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ON THE CONVERGENCE OF SOLUTIONS OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. The main result of this paper shows the asymptotic behaviour of solutions to perturbed linear differential equations in Banach spaces.

1. INTRODUCTION

Let $\mathcal{B}$ be a Banach space equiped with the norm $\|.\|$. Consider in $\mathcal{B}$ the following equations on the finite interval $[0, T]$:

$$\frac{dx}{dt} = A(t)\,x, \qquad x(0) = x_0, \tag{1}$$

$$\frac{dx}{dt} = [A(t) + R(t,\varepsilon)]\,x, \quad x(0) = x_0, \tag{2}$$

$$\frac{dy}{dt} = R(t,\varepsilon)\,y, \qquad y(0) = y_0, \tag{3}$$

$$(0 \le \varepsilon \le \varepsilon_0)\,,$$

where $A(t)$ and $R(t,\varepsilon)$ belong to $[\mathcal{B}]$, the space of all bounded linear operators from $\mathcal{B}$ into itself with the norm $|||.|||$.

A continuous function $x(t) : [0, T] \to \mathcal{B}$ is called a solution of equation (1), ((2), (3)) if $x(t)$ is almost everywhere differentiable and satisfied (1) ((2), (3)). Note that (1) has a unique solution if $A(t)$ is B–integrable (integral in the Bochner sense) on $[0, T]$, in particular, $A(t)$ is strongly continuous on $[0, T]$ (see [6]).

By $X(t)$, $X(t,\varepsilon)$, $Y(t,\varepsilon)$ we denote the Cauchy operators of equations (1), (2), (3), respectively. By $\|.\|_{L_p}$ we denote the norm in $L_p[0, T]$, $1 \le p \le \infty$.

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Assume that $H(t) \in [\mathcal{B}]$ and the function

$$\begin{aligned} &t \to |\|H(t)\||, \\ &[0,T] \to R \end{aligned}$$

belongs to $L_p[0,T]$. We define

$$|\|H(.)\||_p = \|\|\|H(.)\|\|\|_{L_p}.$$

If $|\|H(.,\varepsilon) - H(.)\||_p \to 0$ as $\varepsilon \to 0$ then we say $H(t,\varepsilon) \to H(t)$ in L_p as $\varepsilon \to 0$. Similarly, for $z(t) \in \mathbf{C}([0,T];\mathcal{B})$ we define

$$\|z(.)\|_p = \|\|z(.)\|\|_{L_p},$$

and say $z(t,\varepsilon) \to z(t)$ in L_p as $\varepsilon \to 0$ if $\|z(.,\varepsilon) - z(.)\|_p \to 0$ as $\varepsilon \to 0$.

In the case $\mathcal{B}$ is finite - dimensional, A. Yu. Levin [1] proved the following reducible theorem:

Theorem 1. *The pair of relations*

$$X(t,\varepsilon) \to X(t)\,, \quad X^{-1}(t,\varepsilon) \to X^{-1}(t) \tag{4}$$

in L_p as $\varepsilon \to 0$ is equivalent to the pair of relations

$$Y(t,\varepsilon) \to I\,, \quad Y^{-1}(t,\varepsilon) \to I \tag{5}$$

in L_p as $\varepsilon \to 0$ for any $p \geq 1$.

This theorem showed the approach of solutions of the linear system (2) to the solution of the linear equation (1).

A. Yu. Levin [2], N. T. Hoan [3], P.P. Zabreiko, and N. H. Thai [4] gave several conditions to ensure the relations (5), which describe the behaviour of solutions of system (2).

If $\mathcal{B}$ is an arbitrary Banach space, Levin's theorem and the abovementioned results are still valid with the uniform convergence topology of the operators in $[\mathcal{B}]$. However, it does not give us any information on the convergence of solutions of (2) to the solution of (1). The purpose of this paper is to give some insight on this problem. Althought all operators appeared in our equations are bounded, it seems to us that many results remain valid for several unbounded operators.

