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FRECHET-VALUED MEROMORPHIC FUNCTIONS ON COMPACT SETS IN $\mathbb{C}^n$

NGUYEN VAN DONG

Abstract. Let F be a Frechet space. The main aim of this paper is to prove that $[F^*_{bor}]^* \in (LB_\infty)$ if and only if $M(X, [F^*_{bor}]^*) = M_w(X, [F^*_{bor}]^*)$ for every compact uniqueness subset X of $\mathbb{C}^n$. We also prove that a compact set X in $\mathbb{C}^n$ is pluripolar if and only if X is unique and $M(X, F) = M_w(X, F)$ for every Frechet space $F \in (DN)$.

1. Introduction

Let X be a subset of $\mathbb{C}^n$ and F a sequentially complete locally convex space. A function f defined on a dense open subset X_0 of X with values in F is called meromorphic on X if it can be extended to a meromorphic function on a neighbourhood of X in $\mathbb{C}^n$. This means that there exist a neighbourhood U of X and a meromorphic function $\widehat{f}$ on U such that $X \setminus P(\widehat{f})$ is dense in X and $f(z) = \widehat{f}(z)$ for $z \in X_0 \setminus P(\widehat{f})$. If this holds for $x^* f$, where x^* is an arbitrary element of the dual space F^* of F, we say that f is weakly meromorphic on X. Write $M(X, F)$ and $M_w(X, F)$ for vector spaces of meromorphic and weakly meromorphic functions on X with values in F, respectively. The main aim of the present paper is to find necessary and sufficient conditions for which

$$(\omega) \qquad M(X, F) = M_w(X, F).$$

The case where F is a Banach space and X either is open or compact was proved in [3]. In [2] the authors proved that a Frechet space F has a continuous norm (resp. $F \in (DN)$) if and only if (ω) holds for every open subset (resp. $\widetilde{L}$-regular compact set) X of $\mathbb{C}^n$.

The main results of this paper are the following theorems.

Theorem A. *Let F be a Frechet space. Then $[F^*_{bor}]^* \in (LB_\infty)$ if and only if $M(X, [F^*_{bor}]^*) = M_w(X, [F^*_{bor}]^*)$ for every compact uniqueness set X of $\mathbb{C}^n$. Here F^*_{bor} denotes the space F^* equipped with the bornological topology.*

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Theorem B. *Let X be a compact set of $\mathbb{C}^n$. The following are equivalent:*

i) *X is not pluripolar;*

ii) *$\big[\mathcal{H}(X)\big]' \in \big(LB^\infty\big)$;*

iii) *X is unique and $M(X,F) = M_w(X,F)$ for every Frechet space F with $F \in (DN)$.*

2. Preliminaries

For a Frechet space E we always assume that its locally convex structure is generated by an increasing system $\big\{\|\ \ \|_n\big\}_{n=1}^{\infty}$ of semi-norms.

If M is an absolutely convex subset of E, we define $\|.\|_M^* : E \longrightarrow [0,+\infty]$ by

$$\|u\|_M^* = \sup\big\{|u(x)| : x \in M\big\},$$

where E' denotes the topological dual of E. Instead of $\|u\|_{U_k}^*$ we write $\|u\|_k^*$, where $U_k = \big\{x \in E : \|x\|_k \leq 1\big\}$.

We say that E has the property:

(DN) There exists $p \in \mathbf{N}$ such that for every $q \in \mathbf{N}$ and every $d > 0$ there exist $k \in \mathbf{N}$ and $C > 0$ with

$$\| x \|_q^{1+d} \leq C \ \| x \|_k \ \| x \|_p^d$$

for all $x \in E$.

(LB_∞) For every positive increasing unbounded sequence $\big(\rho_N\big)_{N\in\mathbf{N}}$ there exists $p \in \mathbf{N}$ such that for all $q \in \mathbf{N}$ there exist $N_0 \in \mathbf{N}$ and $C > 0$ such that for all $x \in E$ there exists $k \in \mathbf{N}$ with $q \leq k \leq N_0$ with

$$\| x \|_q^{1+\rho_k} \leq C \ \| x \|_k \ \| x \|_p^{\rho_k}.$$

(LB^∞) For every positive increasing unbounded sequence $\big(\rho_N\big)_{N\in\mathbf{N}}$ and every $p \in \mathbf{N}$ there exists $q \in \mathbf{N}$ such that for all $n_0 \in \mathbf{N}$ there exist $N_0 \in \mathbf{N}$ and $C > 0$ such that for all $u \in E'$ there exists $k \in \mathbf{N}$ with $n_0 \leq k \leq N_0$ with

$$\| u \|_q^{*\,1+\rho_k} \leq C \ \| u \|_k^* \ \| u \|_p^{*\,\rho_k}.$$

Note that $\|.\|_p$ is a continuous norm on E if $E \in (LB_\infty)$.

