN), respectively. Choose an index set I and a Banach space B such that E is isomorphic to a quotient space of $\ell^1(I)\widehat{\otimes}_\pi s$ and F to a subspace of $B\widehat{\otimes}_\pi s$. Note that

$$B\widehat{\otimes}_\pi s = \Big\{(y_p) \subset B : \sup \|y_p\| p^n < \infty\ \forall n \geq 1\Big\}.$$

Thus, by Lemma 2.2 we may assume without loss of generality that $E \cong \ell^1(I)\widehat{\otimes}_\pi s$ and $F \cong B\widehat{\otimes}_\pi s$. To prove the regularity of $\mathcal{H}_\infty(K, F)$ for every compact set K in E it suffices by Lemma 2.3 to assume $K = \{O_E\}$.

First consider the case $\mathbf{I} = \mathbf{N}$. Put

$$\mathbf{M} = \{\alpha = (\alpha_{11}, \dots, \alpha_{ij}, 0, \dots)\}.$$

and $\gamma_{ij,k} = i^k$ for $i, j, k \geq 1$. Let $\sigma \in \mathcal{H}_\infty(O_E, F)$. Choose $q \geq 1$ such that $\sigma \in H_\infty(U_q, F)$, where

$$U_k = \Big\{(z_{ij}) \subset \mathbf{C} : \sum_{i,j\geq 1} |z_{ij}| i^k < \infty\Big\}.$$

By Lemma 2.1, there exists a unique family $\{\xi_{(\alpha,p),n}\} \subset B$ such that

$$\sup\Big\{\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}\Big(\frac{1}{2}\Big)^{|\alpha|}\frac{j^n}{\gamma^\alpha_{.,k}}\big\|\xi_{(\alpha,p),n}\big\| : (\alpha, p) \in \mathbf{M} \times \mathbf{N}\Big\} < \infty$$

for all $n \geq 1$. By the regular characterization of Vogt [14] it remains to check that $\forall \mu\ \exists n, k\ \forall m, K\ \exists N, S\ \forall (\alpha, j)$,

$$\frac{p^m}{\gamma^\alpha_{.,k}} \leq S\Big(\frac{p^n +}{\gamma^\alpha_{.,\mu}} + \frac{p^N}{\gamma^\alpha_{.,K}}\Big)\,. \tag{2}$$

Given $\mu \geq 1$. Take $k = 2\mu$ and $n = 1$. Then (2) has the form: $\forall K, m$ $\exists S, N\ \forall (\alpha, j)$,

$$\frac{p^m}{1^{\mu\alpha_{11}} \dots i^{\mu\alpha_{ij}}} \leq S\Big(p + \frac{p^N}{1^{(k-\mu)\alpha_{11}} \dots i^{(k-\mu)\alpha_{ij}}}\Big)\,. \tag{3}$$

Choose $N > \dfrac{m}{\mu}K + m$. Obviously, (3) holds for p, α with $p^m \leq 1^{\mu\alpha_{11}} \dots i^{\mu\alpha_{ij}}$. If $p^m > 1^{\mu\alpha_{11}} \dots i^{\mu\alpha_{ij}}$ we also (3) have because

$$\begin{aligned}\frac{p^N}{1^{(k-\mu)\alpha_{11}} \dots i^{(k-\mu)\alpha_{ij}}} &\geq \frac{p^m}{1^{\mu\alpha_{11}} \dots i^{\mu\alpha_{ij}}} \times \frac{1^{(\frac{mK}{\mu}+m)\mu\alpha_{11}} \dots i^{(\frac{mK}{\mu}+m)\mu\alpha_{ij}}}{1^{(k-2\mu)\alpha_{11}} \dots i^{(k-2\mu)\alpha_{ij}}}\\ &\geq \frac{p^m}{1^{\mu\alpha_{11}} \dots i^{\mu\alpha_{ij}}}\end{aligned}$$