2. Main results

Assume that $A(t),\ R(t,\varepsilon)$ are strongly continuous (see [5]) and

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \int_0^T |||R(t,\varepsilon)|||\,dt < \infty\,. \tag{6}$$

The main result of this paper is the following

Theorem 2. *Let $x(t)$, $x(t,\varepsilon)$, $y(t)$ be the solutions of* (1), (2), (3), *respectively. Then the condition $x(t,\varepsilon)\to x(t)$ in L_p as $\varepsilon\to 0$, for each $x_0\in\mathcal{B}$ is necessary and sufficient for $y(t,\varepsilon)\to y_0$ in L_p as $\varepsilon\to 0$, for each $y_0\in\mathcal{B}$ $(1\le p\le\infty)$.*

To prove this theorem we need the following lemma.

Lemma. *Suppose that $F(t,\varepsilon)\in[\mathcal{B}]$ for $t\in[0,T]$ and $\varepsilon\in[0,\varepsilon_0]$ satisfies*

a) $\sup\limits_{\varepsilon\in[0,\varepsilon_0]} |||F(.,\varepsilon)|||_p = M<\infty\,,\ 1\le p\le\infty$;

b) $\|F(.,\varepsilon)\,x_0\|_p\to 0$ *for each* $x_0\in\mathcal{B}$ *as* $\varepsilon\to 0\,.$

Then $\|F(.,\varepsilon)\,x(.)\|_p\to 0\,,\ \varepsilon\to 0$, for any continuous function $x(t)$ from $[0,T]$ into $\mathcal{B}$.

Proof. 1) The case $1\le p<\infty$: We first note that for any $a,\,b\ge 0$,

$$(a+b)^p\le 2^{p-1}(a^p+b^p)\,. \tag{7}$$

Suppose that α is an arbitrary positive number. Because $[0,T]$ is compact, we have a finite sequence of points $t_1=0<t_2<\cdots<t_k=T$ such that

$$\sup_{0\le i\le k-1}\|x(t_{i+1})-x(t_i)\|<\frac{\alpha}{2}\,M^{-1}.$$

By Minkowski's inequality and the inequality (7) we obtain

$$\int_{t_i}^{t_{i+1}} \big\|F(t,\varepsilon)\,x(t)\big\|^p dt\ \le$$

$$\left\{\left[\int_{t_i}^{t_{i+1}}\big\|F(t,\varepsilon)[x(t)-x(t_i)]\big\|^p dt\right]^{\frac{1}{p}}+\left[\int_{t_i}^{t_{i+1}}\big\|F(t,\varepsilon)\,x(t_i)\big\|^p dt\right]^{\frac{1}{p}}\right\}^p$$

$$\le 2^{p-1}\left\{\int_{t_i}^{t_{i+1}}\big\|F(t,\varepsilon)[x(t)-x(t_i)]\big\|^p dt+\int_{t_i}^{t_{i+1}}\big\|F(t,\varepsilon)\,x(t_i)\big\|^p dt\right\}.$$

Therefore

$$\int_0^T \left\|F(t,\varepsilon)\,x(t)\right\|^p dt \le \tag{8}$$

$$2^{p-1}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left\|F(t,\varepsilon)[x(t)-x(t_i)]\right\|^p dt + 2^{p-1}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left\|F(t,\varepsilon)\,x(t_i)\right\|^p dt\,.$$

By the condition b) we have

$$2^{p-1}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left\|F(t,\varepsilon)\,x(t_i)\right\|^p dt \le 2^{p-1}\sum_{i=0}^{k-1}\int_0^T \left\|F(t,\varepsilon)\,x(t_i)\right\|^p dt < \frac{\alpha^p}{2} \tag{9}$$

for a sufficiently small ε. On the other hand,

$$\begin{aligned}
&2^{p-1}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left\|F(t,\varepsilon)[x(t)-x(t_i)]\right\|^p dt\\
&\le 2^{p-1}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left|\left\|F(t,\varepsilon)\right\|\right|^p \left\|[x(t)-x(t_i)]\right\|^p dt\\
&< 2^{p-1}\left(\frac{\alpha}{2}\right)^p M^{-1}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \left|\left\|F(t,\varepsilon)\right\|\right|^p dt\\
&= \frac{\alpha^p}{2}\,M^{-1}\int_0^T \left|\left\|F(t,\varepsilon)\right\|\right|^p dt\\
&\le \frac{\alpha^p}{2}. \qquad (10)
\end{aligned}$$

Combining (8), (9), (10) we deduce that $\left\|F(.,\varepsilon)\,x(.)\right\|_p < \alpha$.

