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WEAKLY HOLOMORPHIC EXTENSIONS AND THE CONDITION (L)

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ABSTRACT. In this note we investigate relations between the condition (L) and weakly holomorphic extensions and separately holomorphic functions in infinite dimension.

Let U be an open set in a locally convex space E and A a subset of U. Let f be a map from A into a sequentially complete locally convex space F. We say that f is weakly holomorphically extended to U if for every $u \in F'$, the dual of F, there exists a holomorphic function $\widehat{uf}$ on U such that $\widehat{uf}\big|_A = uf$.

In [1] Ligocka and Siciak have shown that if a holomorphic map $f : A \to F$, where A is an open subset of U and E is a Baire space, is weakly holomorphically extended to U, then f is extended holomorphically to U. Later, T. V. Nguyen [4] extended the above mentioned result to the case where A is an arbitrary subset of U satisfying the condition (L) (A is not necessarily open).

Our main aim in this note is to extend the results of T. V. Nguyen [4] to the general case.

First of all we recall some notations. Let A be a subset of E. We say that A satisfies the condition (L_0) at $a \in E$ if for every sequence of continuous polynomials $\{Q_k\}$ on E satisfying

$$|Q_k(x)| \le M(x), \quad \forall k \in N, \forall x \in A,$$

where $M(x)$ is a constant depending only on x, and for every $\varepsilon > 0$, there exist $C > 0$ and a neighbourhood U of a such that

$$|Q_k(x)| \le C(1+\varepsilon)^{\deg Q_k} \quad \forall x \in U.$$

We say that A satisfies the condition (L) at a if the set $A \cap U$ satisfies the condition (L_0) at a for every neighbourhood U of a.

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Our main results are the following three theorems.

Theorem 1. *Let f be a map from a subset A of an open set U in a Frechet space E into a Frechet space F such that for all $u \in F'$, the function uf is extended to a holomorphic function $\widehat{uf}$ on U. Then f is extended holomorphically to U if*

(i) *$0 \in A$ and $A \cap sU_1$ satisfies (L_0) at 0 for some balanced neighbourhood U_1 of 0 in U and for some $0 < s < 1$;*

(ii) *Every $\widehat{uf}$ is bounded on U_1.*

Theorem 2. *Let f be as in Theorem 1. Then f is extended holomorphically to U if A satisfies the condition (L) at some point $a \in A$.*

Theorem 3. *Let U be an open set in a Frechet space E, V an open set in a Frechet space F. Assume that f is a function defined on $U \times V$ such that*

(i) *$\forall z \in A \subset U, y \to f_z(y) := f(z, y)$ is holomorphic on V;*

(ii) *$\forall y \in V, z \to f^y(z) := f(z, y)$ is holomorphic on U.*

If A satisfies the condition (L) at some point $a \in A$, then f is holomorphic.

The proofs of Theorems 1 and 2 are given in Section 1 and Section 2, respectively. In Section 3 we shall prove Theorem 3 by reducing it to a result of Terada-Nguyen [4].

Remark. In the case where $\widehat{uf}$ (resp. f^y) is bounded on an open set $U_1 \subset U$ which is not dependent on $u \in F'(y \in V)$, Theorem 2 (in which F is a dual of a Frechet space) and Theorem 3 were proved by Nguyen [4].

1. Proof of Theorem 1

We shall need the following.

Lemma 1.1. *Let A be a subset of a Frechet space E. If $A \cap sU$ satisfies the condition (L_0) at 0 for a balanced neighbourhood U of $0 \in E$ and some $0 < s < 1$, then A has the identity property for $H^\infty(U)$, i.e.*

$$f \in H^\infty(U), \quad f\big|_{A \cap U} = 0 \Rightarrow f \equiv 0.$$

Proof. Let $f \in H^\infty(U)$ be such that $f\big|_{A \cap U} = 0$. Consider the Taylor expansion of f at $0 \in E$:

$$f(z) = \sum_{n \geq 0} P_n f(z),$$

where

$$P_n f(z) = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} \frac{f(\lambda z)}{\lambda^{n+1}} d\lambda \quad \text{for} \quad n \geq 0.$$

