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ON THE REGULARITY OF RANDOM MAPPINGS

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Abstract. In this paper, some regular properties of random mappings such as the stochastical continuity, the sample continuity and the measurability are investigated. The relation between the regularity and the problem of substituting the argument of a random mapping by a random variable is discussed.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, (X, d) a separable metric space and Y a separable Banach space. By a random mapping Φ from X into Y we mean a rule that assigns to each element $x \in X$ a unique Y-valued random variable ξ. We call ξ the image of Φ under x and write $\xi = \Phi x$.

It is clear that the notion of random mappings is a natural generalization of the well-known notion of stochastic processes. Random operators and stochastic integrals which have been studied by many authors (see [1], [2], [3], [5]) can be regarded as special cases of random mappings.

Mathematically, we can speak of a random mapping Φ from X into Y as a mapping $\Phi : X \times \Omega \to Y$ such that for each $x \in X$ the mapping $\Phi(x, .)$ from $(\Omega, \mathcal{F})$ into $(Y, \mathcal{B}(Y))$ is measurable. If u is a X-valued random variable, then the mapping Φu from $(\Omega, \mathcal{F})$ into $(Y, \mathcal{B}(Y))$ defined by

$$\Phi u(\omega) = \Phi(u(\omega), \omega)$$

is called a substitution of the argument by the random variable u. In general, Φu need not to be measurable (i.e. it may not a Y-valued r.v.). If Φu is measurable, we say that u is Φ-admissible.

In this paper we are concerned with several regular properties of random mappings such as the stochastical continuity, the sample continuity

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and the measurability and determine conditions for a X-valued random variable to be Φ-admissible. Many other results on the sample continuity of random mappings were presented in [4], [6].

2. Regularity of random mappings

Definition 2.1. (i) A mapping $\Phi : X \times \Omega \to Y$ is called a random mapping from X into Y if for each $x \in X$ the mapping Φx from $(\Omega, \mathcal{F})$ into $(Y, \mathcal{B}(Y))$ defined by

$$\Phi x(\omega) = \Phi(x, \omega)$$

is measurable.
(ii) For each $\omega \in \Omega$ the mapping $\Phi_\omega : x \to \Phi(x, \omega)$ is called a sample path of Φ.
(iii) Two random mappings Φ and Ψ from X into Y are said to be equivalent if

$$P\{\Phi x = \Psi x\} = 1, \quad \forall x \in X.$$

In this case we say that Ψ is a modification of Φ.

We now give several definitions of regularity for a random mappings.

Definition 2.2. (i) Φ is measurable if the mapping $\Phi : X \times \Omega \to Y$ is measurable w.r.t. the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{F}$.
(ii) Φ is stochastically continuous at $x_0 \in X$, if $\forall t > 0$, $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$P\Big\{\|\Phi x - \Phi x_0\| > t\Big\} \le \varepsilon,$$

whenever $d(x, x_0) < \delta$. If Φ is stochastically continuous at every point of X, then we say that Φ is stochastically continuous in X.
(iii) Φ is stochastically uniformly continuous in X, if $\forall t > 0$, $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$P\Big\{\|\Phi x_1 - \Phi x_2\| \ge t\Big\} \le \varepsilon,$$

whenever $d(x_1, x_2) < \delta$.
(iv) Φ is continuous if for almost all $\omega \in \Omega$, the sample path Φ_ω is a continuous mapping from X into Y.
(v) Φ is sample-continuous if Φ has a continuous modification.

It is plain that if Φ is sample continuous, then it is stochastically continuous in X.

By a standard argument it is easy to prove the following.

Proposition 2.3. *A stochastically continuous random mapping Φ on a compact space X is uniformly stochastically continuous.*

A random mapping Φ is called simple if there exists a measurable partition $\big(A_i\big)_{i=1}^{\infty}$ of X and a sequence (ξ_i) of Y-valued r.v.'s such that

$$\Phi x = \xi_i$$

if $x \in A_i$.