2) The case $p=\infty$. The proof is similar.

Proof of Theorem 2. By $X(t)$, $X(t,\varepsilon)$, $Y(t,\varepsilon)$ we denote the Cauchy operators of (1), (2), (3), respectively, or for short we write X, $X(\varepsilon)$, $Y(\varepsilon)$. Because $A(t)$ is strongly continuous and by the uniformly bounded principle, we have $\sup\limits_{t\in[0,T]} |\|A(t)\|| \le N < \infty$ (see [5]). The inequality $\sup\limits_{\varepsilon\in[0,\varepsilon_0]} \int_0^T |\|R(t,\varepsilon)\||dt < \infty$ implies

(11) $|||X(\varepsilon)|||$, $|||X^{-1}(\varepsilon)|||$, $|||Y(\varepsilon)|||$, $|||Y^{-1}(\varepsilon)|||$ are uniformly bounded by $K < \infty$ (see [6]).

1) *The sufficiency.* Assume

$$\|y(.,\varepsilon) - y_0\|_p \to 0 \text{ as } \varepsilon \to 0 \tag{12}$$

for each $y_0 \in \mathcal{B}$. Let $Z(t,\varepsilon) = X^{-1}(\varepsilon)\,Y(\varepsilon)\,X$, then

$$\frac{dZ}{dt} = X^{-1}(\varepsilon)\,Y(\varepsilon)\,A\,X - X^{-1}(\varepsilon)\,A\,Y(\varepsilon)\,X.$$

Setting $F(t,\varepsilon) = X^{-1}(\varepsilon)\,Y(\varepsilon)\,A\,X - X^{-1}(\varepsilon)\,A\,Y(\varepsilon)\,X$ we get

$$\begin{aligned} &\|F(.,\varepsilon)x_0\|_p \leq \\ &K\big\{\,\|Y(\varepsilon)AXx_0 - AXx_0\|_p + |||A|||\,\|Y(\varepsilon)Xx_0 - Xx_0\|_p\big\} \end{aligned} \tag{13}$$

for each $x_0 \in \mathcal{B}$.

The assumption (12) implies $\|(Y(\varepsilon) - I)x_0\|_p \to 0$, $\varepsilon \to 0$ for each $x_0 \in \mathcal{B}$. Applying the above lemma to the continuous functions $h(t) = A(t)\,X(t)\,x_0$, $x(t) = X(t)\,x_0$ we get

$$\begin{aligned} &\|Y(\varepsilon)\,A\,X\,x_0 - A\,X\,x_0\|_p \to 0\,, \\ &\|Y(\varepsilon)\,X\,x_0 - X\,x_0\|_p \to 0 \text{ as } \varepsilon \to 0\,. \end{aligned}$$

Thus, $\|F(.,\varepsilon)\,x_0\|_p \to 0$ as $\varepsilon \to 0$. On the other hand, by the Hölder inequality we obtain

$$\|F(.,\varepsilon)\,x_0\|_1 \leq T^{1-\frac{1}{p}}\,\|F(.,\varepsilon)\,x_0\|_p \to 0 \;\; as\; \varepsilon \to 0 \tag{14}$$

Since $\dfrac{dZ(t,\varepsilon)\,x_0}{dt} = F(t,\varepsilon)\,x_0$ and $Z(0,\varepsilon)x_0 = x_0$, it follows

$$\|Z(t,\varepsilon)\,x_0 - x_0\| \leq \int_0^t \|F(t,\varepsilon)\,x_0\|\,dt\,,$$

$$\sup_t \|Z(t,\varepsilon)\,x_0 - x_0\| \leq \int_0^T \|F(t,\varepsilon)\,x_0\|\,dt\,.$$