These properties were introduced and investigated by Vogt (see [8], [9]). We remark that

$$(LB_\infty) \ \ \not\rightleftarrows \ \ (DN),$$

$$(LB^{\infty}) \quad \not\rightleftarrows \quad (\Omega).$$

Let α be an increasing sequence of positive real numbers with $\lim\limits_{n\to\infty} \alpha_n = \infty$ (α will be called an exponent sequence). For $0 < R \leq \infty$ we define the power series space by

$$\lambda_R(\alpha) = \left\{ x \in \mathbb{C}^N : \|x\|_r = \sum |x_n| r^{\alpha_n} < \infty \text{ for any } 0 < r < R \right\}.$$

Then $\lambda_R(\alpha)$ is a Frechet space under the natural topology induced by the seminorms $\{\| \, \|_r : 0 < r < R\}$. We call $\lambda_R(\alpha)$ a power series space of finite type if $R < \infty$ or of infinite type if $R = \infty$.

Well-known examples of nuclear power series spaces are

$$\begin{aligned} s &\cong \lambda_\infty \left(\log(n+1)_{n\in\mathbf{N}}\right), \\ \mathcal{H}(\mathbb{C}^k) &\cong \lambda_\infty \left(\left(n^{\frac{1}{k}}\right)_{n\in\mathbf{N}} \right), \\ \mathcal{H}(D^k) &\cong \lambda_1 \left(\left(n^{\frac{1}{k}}\right)_{n\in\mathbf{N}} \right), \end{aligned}$$

where D stands for the open unit disk in $\mathbb{C}$ and $\mathcal{H}(\Omega)$ denotes the space of all holomorphic functions on Ω endowed with the compact-open topology.

For locally convex spaces E and F we denote by $\mathcal{L}(E,F)$ the space of all continuous linear mappings, while $\mathcal{LB}(E,F)$ denotes the set of all $A \in \mathcal{L}(E,F)$ for which there exists a zero neighbourhood U in E such that $A(U)$ is bounded.

We recall three following results of [8]

Lemma 2.1. *For a Frechet space F the following assertions are equivalent:*

(i) $\mathcal{L}(\lambda_\infty(\beta), F) = \mathcal{LB}(\lambda_\infty(\beta), F)$ *for all exponent sequences β;*

(ii) *F has property (LB_∞).*

Lemma 2.2. *For a Frechet space E the following assertions are equivalent:*

(i) $\mathcal{L}(E, \lambda_\infty^\infty(\alpha)) = \mathcal{LB}(E, \lambda_\infty^\infty(\alpha))$ *for all exponent sequences α;*

(ii) *E has property (LB^∞).*

Lemma 2.3. *Let E, F be Frechet spaces with $E \in (LB^\infty)$ and $F \in (DN)$. Then*

$$\mathcal{L}(E,F) = \mathcal{LB}(E,F).$$

Unless otherwise specified, we shall write, throughout this paper, $Z(h)$ and $Z(g,\sigma)$ for $h^{-1}(0)$ and $g^{-1}(0)\cap\sigma^{-1}(0)$, respectively, and F^*_{bor} for the bornological space associated to F^*, where F^* is the strong dual of Frechet space F.

3. Proof of Theorem A

To prove Theorem A we first prove the following results:

Lemma 3.1 [2]. *Let X be a pseudoconvex domain in $\mathbb{C}^n$ and f a meromorphic function on X with values in a locally convex space F. Then, for every relatively compact domain Y in X, there exist holomorphic functions h and σ on X such that*

$$f=\frac{h}{\sigma} \text{ and } \operatorname{codim}_y Z(h,\sigma)\geq 2 \text{ for } y\in Y.$$

Lemma 3.2 [2]. *Let F be a locally convex space. Let σ and β be holomorphic functions on an open subset $X\subset\mathbb{C}^n$ and $g: X\longrightarrow F$ a holomorphic function. Assume that $\dfrac{\beta g}{\sigma}$ is holomorphic on X and codim $Z(g,\sigma)\geq 2$. Then $\dfrac{\beta}{\sigma}$ is holomorphic on X.*

Lemma 3.3. *There exists a polar compact uniqueness set X in $\mathbb{C}$.*

Proof. a) First we prove that a compact set X in $\mathbb{C}$ is unique if and only if X is perfect. Indeed, let $f\in\mathcal{H}(U)$, $f\big|_X=0$, where U is a neighbourhood of X in $\mathbb{C}$. Then $f\big|_Z=0$ for every connected component Z of U meeting X and hence $f=0$ on $V=\bigcup\{\,Z\,:\,Z$ is a connected component of U with $Z\cap X\neq\emptyset\}$.