Now we consider the general case. Let $\mathcal{F}(I)$ denote the family of countable sunsets J of I. Let A be a bounded set in $\mathcal{H}_\infty(O_E, F)$. By (i) for each $J \in \mathcal{F}(I)$ we can find a convex neighbourhood W_J of $0 \in \ell^1(J)\widehat{\otimes}_\pi s$ such that $A\big|_{W_J}$ is contained and bounded in $\mathcal{H}_\infty(W_J, F)$. Put

$$W = U\Big\{W_J : J \in \mathcal{F}(I)\Big\}.$$

Then W is a neighbourhood of $0 \in E$. Otherwise, there exists a sequence $\{z_n\} \subset E$, $z_n \notin W$ for $n \geq 1$, converging to $0 \in E$. Choose $J \in \mathcal{F}(I)$ such that $\{z_n\} \subset \ell^1(J)\widehat{\otimes}_\pi s$. Then $z_n \in W_J \subset W$ for n sufficiently large. This is impossible. On the other hand, by the unique principle A is contained in $\mathcal{H}(\overset{0}{W}, F)$. It remains to check that there exists a neighbourhood V of $0 \in E$ in $\overset{0}{W}$ such that A is bounded in $\mathcal{H}_\infty(V, F)$. Otherwise, for each $n \geq 1$ there exists k_n and a countable subset A_n of A such that

$$\sup\Big\{\|f(z)\|_{k_n} : \|z\|_n < 1,\ f \in A_n\Big\} = +\infty.$$

It is easy to see that there exists a countable subset J of I such that for $n \geq 1$,

$$\sup\Big\{\|f(z)\|_{k_n} : \|z\|_n < 1, \quad z \in \ell^1(J)\widehat{\otimes}_\pi s, \quad f \in \bigcup_{j\geq 1} A_n\Big\} = +\infty.$$

This is impossible because for n sufficiently large we have

$$\Big\{z \in \ell^1(J)\widehat{\otimes}_\pi s : \|z\|_n < 1\Big\} \subset W_J$$

and hence

$$\Big\{\|f(z)\|_k;\ z \in W_J,\ f \in A\Big\} < \infty \quad \text{for } k \geq 1.$$

The theorem is proved.

3. Application

In this section we investigate the regularity and completeness of $\mathcal{H}(K, F)$, where

$$\mathcal{H}(K, F) = \lim \operatorname*{ind}_{U \supset K} \big[\mathcal{H}(U, F), \tau_w\big]$$

and τ_w denotes the Nachbin topology on $\mathcal{H}(U, F)$.

3.1. Theorem. *Let E and F be nuclear Frechet spaces with $E \in (\Omega)$ and $F \in (\overline{DN})$. Then $\mathcal{H}(K, F)$ is regular and complete.*

We need the following two lemmas.

3.2. Lemma. *Let E and F be Frechet spaces having (Ω) and $(\overline{DN})$, respectively. Then every F-valued holomorphic function on an open set in E is locally bounded.*

Proof. By [11] we can find a set I and a continuous linear map R from $\ell^1(I)\widehat{\otimes}_\pi s$ onto E, where s is the space of all rapidly decreasing sequences. Note that

$$\ell^1(I)\widehat{\otimes}_\pi s = \Big\{z = \big(z_{ij}\big)_{i\in\mathbf{I}, j\in\mathbf{N}} : \|z\|_\gamma = \sum_{i\in\mathbf{I}, j\in\mathbf{N}} \big|z_{ij}\big| j^\gamma < \infty,\ \forall\gamma\in\mathbf{N}\Big\}$$

and hence $\ell^1(I)\widehat{\otimes}_\pi s \in (\Omega)$.

Let $f \in \mathcal{H}(D, F)$, where D is an open set in E. Since R is open, it suffices to show that $g = f \circ R$ is locally bounded on $R^{-1}(D)$. Let $z_0 \in R^{-1}(D)$. Without loss of the generality we may assume that $R^{-1}(D)$ is balanced and $z_0 = 0$. Consider the Taylor expansion of g at $0 \in \ell^1(I)\widehat{\otimes}_\pi s$

$$g(z) = \sum_{n\geq 0} P_n g(z)$$

where

$$P_n g(z) = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} \frac{g(\lambda z)}{\lambda^{n+1}} d\lambda \quad \text{for} \quad z \in R^{-1}(D).$$