By virtue of (14), $\sup_t \|Z(t,\varepsilon)\, x_0 - x_0\| \to 0$ as $\varepsilon \to 0$. Thus,

$$\|Z(.,\varepsilon)\, x_0 - x_0\|_p \to 0 \ \text{ as } \ \varepsilon \to 0\,. \tag{15}$$

Applying Lemma to $x(t) = X(t)\, x_0$ we get

$$\begin{aligned} &\|Z(.,\varepsilon)\, x_0 - X^{-1}(\varepsilon)\, X\, x_0\|_p \\ &\le |||X^{-1}(\varepsilon)|||\, \|Y(\varepsilon)\, X\, x_0 - X\, x_0\|_p \\ &\le K\, \|Y(\varepsilon)\, X\, x_0 - X\, x_0\|_p \to 0 \ \text{ as } \ \varepsilon \to 0\,. \end{aligned} \tag{16}$$

From (15) and (16) we obtain $\|X^{-1}(\varepsilon)\, X\, x_0 - x_0\|_p \to 0$ and

$$\|X(\varepsilon)\, x_0 - X\, x_0\|_p \le |||X(\varepsilon)|||\, \|X^{-1}(\varepsilon)\, X\, x_0 - x_0\|_p \to 0 \text{ as } \varepsilon \to 0.$$

Thus

$$x(t,\varepsilon) \to x(t) \ \text{ in } L_p \text{ as } \varepsilon \to 0\,.$$

2) *The necessity.* Suppose $\|x(.,\varepsilon) - x(.)\|_p \to 0$ as $\varepsilon \to 0$ for each $x_0 \in \mathcal{B}$. We show that (12) is valid. From $\|[X(.,\varepsilon) - X(.)]\, x_0\|_p \to 0$ for each $x_0 \in \mathcal{B}$ and applying the above lemma to the continuous function $v(t) = X^{-1}(t)\, x_0$ we see that $\|[X(\varepsilon) - X]\, X^{-1}\, x_0\|_p \to 0$. Then

$$\|X(\varepsilon)\, X^{-1} x_0 - x_0\|_p \to 0 \ \text{ as } \varepsilon \to 0\,. \tag{17}$$

Setting $Q(t,\varepsilon) = Y^{-1}(\varepsilon)\, X(\varepsilon)\, X^{-1}$ we obtain

$$\frac{dQ}{dt} = Y^{-1}(\varepsilon)\, A\, X(\varepsilon)\, X^{-1} - Y^{-1}(\varepsilon)\, X(\varepsilon)\, X^{-1} A\,. \tag{18}$$

By $K(t,\varepsilon)$ we denote the right-hand side of (18). Then, for each $x_0 \in \mathcal{B}$,

$$\begin{aligned} \|K(.,\varepsilon)\, x_0\|_p &\le \|Y^{-1}(\varepsilon)\, A\, X(\varepsilon)\, X^{-1} x_0 - Y^{-1}(\varepsilon)\, A\, x_0\|_p \\ &\quad + \|Y^{-1}(\varepsilon)\, X(\varepsilon)\, X^{-1} A\, x_0 - Y^{-1}(\varepsilon)\, A\, x_0\|_p \le \\ & KN\, \|X(\varepsilon)\, X^{-1} x_0 - x_0\|_p + K\, \|X(\varepsilon)\, X^{-1}\, A\, x_0 - A\, x_0\|_p\,. \end{aligned} \tag{19}$$

By the above lemma we deduce $\|X(\varepsilon)\, X^{-1}\, A\, x_0 - A\, x_0\|_p \to 0$. Combining (17) and (19) we get

$$\|K(.,\varepsilon)\, x_0\|_p \to 0 \ \text{ as } \varepsilon \to 0\,.$$

Similarly to the previous proof we obtain $\|Q(.,\varepsilon)\, y_0 - y_0\|_p \to 0$ as $\varepsilon \to 0$ for each $y_0 \in \mathcal{B}$ and $\|Y(.,\varepsilon)\, y_0 - Q(.,\varepsilon)\, y_0\|_p \to 0$, as $\varepsilon \to 0$. This implies $\|Y(.,\varepsilon)\, y_0 - y_0\|_p \to 0$, hence $y(t,\varepsilon) \to y_0$ in L_p, as $\varepsilon \to 0$ for each $y_0 \in \mathcal{B}$. $\square$