Proposition 2.4. *A simple random mapping Φ is measurable.*

Proof. For each $B \in \mathcal{B}(Y)$ we have

$$\Big\{(x,\omega) : \Phi(x,\omega) \in B\Big\} = \bigcup_{i=1}^{\infty} A_i \times \Big\{\omega : \xi_i(\omega) \in B\Big\} \in \mathcal{B}(X) \times \mathcal{F}$$

Theorem 2.5. *A continuous random mapping is measurable.*

Proof. Let $\big(s_i\big)_{i=1}^{\infty}$ be the countable set dense in X. For each n define

$$B_i^{(n)} = \Big\{x \in X : d(x, s_i) < \frac{1}{n}\Big\}$$

and set

$$A_1^{(n)} = B_1^{(n)},$$
$$A_i^{(n)} = B_i^{(n)} \setminus \bigcup_{j=1}^{i-1} B_j^{(n)} \quad (i > 1).$$

Then $\big(A_i^{(n)}\big)_{i=1}^{\infty}$ constitutes a measurable partition of X. Define a simple random mappings Φ_n by

$$\Phi_n x = \Phi s_i \quad \text{for} \quad x \in A_i^{(n)}.$$

By Proposition 2.4, Φ_n is measurable. Let μ be certain probability measure on $(X, \mathcal{B}(X))$ and $\mu \times P$ the complete product measure on $(X \times \Omega, \mathcal{B}(X) \times \mathcal{F})$. Let Ω_1 be the set of ω for which the sample paths Φ_ω are continuous. If we can prove that

$$\lim_{n\to\infty} \Phi_n(x,\omega) = \Phi(x,\omega) \tag{2.1}$$

for each $(x,\omega) \in X \times \Omega_1$, the conclusion will follow since Φ_n are measurable and $(\mu \times P)(X \times \Omega_1) = 1$. Fix $(x_0, \omega_0) \in X \times \Omega_1$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\Phi(x,\omega_0) - \Phi(x_0,\omega_0)\| < \varepsilon$ whenever $d(x,x_0) < \varepsilon$. Taking $n \geq \frac{1}{\delta}$. Since $x_0 \in A_i^{(n)}$ for some i and $d(x_0, s_i) < \frac{1}{n} < \delta$ it follows that

$$\|\Phi_n(x_0,\omega_0) - \Phi(x_0,\omega_0)\| = \|\Phi(s_i,\omega) - \Phi(x_0,\omega_0)\| < \varepsilon$$

which proves (2.1) as desired.

Theorem 2.6. *Let Φ be a uniformly stochastically continuous random mapping. Then Φ has a measurable modification.*

Proof. By the assumption, for each n there exists $\delta_n > 0$ such that

$$P\Big\{\|\Phi x_1 - \Phi x_2\| > \frac{1}{n}\Big\} < 2^{-n},$$

whenever $d(x_1, x_2) < \delta_n$. Let $\big(s_i\big)_{i=1}^{\infty}$ be the countable set dense in X. For each n define

$$B_i^{(n)} = \Big\{x \in X : d(x, s_i) < \delta_n\Big\}$$

and set

$$A_1^{(n)} = B_1^{(n)},$$
$$A_i^{(n)} = B_i^{(n)} \setminus \bigcup_{j=1}^{i-1} B_j^{(n)} \quad (i > 1).$$

Then $\big(A_i^{(n)}\big)_{i=1}^{\infty}$ constitutes a countable partition of X. Define a simple random mapping Φ_n by

$$\Phi_n x = \Phi s_i \quad \text{for} \quad x \in A_i^{(n)}$$

Fix $x \in X$. We shall show that

$$P\Big\{\lim \Phi_n x = \Phi x\Big\} = 1 \tag{2.2}$$

Indeed, for each n there exists i such that $x \in A_i^{(n)}$. Then

$$P\Big\{\|\Phi_n x - \Phi x\| > \frac{1}{n}\Big\} = P\Big\{\|\Phi s_i - \Phi x\| > \frac{1}{n}\Big\} < 2^{-n}$$

(since $d(s_i, x) < \delta_n$). Then (2.2) follows from the Borel-Cantelli Lemma.