Writing every $z \in \ell^1(I)\widehat{\otimes}_\pi s$ in the form

$$z = \sum_{i\in\mathbf{I}, j\in\mathbf{N}} \delta^*_{ij}(z)\delta_{ij},$$

where $\delta^*_{ij}(z) = z_{ij}$ and $\delta_{ij} = \big[\delta^{[i,j]}_{k,\ell} : \mathbf{I}\times\mathbf{N}\big]$, with

$$\delta^{[i,j]}_{k,\ell} = \begin{cases} 1 & (k,\ell) = (i,j) \\ 0 & (k,\ell) \neq (i,j), \end{cases}$$

we get

$$\begin{aligned} g(z) &= \sum_{n\geq 0} P_n g\Big(\sum_{i\in\mathbf{I}, j\in\mathbf{N}} \delta^*_{ij}(z)\delta_{ij}\Big) \\ &= \sum_{n\geq 0} \sum_{\substack{i_1,\dots,i_n\in\mathbf{I} \\ j_1,\dots,j_n\in\mathbf{N}}} P_n g(\delta_{i_1 j_1}, \dots, \delta_{i_n j_n})\delta^*_{i_1 j_1}(z)\dots\delta^*_{i_n j_n}(z) \end{aligned}$$

for $z \in R^{-1}(D)$. Since $\ell^1(I)\widehat{\otimes}_\pi s \in (\Omega)$ we have

$$(\Omega) \qquad \forall\alpha\ \exists\beta\ \forall\gamma \geq \beta\ \exists d_\gamma, C_\gamma > 0 : |||\cdot|||_\beta^{*(1+d_\gamma)} \leq C_\gamma |||\cdot|||_\gamma^* |||\cdot|||_\alpha^{*d\gamma}.$$

Since

$$\left(\frac{|||\cdot|||_\beta^*}{|||\cdot|||_\alpha^*}\right)^{sd_\gamma} \leq \left(\frac{|||\cdot|||_\beta^*}{|||\cdot|||_\alpha^*}\right)^{d_\gamma} \qquad \forall s \geq 1,$$

it follows that

$$|||\cdot|||_\beta^{(*)1+sd_\gamma)} \leq C_\gamma |||\cdot|||_\gamma^* \cdot |||\cdot|||_\alpha^{*sd_\gamma} \quad \forall s \geq 1.$$

On the other hand, by applying (Ω) to δ_{ij}^* we obtain

$$\frac{|||\delta_{ij}|||_\gamma |||\delta_{ij}|||_\alpha^{sd_\gamma}}{C_\gamma} \leq |||\delta_{ij}|||_\beta^{1+sd_\gamma}, \quad \forall i \in \mathbf{I}, j \in \mathbf{N}.$$

By the hypothesis $F \in (\,\overline{DN}\,)$, where

$$(\,\overline{DN}\,)\ \exists p\ \forall q \geq p\ \exists k_q \geq q\ \forall d > 0\ \exists D_d : \|\cdot\|_q^{1+d} \leq D_d \|\cdot\|_{k_q} \|\cdot\|_p^d.$$

First choose α such that

$$M(\alpha, p) = \sup\left\{\|g(z)\|_p : |||z|||_\alpha < 1\right\} < \infty.$$

Next choose $\beta \geq \alpha$ as in (Ω). To complete the proof, it is enough to verify that g is bounded on rU_β for $r > 0$ sufficiently small.

(iii) Given $q \geq p$, choose $k_q \geq q$ as in $(\,\overline{DN}\,)$. Let $\gamma_q \geq \beta$ such that