Conversely, if X is not perfect, then X has an isolated point and hence X is not unique.

b) Given a sequence $\ell=(\ell_n)\downarrow 0$, $\ell_n<2^{-n}$. Define a family of closed intervals $\big(J_{n,j}\big)_{n\geq 0,1\leq j\leq 2^n}$ with

$$\begin{aligned} J_{0,s}&=[0,1]\quad\text{for } s\geq 1,\\ J_{n+1,2j-1}&=[\min\ J_{n,j},\min\ J_{n,j}+\ell_n\ ],\\ J_{n+1,2j}&=[\max\ J_{n,j}-\ell_n,\max\ J_{n,j}\ ],\end{aligned}$$

$$C_n(\ell) = \bigcup_{j=1}^{2^n} J_{n,j},$$
$$C(\ell) = \bigcap_{n\geq 0} C_n(\ell).$$

c) For each $n \geq 0$ define μ_n to be the uniform measure on $\bigcup_{j=1}^{2^n} J_{n,j}$ giving weight 2^{-n} to each $J_{n,j}$, i.e.

$$d\mu_n(x) = \frac{1}{2^n l_n} \chi_{\bigcup_{j=1}^{2^n} J_{n,j}} dx.$$

Define a probability measure on $C(\ell)$ by

$$\mu = \lim_{n\to\infty} \mu_n.$$

Note that this limit exists. Put

$$\varphi(z) = \int_{\mathbb{C}} \log|x - z| d\mu(x).$$

d) We shall prove that with $\ell_n = e^{-2^n}$ and μ defined as above, C ($= C(\ell)$) is a polar compact uniqueness set.

Clearly, C is perfect because C is a set of the Cantor type. We prove $C = \varphi^{-1}(-\infty)$. Obviously $\varphi(z) > -\infty$ for $z \notin C$. Now assume that $x_0 \in C$ and $x_0 \in J_{n,j_n}\ \forall\ n \geq 0$. Since $J_{n,j_{n+1}} \subset J_{n,j_n}$ we have

$$\int_{\mathbb{C}} \log|x - x_0| d\mu(x) = \sum_{n\geq 0}^{\infty} \int_{J_{n,\ j_n} \setminus J_{n,j_{n+1}}} \log|x - x_0| d\mu(x)$$
$$\leq \sum_{n\geq 0}^{\infty} \log(\ell_n)\mu(J_{n,\ j_n} \setminus J_{n,j_{n+1}}) = -\sum_{n\geq 0} 2^{-n}2^{n-1} = -\sum_{n\geq 0} \frac{1}{2} = -\infty.$$

Finally, it remains to check that φ is subharmonic on $\mathbb{C}$.

Let $z_0 \notin C$. Then $\operatorname{dist}(C, z_0) > 0$ and hence $\log|z - z_0|$ is bounded on a neighbourhood of z_0. By Lebesgue dominated convergence theorem, φ is continuous at z_0. Let $z_0 \in C$. Given $A > 0$. Choose $\varepsilon > 0$ such that

$$\int_{\mathbb{C}\setminus\triangle(z_0,\varepsilon)} \log|x - x_0|\ d\mu(x) < -2A,$$

with

$$\triangle(z_0,\varepsilon)=\{z\in\mathbb{C}:|z-z_0|<\varepsilon\}.$$

Since

$$\log|x-z|\longrightarrow\log|x-z_0|\quad\text{as } z\longrightarrow z_0$$

uniformly on $\mathbb{C}\setminus\triangle(z_0,\varepsilon)$, we have

$$\int\limits_{\mathbb{C}}\log|x-z|\ d\mu(x)\le\int\limits_{\mathbb{C}\setminus\triangle(z_0,\varepsilon)}\log|x-z|\ d\mu(x)\le -A$$

for $|z-z_0|<\eta$ with $\eta>0$ sufficiently small. Thus φ is upper-semicontinuous. Hence, from the inequality

$$\begin{aligned}\frac{1}{\pi r^2}\int\limits_{\triangle(a,r)}\varphi(z)\ d\lambda(z)&=\int\limits_{\mathbb{C}}\frac{1}{\pi r^2}\int\limits_{\triangle(a,r)}\log|x-z|\ d\,\mu(x)\\&\ge\int\limits_{\mathbb{C}}\log|a-x|\ d\mu(x)=\varphi(a)\end{aligned}$$

it follows that φ is subharmonic.