By Cauchy inequalities we have

$$|P_n f(z)| \leq \|f\|_U, \quad \|z\|_U < 1, \quad n \geq 0,$$

where $\|z\|_U$ is the evaluation of Minkowski functional associated to U at z. Hence

$$|P_n f(z)| \leq s^n \|f\|_U, \quad \|z\|_U < s, \quad n \geq 0.$$

Put

$$Q_j(z) = \sum_{n=0}^{j} P_n f(z).$$

Then

$$|Q_j(z)| \leq |f(z)| + C_0 s^j \|f\|_U, \quad \|z\|_U < s, \quad j \geq 0,$$

where $C_0 = \sum\limits_{n \geq 1} s^n$. Therefore,

$$|Q_j(z)| \leq C_0 s^j \|f\|_U, \quad z \in A, \quad \|z\|_U < s, \quad j \geq 0.$$

Since $A \cap sU$ satisfies the condition (L_0) at 0 for every $\varepsilon > 0$, there exist $\delta > 0$ and $C > 0$ such that

$$|Q_j(z)| \leq C s^j e^{j\varepsilon} \|f\|_U, \quad \|z\|_U < \delta, \quad j \geq 0.$$

If $0 < \varepsilon < \log \dfrac{1}{s}$, then

$$|Q_j(z)| \leq C(se^{\varepsilon})^j \|f\|_U \to 0 \quad \text{as} \quad j \to \infty.$$

Thus $f(z) = \lim\limits_{j \to \infty} Q_j(z) = 0, \quad \|z\|_U < \delta$. This yields $f \equiv 0$ on U.

Now we can prove Theorem 1 as follows.

Let $A \cap sU_1$ satisfy the condition (L_0) at 0 with $0 < s < 1$. By Lemma 1.1, the graph of the linear map $S : F'_{\text{bor}} \to H^\infty(U_1)$ which is given by

$$(Su)(z) = \widehat{uf}, \quad u \in F'_{\text{bor}}, z \in U_1,$$

is closed. The closed graph theorem yields the continuity of S. Define $g : U_1 \to [F'_{\text{bor}}]' \supset F$ by

$$g(z)(u) = (Su)(z) \quad \text{for} \quad z \in U_1, u \in F'_{\text{bor}}.$$

Since the Dirac map $\delta : U_1 \to [H^\infty(U_1)]'$ with

$$(\delta z)(\varphi) = \varphi(z) \quad \text{for} \quad z \in U_1, \varphi \in H^\infty(U_1)$$

is holomorphic and

$$(\omega g)(z) = (S"\omega)(\delta z) \quad \text{for} \quad \omega \in [F'_{\text{bor}}]", z \in U_1,$$

it follows that g is holomorphic.

Again by Lemma 1.1, we deduce that $g(U_1) \subset F$ and hence $g : U_1 \to F$ is also holomorphic. Moreover, by the identity property of $A \cap sU_1$ for $H^\infty(U_1)$ and by $\{\widehat{uf} : u \in F'\} \subset H^\infty(U_1)$, we have

$$ug = \widehat{uf}\big|_{U_1} \quad \text{for} \quad u \in F'.$$

By applying Ligocka-Siciak's result [1], it follows that g and hence f is extended holomorphically to U. Theorem 1 is now proved.

2. Proof of Theorem 2

Consider the linear map $S : F'_{\text{bor}} \to \lim\limits_{a \in K \nearrow U} \text{proj}\ H(K)$ induced by f as in Theorem 1.