$$M(\gamma_q, k_q) < \infty$$

and let $s > 0$ sufficiently large for which

$$C_{\gamma_q}^{\frac{1}{(1+sd_{\gamma_q})}} \leq 2.$$

Then we have

$$\sum_{n\geq 0}\|P_n g(z)\|_q \leq \sum_{n\geq 0}\sum_{\substack{i_1,\dots,i_n\in\mathbf{I}\\ j_1,\dots,j_n\in\mathbf{N}}}\frac{\|P_n g(\delta_{i_1j_1},\dots,\delta_{i_nj_n})\|_q}{|||\delta_{i_1j_1}|||_\beta\cdots|||\delta_{i_nj_n}|||_\beta}\times$$
$$\times\, |||\delta_{i_1j_1}|||_\beta\cdots|||\delta_{i_nj_n}|||_\beta|\delta^*_{i_1j_1}(z)|\cdots|\delta^*_{i_nj_n}(z)|$$
$$\leq \sum_{n\geq 0}\sum_{\substack{i_1,\dots,i_n\in\mathbf{I}\\ j_1,\dots,j_n\in\mathbf{N}}} D_{sd_{\gamma_q}}^{\frac{1}{1+sd_{\gamma_q}}}\Big(C_{\gamma_q}^{\frac{1}{1+sd_{\gamma_q}}}\Big)^n\times\frac{\big\|P_n g(\delta_{i_1j_1},\dots,\delta_{i_nj_n})\big\|_{k_q}^{\frac{1}{1+sd_\gamma}}}{\big|\big|\big|\delta_{i_1j_1}\big|\big|\big|_{\gamma_q}^{1+sd_\gamma}\cdots\big|\big|\big|\delta_{i_nj_n}\big|\big|\big|_{\gamma_q}^{1+sd_\gamma}}\times$$
$$\times\,\frac{\big\|P_n g(\delta_{i_1j_1},\dots,\delta_{i_nj_n})\big\|_p^{\frac{sd_{\gamma_q}}{1+sd_{\gamma_q}}}}{\big|\big|\big|\delta_{i_1j_1}\big|\big|\big|_\alpha^{\frac{sd_{\gamma_q}}{1+sd_\gamma}}\cdots\big|\big|\big|\delta_{i_nj_n}\big|\big|\big|_\alpha^{\frac{sd_{\gamma_q}}{1+sd_\gamma}}}$$
$$\times\, |||\delta_{i_1j_1}|||_\beta\cdots|||\delta_{i_nj_n}|||_\beta|\delta^*_{i_1j_1}(z)|\cdots|\delta^*_{i_nj_n}(z)|$$
$$\leq D_{sd_{\gamma_q}}^{\frac{1}{1+sd_\gamma}} M\big(\gamma_q,k_q\big)^{\frac{1}{1+sd_\gamma}} M\big(\alpha,p\big)^{\frac{sd_{\gamma_q}}{1+sd_{\gamma_q}}}\sum_{n\geq 0}2^n\frac{n^n}{n!}\Big(\sum_{\substack{i\in\mathbf{I}\\ j\in\mathbf{N}}}|\delta^*_{ij}(z)|\|\delta_{ij}\|_\beta\Big)^n$$
$$\leq D_{sd_{\gamma_q}}^{\frac{1}{1+sd_\gamma}} M\big(\gamma_q,k_q\big)^{\frac{1}{1+sd_\gamma}} M\big(\alpha,p\big)^{\frac{sd_{\gamma_q}}{1+sd_{\gamma_q}}}\times\sum_{n\geq 0}2^n\frac{n^n}{n!}\|z\|_\beta^n$$
$$\leq D_{sd_{\gamma_q}}^{\frac{1}{1+sd_\gamma}} M\big(\gamma_q,k_q\big)^{\frac{1}{1+sd_\gamma}} M\big(\alpha,p\big)^{\frac{sd_{\gamma_q}}{1+sd_{\gamma_q}}}\times\sum_{n\geq 0}2^n\frac{n^n}{n!}\Big(\frac{1}{4e}\Big)^n<\infty$$

for $\|z\|_\beta < \dfrac{1}{4e}\cdot$ Consequently, g and hence f is locally bounded. □

3.3. Lemma. *Let E and F be as in Theorem 3.1. Then the canonical map*

$$\mu:\mathcal{H}(K)\widehat{\otimes}_\pi F\to\big[\mathcal{H}'(K)\widehat{\otimes}_\pi F'\big]'$$

is an isomorphism.