Remark. The author thanks P. Thomas, who showed us the construction of the set C as in Lemma 3.3.

Now we are able to prove Theorem A.

Let F be a Frechet space with $\left[F^*_{bor}\right]^*\in(LB_\infty)$ and $f\in M_w\big(X,\left[F^*_{bor}\right]^*\big)$, where X is a compact uniqueness set in $\mathbb{C}^n$. First note that $F^*_{bor}\cong\lim\operatorname{ind}F^*_p$, where $\{\|.\|_p\}$ is a fundamental system of semi-norms of F and F_p denotes the Banach space associated to $\|.\|_p$ for each $p\ge 1$. Hence $\left[F^*_{bor}\right]^*\cong\lim\operatorname{proj}F^{**}_p$. Since $\left[F^*_{bor}\right]^*\in(LB_\infty)$, we may assume without loss of generality that $\|.\|^*_{U_1^0}$ is a norm on $\left[F^*_{bor}\right]^*$. By [3], for each $p\ge 1$, there exist a neighbourhood U_p of X in $\mathbb{C}^n$ and a meromorphic function $\widehat{f_p}:U_p\longrightarrow F^{**}_p$ such that $f_p\big|_{X\setminus P(\widehat{f}_p)}=\omega_p f\big|_{X\setminus P(\widehat{f}_p)}$, where $\omega_p:\left[F^*_{bor}\right]^*\longrightarrow F^{**}_p$ is the canonical map.

The meromorphic function f_p is extended uniquely to a meromorphic function $\widehat{f_p}$ on $\widehat{U}_p$, the envelope of holomorphy of U_p. By Lemma 3.1, $\widehat{f_p}$ and hence f_p can be written in the form $f_p=\dfrac{h_p}{\sigma_p}$, where $h_p:U_p\longrightarrow F^{**}_p$, $\sigma_p:U_p\longrightarrow\mathbb{C}$ are holomorphic functions and $\sigma_p\ne 0$ such that

$$\operatorname{codim}Z(h_p,\sigma_p)\ge 2.$$

Since $\|.\|^*_{U_1^0}$ is a norm on $\left[F^*_{bor}\right]^*$, ω_1^p is injective, where $\omega_1^p : F_p^{**} \longrightarrow F_1^{**}$ is the canonical map. Since $\omega_1 = \omega_1^p.\omega_p$ and by the uniqueness of X, shrinking U_p we get

$$\frac{h_1}{\sigma_1}\Big|_{U_p} = \frac{\omega_1^p.h_p}{\sigma_p} \ .$$

From the injectivity of ω_1^p we have

$$Z(\omega_1^p h_p, \sigma_p) = Z(h_p, \sigma_p),$$

and hence

$$\operatorname{codim} Z(\omega_1^p h_p, \sigma_p) = \operatorname{codim} Z(h_p, \sigma_p) \geq 2.$$

Indeed, we have

$$\frac{h_1}{\sigma_1}\Big|_{X\setminus P(f_p)} = \frac{\omega_1^p.h_p}{\sigma_p}\Big|_{X\setminus P(f_p)}$$

or

$$\sigma_p h_1\big|_{X\setminus P(f_p)} = \omega_1^p.\sigma_1 h_p\big|_{X\setminus P(f_p)}.$$

Since $X \setminus P(f_p)$ is dense in X,

$$\sigma_p h_1\big|_X = \omega_1^p.\sigma_1 h_p\big|_X.$$

By Lemma 3.2, it follows that $\dfrac{\sigma_1}{\sigma_p}\Big|_{U_p}$ is holomorphic for $p \geq 1$. Again by the uniqueness of X we can define a linear map

$$\widetilde{h} \ : \ F^*_{bor} \longrightarrow \mathcal{H}(X)$$

by

$$\widetilde{h}\big|_{F_p^{**}} = \left(\frac{\sigma_1}{\sigma_p}\right) \ \widetilde{h}_p \qquad \text{for } p \geq 1,$$

where $\widetilde{h}_p(x^*)(z) = x^*\left(h_p(z)\right)$ for $z \in U_p$ and $x^* \in F_p^*$. Obviously $\widetilde{h}$ has a closed graph. The open mapping theorem of Grothendieck [6] yields the continuity of $\widetilde{h}$.