(i) First we will show that S is continuous. It suffices to check the continuity of the map

$$S_{K,n} : F'_n \to H(K)$$

for all $n \geq 1$ and all compact sets K in U containing the point a. We now prove that $S_{K,n}$ has closed graph. Indeed, let $\{u_k\} \subset F'_n, u_k \to u$ be such that $S_{K,n}(u_k) \to \sigma \in H(K)$. Since $H(K) = \lim\limits_{V \searrow K} \text{ind}\ H^\infty(V)$ is regular [2], we can find a neighbourhood V of K such that $\{S_{K,n}(u_k)\}$ is contained and bounded in $H^\infty(V)$. By the Monteless of $H(V)$ we may assume, without loss of generality, that $S_{K,n}(u_k) \to \sigma$ in $H(V)$. From the relations

$$S_{K,n}(u)(z) = uf(z) = \lim_k u_k f(z) = \lim_k S_{K,n}(u_k)(z) = \sigma(z)$$

for all $z \in A \cap V$, we deduce that

$$S_{K,n}(u) = \sigma.$$

Hence $S_{K,n}$ has a closed graph. By using the closed graph theorem of Grothendieck [5], $S_{K,n}$ is continuous.

(ii) Consider the map

$$\hat{f} : U \to [F'_{\text{bor}}]' \supset F$$

given by

$$\hat{f}(z)(u) = S(u)(z) \quad \text{for} \quad z \in U \quad \text{and} \quad u \in F'_{\text{bor}}.$$

Then the maps $\hat{f}_n := \hat{f} : U \to F_n$" are holomorphic by the holomorphicity of $u\hat{f}_n$ for all $u \in F'_n$. This yields the holomorphicity of f.

(iii) From the inclusion $\hat{f}(A) \subset F$ and the identity property of A, we deduce that $\hat{f}(U) \subset F$.

The proof of Theorem 2 is now complete.

3. Proof of Theorem 3

Without loss of generality we may assume that $a = 0 \in A$. For $k, n \geq 1$, put

$$F_k^n = \{y \in V : \|f^y\|_{U_n} \leq k\},$$

where $\{U_n\}$ is a decreasing neighbourhood basis of $0 \in A$.

Let us check that F_k^n are closed. Let $\{y_j\} \subset F_k^n$ be such that $y_j \to y \in V$. Since $y_j \in F_k^n$, we have $\|f^{y_j}\|_{U_n} \leq k$. Thus, the family $\{f^{y_j}\}$ is bounded in $H(U_n)$. By the Monteless of $H(U_n)$, we may assume that

$$f^{y_j} \to \varphi \quad \text{in} \quad H(U_n).$$

From the relations

$$\varphi(z) = \lim_j f^{y_j}(z) = \lim_j f(z, y_j) = f(z, y)$$

for $z \in A \cap U_n$ and from the condition (L) of A we get

$$\varphi(z) = f^y(z) \quad \text{for} \quad z \in U_n.$$

This implies

$$\|\varphi\|_{U_n} = \|f^y\|_{U_n} \leq k, \quad \text{i.e} \quad y \in F_k^n.$$

By Baire theorem we can find k_0, n_0 such that

$$W = \text{Int } F_{n_0}^{k_0} \neq \phi.$$

We now consider $f\big|_{U\times W}$. Since $A \cap U$ satisfies the condition (L) at $0 \in A \cap U$ and f^y is bounded on $U_{n_0} \subset U$ for $y \in W$, the theorem of Terada-Nguyen [4] implies that f is holomorphic on $U \times W$.

Finally, for each $B \in \mathcal{K}(E)$, the family of all non zero balanced convex compact sets in E, consider the function

$$f_B = f\big|_{U_B\times V}, \quad \text{where} \quad U_B = U \cap B.$$

Since $1/2U_B$ is relatively compact in U and $f^y \in H(U)$ for $y \in V$, f^y is bounded on $1/2U_B$. On the other hand, since A satisfies the condition (L) at $0 \in A \cap U_B$, the theorem of Terada-Nguyen [4] again implies that f_B is holomorphic on $U_B \times V$. Then f is holomorphic on $U \times V$. Hence Theorem 3 is proved.

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