Proof. By Meise and Vogt [5] $\mathcal{H}'(K)\in(\Omega)$. Hence by Vogt [12], $\mathcal{H}'(K)\widehat{\otimes}_\pi F'$ is bornological. By Lemma 3.2 this yields

$$\mathcal{H}'(K)\widehat{\otimes}_\pi F'\cong\lim_{k}\operatorname{ind}\big(\mathcal{H}'(K)\widehat{\otimes}_\pi F'_k\big).$$

Moreover the inductive limit is regular. Hence

$$[\mathcal{H}'(K)\widehat{\otimes}_\pi F']\cong\lim_{k}\operatorname{proj}[\mathcal{H}'(K)\widehat{\otimes}_\pi F'_k]\cong\mathcal{H}(K)\widehat{\otimes}_\pi F'$$

Proof of Theorem 3.1. Let $\{U_n\}$ be a neighbourhood basis of K. Since $E \in (\Omega)$ and $F \in (\overline{DN})$ we have

$$\begin{aligned}
\mathcal{H}_\infty(K,F) &= \mathcal{H}(K,F) \quad \text{(Lemma 3.2)}\\
&= \mathcal{H}(K)\widehat{\otimes}_\pi F \text{ (by [5] } [\mathcal{H}(K)]' \in (\Omega) \text{ and by Lemma 3.2)}\\
&\cong [\mathcal{H}'(K)\widehat{\otimes}_\pi F'] \text{ (Lemma 3.3)}\\
&\cong [(\lim\ \text{proj}\ (\mathcal{H}'_\infty(U_n))\widehat{\otimes}_\pi F']'\\
&\cong [\lim\ \text{proj}\ (\mathcal{H}'_\infty(U_n)\widehat{\otimes}_\pi F')]'\\
&\cong [\lim\ \text{proj}\ (\mathcal{H}_\infty(U_n)\widehat{\otimes}_\pi F)']'\\
&\cong [\lim\ \text{proj}\ (\mathcal{H}_\infty(U_n,F)']'\\
&\cong [\lim\ \text{proj}\ (\mathcal{H}_\infty(U_n),F)]''\\
&\cong [\mathcal{H}_\infty(K,F)]''.
\end{aligned}$$

Thus, $\mathcal{H}_\infty(K,F) \cong [H_\infty(K,F)]''$ is complete. It remains to check that the canonical map $\mathcal{H}(K,F) \to \mathcal{H}(K)\widehat{\otimes}_\pi F \cong \mathcal{H}(K)\widehat{\otimes}_\varepsilon F$ is continuous and hence $\mathcal{H}(K,F) \cong \mathcal{H}_\infty(K,F)$. For each $n \geq 1$ the restriction map

$$\begin{aligned}
R_n : [\mathcal{H}(U_n,F),\tau_w] \to \mathcal{H}(K)\widehat{\otimes}_\pi F &\cong \mathcal{H}(K)\widehat{\otimes}_\varepsilon F\\
&\cong [\lim \operatorname*{ind}_m \mathcal{H}(U_m),\tau_w]\widehat{\otimes}_\varepsilon F
\end{aligned}$$

is continuous. Indeed let ρ be a continuous semi-norm on $\mathcal{H}(K)\widehat{\otimes}_\varepsilon F$. Choose two continuous semi-norms α and β on $\mathcal{H}(K)$ and F, respectively, such that

$$\rho\Big(\sum_{k=1}^m g_k \otimes x_k\Big) \leq \sup\Big\{\alpha\Big(\sum_{k=1}^m x^*(x_k)g_k\Big) : x^* \in U_\beta^0\Big\}$$

for $g_k \in \mathcal{H}(K)$ and $x_k \in F$, $k = \overline{1,m}$. Let V be an arbitrary neighbourhood of K in U. Take $C_V > 0$ such that

$$\alpha(g) \leq C_V \|g\|_V \quad \text{for} \quad g \in \mathcal{H}_\infty(V).$$

Then

$$\begin{aligned}
\rho\Big(R_n\Big(\sum_{k=1}^m g_k x_k\Big)\Big) &= \rho\Big(\sum_{k=1}^m g_k \otimes x_k\Big)\\
&\leq \sup\Big\{\alpha\Big(\sum_{k=1}^m x^*(x_k)g_k\Big) : x^* \in U_\beta^0\Big\}\\
&\leq C_V \sup\Big\{\Big\|\sum_{k=1}^m x^*(x_k)g_k\Big\|_V : x^* \in U_\beta^0\Big\}.
\end{aligned}$$

This means that ρR_n is τ_w-continuous. Hence

$$\mathcal{H}(K,F) \cong \lim\operatorname*{ind}_n [\mathcal{H}(U_n,F),\tau_w] \cong \mathcal{H}_\infty(K,F)$$

is regular and complete □

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