By [5], $[\mathcal{H}(X)]^*$ is isomorphic to a quotient space of $\mathcal{H}(\mathbb{C}^n) \cong \lambda_\infty\left(j^{\frac{1}{n}}\right)$ and hence by [8]:

$$\mathcal{LB}\left(F^*, [\mathcal{H}(X)]\right) = \mathcal{L}\left(F^*, [\mathcal{H}(X)]\right).$$

It follows that we can find a neighbourhood W of $0 \in F^*_{bor}$ such that $\widetilde{h}(W)$ is bounded in $\mathcal{H}(X)$. This yields p for which $\widetilde{h}(W)$ is contained and bounded in $\mathcal{H}^\infty(U_p)$, the Banach space of bounded holomorphic functions on U_p. Thus, the form

$$\widehat{h}(z)(x^*) = \widetilde{h}(x^*)(z) \quad \text{for } z \in U_p, \quad x^* \in F^*_{bor}$$

defines a holomorphic function $\widehat{h}$ from U_p into $\left[F^*_{bor}\right]^*$ such that $\dfrac{\widehat{h}}{\sigma_1}\big|_X = f$. Hence $f \in M\big(X, [F^*_{bor}]^*\big)$.

Conversely, by Lemma 3.3 we can choose a compact polar uniqueness set X in $\mathbb{C}$. It is known that $[\mathcal{H}(X)]^* \cong \mathcal{H}(\mathbb{C} \setminus X) \cong \mathcal{H}(\mathbb{C})$. Here the first isomorphism follows from the dual Grothendieck theorem and the second one was proved by Zaharjuta [10].

By Vogt [8] it suffices to show that

$$\mathcal{LB}\left([\mathcal{H}(X)]^*, [F^*_{bor}]^*\right) = \mathcal{L}\left([\mathcal{H}(X)]^*, [F^*_{bor}]^*\right).$$

Given $T \in \mathcal{L}\big([\mathcal{H}(X)]^*, [F^*_{bor}]^*\big)$. Consider $T^* : [F^*_{bor}]^{**} \longrightarrow [\mathcal{H}(X)]^{**}$. Since $[\mathcal{H}(X)]^{**} = \mathcal{H}(X)$, we can define a map $f : X \longrightarrow [F^*_{bor}]^*$ by

$$f(z)(x^*) = (T^*x^*)(z) \quad \text{for } x^* \in [F^*_{bor}]^{**},\ z \in X.$$

By the $\sigma[(F^*_{bor}]^{**}, [F^*_{bor}]^*)$-continuity of $f(z)$, $f(z) \in [F^*_{bor}]^*$. Moreover, $f \in M_w(X, [F^*_{bor}]^*)$. By the hypothesis we can find a neighbourhood U of X in $\mathbb{C}^n$ and a $[F^*_{bor}]^*$-valued meromorphic function $\widehat{f}$ on U such that

$$\widehat{f}\big|_{X \setminus P(\widehat{f})} = f\big|_{X \setminus P(\widehat{f})}.$$

Hence we have

$$\widehat{\sigma}(z)(T^*x^*)(z) = \widehat{\sigma}(z)\widehat{f}(z)(x^*) = \widehat{h}(z)(x^*),$$

where $\widehat{h} : U \longrightarrow [F^*_{bor}]^*$ and $\widehat{\sigma} : U \longrightarrow \mathbb{C}$ are bounded holomorphic functions and $\widehat{\sigma} \neq 0$ such that $\widehat{f} = \dfrac{\widehat{h}}{\widehat{\sigma}}$ and $Z(\widehat{h}, \widehat{\sigma}) = \emptyset$.

We have that $\widehat{\sigma}T^*(B^0)$ is contained and bounded in $\mathcal{H}^\infty(U)$, where $B = \widehat{h}(U)$. It follows that $T^*(B^0)$ is contained and bounded in $\mathcal{H}(U\setminus$

$Z(\widehat{\sigma})$). Shrinking U we may assume that $P(\widehat{f}) = Z(\widehat{\sigma}) \subset X$ and has only finite many points.

Since X is unique, X does not have an isolated point. On the other hand, from the continuity of f on X it follows that $Z(\widehat{\sigma}) = \emptyset$ and hence $\widehat{f}$ is holomorphic on U. Choose a relatively compact neighbourhood V of X in U. We have

$$\inf_V |\widehat{\sigma}| > 0.$$

Hence T^* is bounded on B°.