Set $S = \Big\{(x,\omega) : \lim \Phi_n(x,\omega) \text{ exists}\Big\}$. The random mapping Ψ defined as

$$\Psi(x,\omega) = \begin{cases} \lim \Phi_n(x,\omega) & \text{if } (x,\omega) \in S, \\ 0 & \text{otherwise.} \end{cases}$$

is measurable. Now we shall show that Ψ is the required modification. Indeed, by using (2.2) we get

$$P\Big\{\Phi x = \Psi x\Big\} \geq P\Big\{\lim \Phi_n x = \Phi x\Big\} = 1.$$

Example 1. Take $\Omega = X$, $\mathcal{F} = \mathcal{B}(X)$ and assume that P is a non-atom probability measure. Let a, b be two different elements of Y and D a non-Borel subset of X. Define a random mapping Φ from X into Y as follows.

If $x \in D$, then

$$\Phi(x,\omega) = \begin{cases} a & \text{if } \omega = x, \\ b & \text{if } \omega \neq x, \end{cases}$$

and, if $x \notin D$, then $\Phi(x,\omega) = b$ for all $\omega \in \Omega$.

Now we shall show that Φ is sample continuous, uniformly stochastically continuous and non-measurable.

Let Ψ be the random mapping given by $\Psi(x,\omega) = b$ for all $x \in X$, $\omega \in \Omega$. Then Ψ is continuous and

$$P\Big\{\Phi x \neq \Psi x\Big\} = \begin{cases} P\{\emptyset\} = 0 & \text{if } x \notin D, \\ P\Big\{\omega : \omega = x\Big\} = 0 & \text{if } x \in D. \end{cases}$$

which proves that Φ is sample continuous.

Φ is not measurable since

$$\Big\{(x,\omega) : \Phi(x,\omega) = a\Big\} = D \times D \notin \mathcal{B}(X) \times \mathcal{F}.$$

Now we shall show that $\forall x_1, x_2 \in X$,

$$P\Big\{\Phi x_1 = \Phi x_2\Big\} = 1.$$

which implies that Φ is uniformly stochastically continuous. Indeed, if $x_1, x_2 \notin D$, then $\Phi x_1(\omega) = \Phi x_2(\omega)$ for all ω. If $x_1 \notin D$, $x_2 \in D$, then

$$P\Big\{\Phi x_1 = \Phi x_2\Big\} = P\Big\{\Phi x_2 = b\Big\} = P\Big\{\omega : \omega \neq x_2\Big\} = 1.$$

If $x_1, x_2 \in D$, then

$$P\Big\{\Phi x_1 = \Phi x_2\Big\} \geq P\Big\{\omega : \omega \neq x_1, x_2\Big\} = 1.$$

Example 2. In this example we construct a random mapping which is measurable, uniformly stochastically continuous and not sample-continuous.

Let H be a separable Hilbert space with the orthogonal basis (e_n) and (α_n) be the canonical Gaussian sequence of i.i.d. $N(0,1)$ r.v.'s. Let μ be the Gaussian symmetric probability measure on $(H, \mathcal{B}(H))$. For each n define

$$S_n(x,\omega) = \sum_{i=1}^{n} \alpha_i(\omega)(x, e_i)e_i.$$

Clearly, S_n is a measurable random mapping on H. We have

$$\int_H \int_\Omega \Big\|S_{n+m}(x,\omega) - S_n(x,\omega)\Big\|^2 dP d\mu = \int_H E\|S_{n+m} - S_n\|^2 d\mu$$

$$= \int_H \sum_{n+1}^{n+m} |(x, e_i)|^2 d\mu(x) \leq \int_H \sum_{n+1}^{\infty} |(x, e_i)|^2 d\mu(x).$$

Since $\sum\limits_{n+1}^{\infty} |(x, e_i)|^2 \leq \|x\|\|^2$ and $\int\limits_H \|x\|^2 d\mu < \infty$, by the dominated convergence theorem we conclude that $\{S_n(x,\omega)\}$ is a Cauchy sequence. Hence $\lim S_n(x,\omega)$ exists $(\mu \times P)$-a.s. Put

$$\Phi(x,\omega) = \begin{cases} \lim\limits_{n\to\infty} S_n(x,\omega) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Phi(x,\omega)$ is the measurable random mapping. For each fixed $x \in X$, the series $\sum\limits_{i=1}^{\infty} \alpha_i(\omega)(x, e_i)e_i$ converges a.s. So

$$\Phi x = \sum_{i=1}^{\infty} \alpha_i(\omega)(x, e_i)e_i \quad \text{a.s. .}$$

From this it follows that

$$E\|\Phi x_1 - \Phi x_2\|^2 = \|x_1 - x_2\|^2,$$

which prove the uniform stochastical continuity of Φ. Finally, it is easy to see that Φ is stochastically linear. Since

$$\sup_n \|\Phi e_n\| = \sup_n \|\alpha_n e_n\| = \sup_n |\alpha_n| = \infty,$$

by theorem 3.2 [6], Φ is not sample continuous.