Theorem 3. *Suppose* $\sup_{\varepsilon\in[0,\varepsilon_0]} |||R(.,\varepsilon)|||_q < \infty$, *where* $\frac{1}{p}+\frac{1}{q}=1$ *(*$p=\infty$ *if* $q=1$ *and* $q=\infty$ *if* $p=1$*). Then the condition* $x(t,\varepsilon)\to x(t)$ *in* L_p *as* $\varepsilon\to 0$*, for each* $x_0\in\mathcal{B}$*, is necessary and sufficient for*

$$\Big\| \int_0^t R(s,\varepsilon)\,y_0\,ds\Big\|_p \to 0 \quad \text{as } \varepsilon\to 0. \tag{20}$$

Proof. The inequality

$$|||R(.,\varepsilon)|||_1 \le \|R(.,\varepsilon)\|_q\,.\,T^{1-\frac{1}{q}}$$

implies $\sup_{\varepsilon\in[0,\varepsilon_0]} |||R(t,\varepsilon)|||_1 < \infty$. By Theorem 2, to prove Theorem 3 it suffices to show that condition (20) is equivalent to $y(t,\varepsilon)\to y_0$ in L_p as $\varepsilon\to 0$ for each $y_0\in\mathcal{B}$. Consider the following Volterra integral equation

$$z(t,\varepsilon) = \int_0^t \|R(s,\varepsilon)\,z(s,\varepsilon)\|\,ds + f(t)\,,$$

where $f\in L_p[0,T]$. As in [3] we have

$$\|z\|_{L_p} \le \big(1+T^{\frac{1}{p}}|||R(.,\varepsilon)|||_q \exp|||R(.,\varepsilon)|||_1\big)\,\|f\|_{L_p}\,. \tag{21}$$

Assume now (20) is true. Then

$$y(t,\varepsilon) = y_0 + \int_0^t R(s,\varepsilon)\,y_0\,ds + \int_0^T R(s,\varepsilon)\,[y(s,\varepsilon)-y_0]\,ds\,. \tag{22}$$

Thus

$$\|y(t,\varepsilon)-y_0\| \le \Big\|\int_0^t R(s,\varepsilon)\,y_0\,ds\Big\| + \int_0^t \|R(s,\varepsilon)\,[y(s,\varepsilon)-y_0]\|\,ds\,.$$

By the theorem on integral inequalities (see [6], p. 154) we obtain $\|y(t,\varepsilon)-y_0\|\le z(t,\varepsilon)$, where $z(t,\varepsilon)$ is the solution of the equation

$$z(t,\varepsilon) = \int_0^t \|R(s,\varepsilon)\,z(s,\varepsilon)\,ds\| + \Big\|\int_0^t R(s,\varepsilon)\,y_0\,ds\Big\|\,. \tag{23}$$

Applying the estimate (21) with respect to equation (23), where $f(t) = \big\|\int\limits_0^t R(s,\varepsilon)\, y_0\, ds\big\|$, we get

$$\|y(.,\varepsilon) - y_0\|_p \le \|z(.,\varepsilon)\|_{L_p} \le$$

$$\big(1 + T^{\frac{1}{p}}\, |||R(.,\varepsilon)|||_q \exp |||R(.,\varepsilon)|||_1\big) \Big\| \int\limits_0^t R(s,\varepsilon)\, y_0\, ds \Big\|_p \to 0 \text{ as } \varepsilon \to 0\,.$$

Therefore, $y(t,\varepsilon) \to y_0$ in L_p as $\varepsilon \to 0$.