Put $W = T^*(B^\circ)$. Then $V = W^\circ$ is a neighbourhood of $O \in [\mathcal{H}(X)]^*$ and $T(V) \subset B^{\circ\circ}$ is bounded in $[F^*_{bor}]^*$. Hence $[F^*_{bor}]^* \in (LB_\infty)$. The theorem is now proved.

4. Proof of Theorem B

We need the following results.

Lemma 4.1. *Let K be a compact set in $\mathbb{C}^n$ such that $[\mathcal{H}(K)]' \in (LB^\infty)$. Then K is a unique set.*

Proof. Given $f \in \mathcal{H}(K)$ with $f\big|_K = 0$. Let (U_k) be a neighbourhood basis of K in $\mathbb{C}^n$. For each $k \geq 1$, put

$$\varepsilon_k = \|f\|_{U_k} = \sup\{|f(x)| : x \in U_k\}.$$

Then $\varepsilon_k \downarrow 0$. By applying (LB^∞) to $\rho_N = \sqrt{-\log \varepsilon_N} \uparrow +\infty$ we have for $p \geq 1$, $f \in \mathcal{H}^\infty(U_p)$, there exists $q \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exist $\widetilde{N} \in \mathbf{N}$ and $C_N > 0$ such that for all $n \in \mathbf{N}$ there exists $k_n \in \mathbf{N}$ with $N \leq k_n \leq \widetilde{N}$ and

$$\|f^n\|_q^{1+\rho_{k_n}} \;\leq\; C_N \, \|f^n\|_{k_n} \, \|f^n\|_p^{\rho_{k_n}}.$$

This yields

$$\|f\|_q^{1+\rho_{k_n}} \;\leq\; C_N^{\frac{1}{n}} \, \|f\|_{k_n} \, \|f\|_p^{\rho_{k_n}}.$$

Choose $N \leq k \leq \widetilde{N}$ such that $\#A = \infty$, where $A = \{n : k_n = k\}$. Then

$$\begin{aligned}
\|f\|_q \;&\leq\; \lim_{n \in A} C_N^{\frac{1}{n}} \, \|f\|_{k_n}^{\frac{1}{1+\rho_{k_n}}} \, \|f\|_p^{\frac{\rho_{k_n}}{1+\rho_{k_n}}} \\
&= \|f\|_k^{\frac{1}{1+\rho_k}} \, \|f\|_p^{\frac{\rho_k}{1+\rho_k}} \\
&= \big(\varepsilon_k\big)^{\frac{1}{1+\rho_k}} \, \big(\varepsilon_p\big)^{\frac{\rho_k}{1+\rho_k}} \;\longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$ because

$$\frac{1}{1+\rho_k}\,\log \varepsilon_k = \frac{\log \varepsilon_k}{1+\sqrt{-\log \varepsilon_k}} \longrightarrow -\infty$$

as $k \longrightarrow \infty$ and

$$\lim_{k\to\infty} \left(\varepsilon_p\right)^{\frac{\rho_k}{1+\rho_k}} = \varepsilon_p.$$

Hence $f = 0$ on V_q. This means that K is a uniqueness set.

Lemma 4.2 [2]. *Let F be a Frechet space with $F \in (DN)$. Then $\left[F^*_{\mathrm{bor}}\right]* \in (DN)$.*

Now we are able to prove Theorem B.

(i) $\Rightarrow$ (ii). By Vogt [8], to prove $[\mathcal{H}(X)]' \in (LB^\infty)$ it suffices to show that every continuous linear map $T : [\mathcal{H}(X)]' \longrightarrow (\mathbf{C})$ is bounded on some neighbourhood of $0 \in [\mathcal{H}(X)]'$.

Define the function

$$f_T(x, \lambda) = T(\delta_x)(\lambda) \quad \text{for } x \in X,\ \lambda \in \mathbb{C},$$

where δ_x is the Dirac functional defined by x:

$$\delta_x(\varphi) = \varphi(x) \quad \text{for} \quad \varphi \in \mathcal{H}(X).$$

Let $\{V_p\}$ be a neighbourhood basis of X in $\mathbb{C}^n$. For each $p \geq 1$, put

$$A_p = \Big\{\lambda \in \mathbb{C} : f_T^\lambda \in \mathcal{H}(V_p);\ \|f_T^\lambda\|_{V_p} \leq p\Big\},$$

where

$$f_T^\lambda(x) = f_T(x, \lambda).$$

Then A_p is closed in $\mathbb{C}$ for $p \geq 1$ because $\mathcal{H}(V_p)$ is Montel. Moreover, $\mathbb{C} = \bigcup\limits_{p\geq 1} A_p$. The Baire Theorem yields p_0 such that $\operatorname{Int} A_{p_0} \neq \emptyset$.