3. Substitution of the argument by a random variable

Let u be a X-valued random variable. The mapping $\Phi u : \Omega \to Y$ given by $\Phi u(\omega) = \Phi(u(\omega), \omega)$ is called a substitution of the argument by u.

The following simple example shows that Φu may not be a Y-valued r.v.

Example 3. Let Φ be the random mapping constructed in Example 1. Define a random variable $u : \Omega \to X$ by $u(\omega) = \omega$. Then

$$\Big\{\omega : \Phi u(\omega) = a\Big\} = \Big\{\omega : \Phi(\omega, \omega) = a\Big\} = D \notin \mathcal{F}.$$

Hence Φu is not measurable.

Definition 3.1. An X-valued random variable u is said to be Φ-admissible if Φu is a Y-valued r.v.

Proposition 3.2. *If u is a countably-valued r.v. then u is Φ-admissible for all random mappings Φ.*

Proof. Assume that $u = x_i$ on the set E_i $(i = 1, 2, \dots)$. Then, for each $B \in \mathcal{B}(Y)$,

$$\Big\{\omega : \Phi u \in B\Big\} = \bigcup_{i=1}^{\infty} \Big\{\omega : \Phi x_i \in B\Big\} \cap E_i \in \mathcal{F}.$$

Proposition 3.3. *If Φ is measurable (in particular, if Φ is continuous) then each X-valued random variable u is Φ-admissible.*

Proof. Indeed, Φu can be represented as the composition of two measurable mappings

$$\Phi u = \Phi \circ S$$

where S is a measurable mapping from Ω into $X \times \Omega$ given by

$$S(\omega) = (u(\omega), \omega).$$

It is sometimes useful to think of a random mapping Φ from X into Y as a mapping from X into the space $L_0^Y(\Omega)$ of Y-valued random variables. From this point of view, two random mappings Φ and Ψ which are equivalent should be considered to be identical. However, as we will see in the following example, it may occur that for certain X-valued random variable u,

$$P\{\Phi u \neq \Psi u\} = 1,$$

even though Φ and Ψ are equivalent.

Example 4. Let $\Omega = X = [0,1]$, $\mathcal{F} = \mathcal{B}(X)$ and P the Lebesgue measure on $[0,1]$. Taking two different elements a, b in Y we define two random mappings Φ and Ψ as follows:

$$\Phi(x,\omega) = b \quad \forall (x,\omega) \in X \times \Omega$$

and

$$\Psi(x,\omega) = \begin{cases} a & \text{if } \omega = x, \\ b & \text{if } \omega \neq x. \end{cases}$$

It is easily seen that Φ, Ψ are equivalent. Consider the X-valued random variable u given by $u(\omega) = \omega$. Then $\Phi u(\omega) = b$ and $\Psi u(\omega) = a$ for all $\omega \in \Omega$.

Now our goal is to examine random variables u for which the above "paradox" does not occur.

Proposition 3.4. *Suppose that u is a X-valued countable random variable. Then $\Phi u = \Psi u$ a.s. provided that Φ and Ψ are equivalent.*

Proof. Assume that $u(\Omega) = \{x_i\}_{i=1}^{\infty}$. Since $\Phi x_i = \Psi x_i$ a.s., we can find a set D of probability one such that $\Phi(x_i,\omega) = \Psi(x_i,\omega)$ for all i and every $\omega \in D$. Now, for each $\omega \in D$, if $u(\omega) = x_i$ then

$$\Phi u(\omega) = \Phi(x_i,\omega) = \Psi(x_i,\omega) = \Psi u(\omega).$$

Hence $\Phi u = \Psi u$ a.s.