Conversely, suppose $y(t,\varepsilon) \to y_0$ in L_p as $\varepsilon \to 0$ for each $y_0 \in \mathcal{B}$. From (22) we get

$$\int\limits_0^t R(s,\varepsilon)\, y_0\, ds = \big(y(t,\varepsilon) - y_0\big) + \int\limits_0^t R(s,\varepsilon)[y(s,\varepsilon) - y_0]\, ds\,. \tag{24}$$

By the Hölder inequality we have

$$\begin{aligned} \int\limits_0^t \|R(s,\varepsilon)[y(s,\varepsilon) - y_0]\|\, ds &\le \int\limits_0^T |||R(s,\varepsilon)|||\; \|y(s,\varepsilon) - y_0\|\, ds \\ &\le |||R(.,\varepsilon)|||_q \,.\, \|y(.,\varepsilon) - y_0\|_p\,. \end{aligned} \tag{25}$$

From (24) and (25) we obtain

$$\Big\| \int\limits_0^t R(s,\varepsilon)\, y_0\, ds \Big\|_p \le \|y(.,\varepsilon) - y_0\|_p + T^{\frac{1}{p}}\, |||R(.,\varepsilon)|||_q\, \|y(.,\varepsilon) - y_0\|_p\,.$$

This implies $\big\|\int\limits_0^t R(s,\varepsilon)\, y_0\, ds\big\|_p \to 0$ as $\varepsilon \to 0$ for each $y_0 \in \mathcal{B}$. The theorem is completely proved. $\square$

Now consider the equation

$$\frac{dx}{dt} = [A(t) + R(t,\varepsilon)]\, x + f(t,\varepsilon)\,, \quad x(0,\varepsilon) = x_0^{\varepsilon} \tag{26}$$

and

$$\frac{dx}{dt} = A(t)\, x + f(t)\,, \quad x(0) = x_0\,, \tag{27}$$

Denote the solution of (26), (27) by $x(t,\varepsilon)$ and $x(t)$, respectively. We have the following result

Theorem 4. *Suppose* $\sup_{\varepsilon\in[0,\varepsilon_0]} |\|R(.,\varepsilon)\||_q < \infty$, *where* $\frac{1}{p}+\frac{1}{q}=1$ *(*$p=\infty$ *if* $q=1$ *and* $q=\infty$ *if* $p=1$*) and*

$$\Big\|\int_0^t R(s,\varepsilon)\,y_0\,ds\Big\|_p \to 0 \quad as \quad \varepsilon\to 0.$$

If $x_0^\varepsilon \to x_0$ *in* $\mathcal{B}$ *and* $f(t,\varepsilon)$, $f(t)$ *belong to* $\mathbf{C}([0,T];\mathcal{B})$ *such that* $f(t,\varepsilon)\to f(t)$ *in* L_p *as* $\varepsilon\to 0$, *then* $x(t,\varepsilon)\to x(t)$ *in* L_p *as* $\varepsilon\to 0$.

Proof. It is well-known that

$$x(t,\varepsilon) = X(t,\varepsilon)\,x_0^\varepsilon + \int_0^t X(t,\varepsilon)\,X^{-1}(s,\varepsilon)\,f(s,\varepsilon)\,ds\,,$$

$$x(t) = X(t)\,x_0 + \int_0^t X(t)\,X^{-1}(s)\,f(s)\,ds\,.$$

Hence

$$\|x(t,\varepsilon)-x(t)\|_p \le \|X(t,\varepsilon)\,x_0^\varepsilon - X(t)\,x_0\|_p + \Big\|\int_0^t X(t,\varepsilon)\,X^{-1}(s,\varepsilon)\,f(s,\varepsilon)\,ds - \int_0^t X(t)\,X^{-1}(s)\,f(s)\,ds\Big\|_p\,. \tag{28}$$

By virtue of Theorem 3 and $x_0^\varepsilon\to x_0$ in $\mathcal{B}$ we have

$$\|X(\varepsilon)\,x_0^\varepsilon - X\,x_0\|_p \le \|X(\varepsilon)x_0^\varepsilon - X(\varepsilon)x_0\|_p + \|X(\varepsilon)x_0 - X\,x_0\|_p \to 0 \;as\; \varepsilon\to 0. \tag{29}$$