Consider the separate holomorphic function

$$\widetilde{f}_T : (X \times \mathbb{C}) \cup (V \times \operatorname{Int} A_{p_0}) \longrightarrow \mathbb{C}$$

given by

$$\widetilde{f}_T(x, \lambda) = \begin{cases} f_T(x, \lambda) & \text{if } (x, \lambda) \in X \times \mathbb{C}, \\ f_T^\lambda(x) & \text{if } (x, \lambda) \in V \times \operatorname{Int} A_{p_0}, \end{cases}$$

where $V = V_{p_0}$. By Nguyen T. Van and Zeriahi [7], there exists a holomorphic extension $\widehat{f}_T$ of $\widetilde{f}_T$ to a neighbourhood $V \times \mathbb{C}$ of $X \times \mathbb{C}$. Since

$$\mathcal{H}(V, \mathcal{H}(\mathbb{C})) \cong \mathcal{H}(V) \widehat{\otimes}_\pi \mathcal{H}(\mathbb{C}) \cong \mathcal{L}\left([\mathcal{H}(V)]', \mathcal{H}(\mathbb{C})\right),$$

the form

$$S(\delta_z)(\lambda) = \widehat{f}_T(z)(\lambda) \quad \text{for } z \in V,\ \lambda \in \mathbb{C}$$

defines a continuous linear map from $[\mathcal{H}(V)]'$ into $\mathcal{H}(\mathbb{C})$. By the uniqueness of X, from the relations

$$\begin{aligned} T\Big(\sum_j \lambda_j\, \delta_{z_j}\Big) &= \sum_j \lambda_j\, T(\delta_{z_j}) = \sum_j \lambda_j\, f_T(z_j) \\ &= \sum_j \lambda_j\, S(\delta_{z_j}) = S\Big(\sum_j \lambda_j\, \delta_{z_j}\Big) \end{aligned}$$

it follows that $T = S$. Hence T is compact.

(ii) $\Rightarrow$ (iii). Let $F \in (DN)$ and $f \in M_w(X, F)$, where X is a compact set in $\mathbb{C}^n$ with $[\mathcal{H}(X)]' \in (LB^\infty)$. By Lemma 4.1, X is a unique set. As in the proof of Theorem A, we can define a linear continuous mapping

$$\widetilde{h}\ :\ F^*_{\text{bor}} \longrightarrow \mathcal{H}(X).$$

Since $[\mathcal{H}(X)]' \in (LB^\infty)$ and $[F^*_{bor}]^* \in (DN)$ (Lemma 4.2), by Vogt [8] we have

$$\mathcal{L}\left(F^*_{bor}, \mathcal{H}(X)\right) = \mathcal{LB}\left(F^*_{bor}, \mathcal{H}(X)\right).$$

By an argument analogous to that used in the proof of Theorem A, we can find a neighbourhood W of $0 \in F^*_{bor}$ and a neighbourhood U_p of W such that $\widetilde{h}(W)$ is contained and bounded in $\mathcal{H}^\infty(U_p)$, the Banach space of bounded holomorphic functions on U_p. Thus, the form

$$\widehat{h}(z)(x^*) = \tilde{h}(x^*)(z) \quad \text{for } z \in U_p,\ x^* \in F^*$$

defines a holomorphic function $\widehat{h}$ from U_p into F. From this it follows that $f \in M(X, F)$.

(iii) $\Rightarrow$ (i) Assume that X is pluripolar. Consider a plurisubharmonic function φ on $\mathbb{C}^n$ for which $\varphi\big|_X - \infty$, $\varphi \not\equiv -\infty$ and the Hartogs domain

$$\Omega_\varphi = \Big\{(z, \lambda) \in \mathbb{C}^n \times \mathbb{C}\ :\ |\lambda| < e^{-\varphi(z)}\Big\}.$$

Let f be holomorphic function with Ω_φ being a domain of existence of f [4]. Since $X \times \mathbb{C} \subset \Omega_\varphi$, f induces $\widehat{f} \in \mathcal{H}_w(X, \mathcal{H}(\mathbb{C}))$, where $\mathcal{H}_w(X, \mathcal{H}(\mathbb{C}))$ is the space of weakly holomorphic functions on X with values in $\mathcal{H}(\mathbb{C})$.