Theorem 3.5. *Suppose that Φ and Ψ are continuous and equivalent. Then, for each X-valued r.v. u, we have $\Phi u = \Psi u$ a.s.*

Proof. Let $\left(s_i\right)_{i=1}^{\infty}$ be the countable set dense in X. For each n define

$$B_i^{(n)} = \left\{x \in X : d(x, s_i) < \frac{1}{n}\right\}$$

and set

$$A_1^{(n)} = B_1^{(n)},$$
$$A_i^{(n)} = B_i^{(n)} \setminus \bigcup_{j=1}^{i-1} B_j^{(n)} \quad (i > 1).$$

and

$$\Omega_i^{(n)} = u^{-1}\left(A_i^{(n)}\right).$$

Then $\left(\Omega_i^{(n)}\right)_{i=1}^{\infty}$ constitutes a measurable partition of Ω. Define countably valued random variables u_n by

$$u_n(\omega) = s_i, \quad \text{if} \quad \omega \in \Omega_i^{(n)}.$$

At first we show that $u_n(\omega) \to u(\omega)$ for all $\omega \in \Omega$. Indeed, let $\omega \in \Omega$. For each n there exists $i = i(n)$ such that $\omega \in \Omega_i^{(n)}$. Then

$$d(u_n(\omega), u(\omega)) = d(s_i, u(\omega)) < \frac{1}{n} \to 0, \quad \text{as } n \to \infty.$$

By the assumption there exists a set Ω_0 with $P(\Omega_0) = 1$ such that the sample path Φ_ω is continuous for every $\omega \in \Omega$. Therefore

$$\lim_n \Phi(u_n(\omega), \omega) = \Phi(u(\omega), \omega),$$

for every $\omega \in D$, i.e.

$$\lim_n \Phi u_n = \Phi u \quad \text{a.s.}\ .$$

Similarly, $\lim \Psi u_n = \Psi u$ a.s.

Since $\Phi u_n = \Psi u_n$ a.s. (Proposition 3.4), we infer that $\Phi u = \Psi u$ a.s.

For each random mapping Φ, $\mathcal{F}(\Phi)$ denotes the σ-algebra generated by the family $\left\{\Phi x\right\}_{x \in X}$. The random variable u is said to be independent with respect to Φ if the σ-algebra $\mathcal{F}(u)$ generated by u and $\mathcal{F}(\Phi)$ are independent.

Theorem 3.6. *Suppose that Φ is uniformly stochastically continuous and independent of u. If Ψ is a modification of Φ, then*

$$\Phi u = \Psi u \quad a.s.$$

provided that u is Φ-admissible and Ψ-admissible as well.

Proof. Let u_n be the sequence of countably valued random variables defined in the proof of Theorem 3.5.

Let $t > 0$, $\varepsilon > 0$. Since Φ is uniformly stochastically continuous, there exists $\delta > 0$ such that

$$P\Big\{\|\Phi x_1 - \Phi x_2\| > t\Big\} < \varepsilon, \tag{3.1}$$

whenever $d(x_1, x_2) < \delta$.

Using the independence of u and Φ we get

$$P\Big\{\|\Phi u_n - \Phi u\| > t,\ u \in A_i^{(n)}\Big\} = \int\limits_{A_i^{(n)}} P\Big\{\|\Phi s_i - \Phi x\| > t\Big\} d\mu(x), \tag{3.2}$$

where μ is the distribution of u. Taking $n > \dfrac{1}{\delta}$. Since $d(x, s_i) < \dfrac{1}{n} < \delta$ for each $x \in A_i^{(n)}$, from (3.1) and (3.2) we get

$$P\Big\{\|\Phi u_n - \Phi u\| > t, u \in A_i^{(n)}\Big\} < \varepsilon\mu(A_i^{(n)}).$$

Consequently, for $n > \dfrac{1}{\delta}$ we have

$$\begin{aligned} P\Big\{\|\Phi u_n - \Phi u\| > t\Big\} &= \sum_{i=1}^{\infty} P\Big\{\|\Phi u_n - \Phi u\| > t, u \in A_i^{(n)}\Big\} \\ &\le \varepsilon \sum_{i=1}^{\infty} \mu(A_i^{(n)}) = \varepsilon. \end{aligned}$$

This proves that $P - \lim \Phi u_n = \Phi u$.

Evidently, if Ψ is a modification of Φ, then Ψ is also uniformly stochastically continuous and independent of u. Hence $P - \lim \Psi u_n = \Psi u$. Now the assertion of Theorem 3.6 follows from Proposition 3.4

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