On the other hand,

$$\Big\|\int_0^t X(t,\varepsilon)\,X^{-1}(s,\varepsilon)\,f(s,\varepsilon)\,ds - \int_0^t X(t)\,X^{-1}(s)\,f(s)\,ds\Big\|_p \le \tag{30}$$

$$\Big\|\int_0^t X(t,\varepsilon)\,X^{-1}(s,\varepsilon)\,[f(s,\varepsilon)-f(s)]\,ds\Big\|_p +$$

$$+\left\|X(t,\varepsilon)\int_0^t [X^{-1}(s,\varepsilon)\, f(s) - X^{-1}(s)\, f(s)]\, ds\right\|_p +$$
$$\left\|X(t,\varepsilon)\int_0^t X^{-1}(s)\, f(s)\, ds - X(t)\int_0^t X^{-1}(s)\, f(s)\, ds\right\|_p .$$

From (11) we deduce that

$$\begin{aligned}(31)\qquad &\Big\| \int_0^t X(t,\varepsilon)\, X^{-1}(s,\varepsilon)\, [f(s,\varepsilon) - f(s)]\, ds\Big\|_p \le \\ &\Big\| \int_0^T |||X(t,\varepsilon)|||\, |||X^{-1}(s,\varepsilon)|||\, \|f(s,\varepsilon) - f(s)\|\, ds\Big\|_p \le \\ &T^{\frac{1}{q}}\, K^2\, \|f(.,\varepsilon) - f(.)\|_p\, T^{\frac{1}{p}} = \\ &K^2\, T\, \|f(.,\varepsilon) - f(.)\|_p \to 0 \ \text{ as } \varepsilon \to 0\, .\end{aligned}$$

From (17) we get

$$\|X^{-1}(\varepsilon)\, x_0 - X^{-1}\, x_0\|_p \le |||X^{-1}(\varepsilon)|||\, \|x_0 - X(\varepsilon)\, X^{-1}(s)\, x_0\|_p \to 0 \text{ as } \varepsilon \to 0$$

for each $x_0 \in \mathcal{B}$. Applying the above lemma we obtain $\|X^{-1}(\varepsilon)\, f - X^{-1}\, f\|_p \to 0$ as $\varepsilon \to 0$. This implies

$$\begin{aligned}(32)\qquad &\Big\|X(t,\varepsilon) \int_0^t [X^{-1}(s,\varepsilon)\, f(s) - X^{-1}(s)\, f(s)]\, ds\Big\|_p \le \\ &K \Big\| \int_0^T \|X^{-1}(s,\varepsilon)\, f(s) - X^{-1}(s)\, f(s)\|\, ds\Big\|_p = \\ &K\, T\, \|X^{-1}(\varepsilon)\, f - X^{-1} f\|_p \to 0\, .\end{aligned}$$

Now applying the above lemma to $h(t) = \int_0^t X^{-1}(s)\, f(s)\, ds$ we get

$$(33)\qquad \Big\|X(t,\varepsilon) \int_0^t X^{-1}(s)\, f(s)\, ds - X(t) \int_0^t X^{-1}(s)\, f(s)\, ds\Big\|_p \to 0\, .$$

From (30), (31), (32), (33) we deduce

$$(34)\quad \Big\| \int_0^t X(t,\varepsilon)\, X^{-1}(s,\varepsilon)\, f(s,\varepsilon)\, ds - \int_0^t X^{-1}(s)\, f(s)\, ds \Big\|_p \to 0 \text{ as } \varepsilon \to 0\,.$$

Thus $\|x(.,\varepsilon) - x(.)\|_p \to 0$, by (28), (29), (34). The theorem is proved. $\square$

Note that condition $\int_0^T |\|R(t,\varepsilon)\||\, dt \to 0$ implies condition $\big\| \int_0^t R(t,\varepsilon)\, x_0 \big\|_p \to 0$ for each $x_0 \in \mathcal{B}$. We give below some examples where $\|R(s,\varepsilon)x_0 ds\|_p \to 0$ as $\varepsilon \to 0$ but $\int_0^T |\|R(t,\varepsilon)\||\, dt \not\to 0$ as $\varepsilon \to 0$.