Indeed, let $\mu \in [\mathcal{H}(\mathbb{C})]'$. Choose $r > 0$ such that μ can be considered as a contiuous linear functional on $\mathcal{H}(r\triangle)$. Let V be a neighbourhood of X for which $V \times r\triangle$ is a compact subset of Ω_φ. Then $\widehat{f} : V \longrightarrow \mathcal{H}(r\triangle)$ is a holomorphic and hence $\mu\widehat{f}$ is holomorphic on V. By the hypothesis there exist a neighbourhood W of X in $\mathbb{C}^n$ and a meromorphic function $\widehat{g}$ on W with values in $\mathcal{H}(\mathbb{C})$ such that

$$\widehat{f}\big|_{X\setminus P(\widehat{g})} = \widehat{g}\big|_{X\setminus P(\widehat{g})}.$$

Write $\widehat{g} = \dfrac{\widehat{h}}{\widehat{\sigma}}$, where $\widehat{h} \in \mathcal{H}(W, \mathcal{H}(\mathbb{C}))$, $\widehat{\sigma} \in \mathcal{H}(W)$, $\widehat{\sigma} \neq 0$, such that $\operatorname{codim} Z(\widehat{h}, \widehat{\sigma}) \geq 2$. It follows that $\widetilde{f} : W \times \mathbb{C} \to \mathbb{C}$ is meromorphic and $P(\widetilde{f}) = P(\widehat{g}) \times \mathbb{C}$, where $\widetilde{f}$ is induced by $\widehat{g}$. Moreover,

$$\widetilde{f}\big|_{[X\setminus P(\widehat{g})]\times\mathbb{C}} = f\big|_{[X\setminus P(\widehat{g})]\times\mathbb{C}}.$$

Write the Hartogs expansion of f on Ω_φ as

$$f(z,\lambda) = \sum_{n\geq 0} f_n(z)\lambda^n,$$

where

$$f_n(z) = \frac{1}{2\pi i} \int\limits_{|\lambda| = e^{-\delta\varphi}} \frac{f(z,\lambda)}{\lambda^{n+1}} d\lambda \quad (\delta > 1).$$

Since the sequence $\left\{\dfrac{1}{n}\log|f_n(z)|\right\}$ is locally bounded from above, for each $m \geq 1$ we can define

$$\begin{aligned}\psi_m(z) &= \sup\left\{\frac{1}{n}\log|f_n(z)| : n \geq m\right\},\\ \psi_m^*(z) &= \limsup_{z'\to z} \psi_m(z').\end{aligned}$$

By Bedford-Taylor [1] ψ_m^* is plurisubharmonic and the set $\left\{\psi_m < \psi_m^*\right\}$ is pluripolar.

Let

$$\widehat{\psi} = \lim_{m\to\infty} \psi_m^*.$$

Since Ω_φ is the domain of existence of f, $\widehat{\psi}$ is not equal to $-\infty$ on every non-empty open set in $\mathbb{C}^n$. Indeed, if $\widehat{\psi} = -\infty$ on a non-empty open subset U of $\mathbb{C}^n$, then the Hartogs lemma implies that the series $\sum_n f_n(z)\lambda^n$ converges to a holomorphic function on $U\times\mathbb{C}$. This yields that $U\times\mathbb{C} \subset \Omega_\varphi$ and hence $\varphi\big|_U = -\infty$. It follows that $\widehat{\psi}$ is plurisubharmonic and $\left\{\psi < \widehat{\psi}\right\}$ is pluripolar, where $\psi = \lim_{m\to\infty} \psi_m$. Choose a neighbourhood V of $X\setminus P(\widehat{f})$ in W such that $V\times\triangle \subset \Omega_\varphi$. Consider the Hartogs expansion of $\widetilde{f}$ on $V\times\triangle$ with

$$\widetilde{f}(z,\lambda) = \sum_{n\geq 0} \widetilde{f_n}(z)\lambda^n.$$

Then

$$\widetilde{f_n}\big|_{X\setminus P(\widehat{g})} = \widehat{f}\big|_{X\setminus P(\widehat{g})} \quad \text{for } n\geq 0,$$

and hence

$$\widetilde{f_n}\big|_V = f_n\big|_V \quad \text{for } n\geq 0.$$

This yields

$$\begin{aligned}-\infty &= \limsup \frac{1}{n}\log|\widetilde{f_n}(z)| = \limsup \frac{1}{n}\log|f_n(z)|\\ &= \widehat{\psi}(z)\end{aligned}$$

for $z\in V\setminus\left(\psi < \widehat{\psi}\right)$, which is impossible.

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Department of Mathematics
Ho Chi Minh City University of education
280 An Duong Vuong, District 5.