Example 1. Let $T = 1$ and $R(t,\varepsilon) = \sin \frac{t}{\varepsilon} I$. It is easy to check that $|\|R(t,\varepsilon)\|| = \big|\sin \frac{t}{\varepsilon}\big|$. This implies

$$\sup_{\varepsilon \in [0,1]} |\|R(t,\varepsilon)\||_q < \infty\,, \ t \in [0,1],$$

$$\Big\| \int_0^t R(s,\varepsilon)\, x ds \Big\|_p = \varepsilon\, \|x\|\, \big\|1 - \cos \frac{t}{\varepsilon}\big\|_{L_p} \to 0 \text{ as } \varepsilon \to 0$$

for each $x \in \mathcal{B}$. Thus, $R(t,\varepsilon)$ satisfies Theorem 2. But

$$\lim_{\varepsilon \to 0} \int_0^T |\|R(t,\varepsilon)\||\, dt = \lim_{\varepsilon \to 0} \int_0^1 \big|\sin \frac{t}{\varepsilon}\big|\, dt = \lim_{\varepsilon \to 0} \varepsilon \int_0^{\frac{1}{\varepsilon}} |\sin s|\, ds.$$

It is well known that $\lim_{\varepsilon \to 0} \varepsilon \int_0^{\frac{1}{\varepsilon}} |\sin s|\, ds = k > 0$. (see [7], p. 384).

Example 2. Let $\mathcal{B} = \ell_1$, $T = 1$. We consider $R(t,\varepsilon)$ in a form of a sequence $R\Big(t, \dfrac{1}{n}\Big)$, $n = 1, 2, 3, \dots$. Assume that $r_{ij}(t)$, i, $j = 1, 2, 3, \dots$ are continuous functions on [0,1] such that

$$\sup_{t \in [0,1]} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |r_{ij}(t)| = M < \infty\,.$$

Then $R\Big(t,\dfrac{1}{n}\Big)$ is the infinite matrix $\big(r_{ij}(t,n)\big)$, $n = 1, 2, ...,$ where

$$r_{ij}(t,n) = \begin{cases} 0 & \text{for } j \le n, \\ r_{i\,j-n}(t) & \text{for } j > n, \end{cases}$$

and $R\Big(t,\dfrac{1}{n}\Big)$ acts in ℓ_1 by the multiplication. It is easy to check that $R\Big(t,\dfrac{1}{n}\Big) \in [\ell_1]$ and

$$\sup_{t\in[0,1]} \Big\|R\Big(t,\frac{1}{n}\Big)\,x\Big\| \le M \sum_{i=n+1}^{\infty} |x_i| \qquad n = 1, 2, 3, ..,$$

where $x = (x_1, x_2, \dots) \in \ell_1$. This implies that $\sup\limits_{n} \big|\big|\big|R\big(.,\dfrac{1}{n}\big)\big|\big|\big|_q < \infty$ for any $q \ge 1$. Also, we have $\sup\limits_{t\in[0,1]} \big\|R\big(t,\dfrac{1}{n}\big)\,x\big\| \to 0$ for each $x \in \ell_1$ as $\dfrac{1}{n} \to 0$. Thus $\Big\|\int\limits_0^t R\big(s,\dfrac{1}{n}\big)\,x\,ds\Big\|_p \to 0$ for each $x \in \ell_1$ as $\dfrac{1}{n} \to 0$. Hence, $R\big(t,\dfrac{1}{n}\big)$ satisfies Theorem 2. It is easy to show that

$$\Big\|R\big(t,\frac{1}{n}\big)\Big\| \ge |r_{11}(t)|.$$

Hence we can choose $r_{11}(t)$ such that $\int\limits_0^1 \big\|R\big(t,\dfrac{1}{n}\big)\big\|\,dt \not\to 0$ as $\varepsilon = \dfrac{1}{n} \to 0$.

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