ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 2, 1998, pp. 263–267

RELATIVE PRIMENESS OF ENTIRE FUNCTIONS

JIAZHANG ZHOU AND XIN-HOU HUA

Abstract. In this note, we will discuss the relative primeness of entire functions. In particular, we will answer a problem due to C. C. Yang.

1. Introduction

In what follows, unless specified, all the functions in this paper are nonconstant entire functions. Let $F(z)$ be a nonconstant entire function. A decomposition

$$F(z) = f(g(z)) = f \circ g(z)$$

will be called a factorization of F with f and g being the left and right factor of F, respectively, where f is meromorphic and g is entire (g may be meromorphic when f is rational). In particular, we shall use the notion $g|F$ iff $F = f(g)$ for some f.

Since 1968, the factorization theory has been studied zealously by many complex analysts and moderate progress has been achieved by utilizing Nevanlinna theory (see, e.g., [1, 2, 3]). Moreover, the study of factorization has utilized and enriched the value distribution theory. In addition, it has some applications to complex dynamics. In past decades, investigations have focused on criteria of factorizability and the unique factorizable property of entire and meromorphic functions. Recently a new type of question relating to these developments has been tackled; namely whether greatest common right factor (GCRF) or least common right multiplier (LCRM) exists for a pair or a set of entire functions (see Yang [4]).

Definition 1. Let F, G be two given functions having h as their common right factors, i.e. $h|F$ and $h|G$. h is called a GCRF of F and G iff $h_1|h$ to any common right factor h_1 of F and G.

Definition 2. F and G are called relatively prime, if their GCRF is linear.

Received December 12, 1996.
1991 Mathematics Subject Classification. 30D05 and 30D35.
Key words and phrases. Entire and meromorphic function, factorization, primeness.
The second author was supported by NSF of China and NSF of Jiangsu Province.

The following result was proved by Yang [4].

Theorem 1. *Let F and G be two transcendental entire functions such that FG is prime. If*

$$T(r,F) = S(r,G) \quad or \quad T(r,G) = S(r,F). \tag{1}$$

Then F and G are relatively prime.

Remark. The same conclusion holds if we replace FG by F/G or $F \pm G$.

Example. $\cos z$ and $e^{\sin z}$ are relatively prime.

In this note, we shall cancel the condition (1) and prove the following result.

Theorem 2. *Let F and G be two transcendental entire functions such that FG is prime. Then F and G are relatively prime, unless*

$$F = f(h), \quad G = g(h), \tag{2}$$

where h is a nonlinear prime entire function, f and g satisfy one of the following cases:
(i) *$f = Le^{\alpha}$ and $g = e^{-\alpha}$;*
(ii) *$f = e^{\alpha}$ and $g = Le^{-\alpha}$,*
here $\alpha(z)$ is entire, and L is linear.

Corollary 1. *Under the assumption of Theorem 2, if*

$$T(r,F) \not\sim T(r,G)$$

on a set of r with infinite linear measure, then F and G are relatively prime.

Remark. The converse of Theorem 2 does not hold.

Example. Let $F(z) = \cos z$, $G(z) = \cos z e^{\sin z}$. Then G is prime, and hence F and G are relatively prime. However

$$FG = (1 - w^2)e^w \circ \sin z.$$

For any two prime functions f and g, we denote by $(f, g) = z$ the property that f and g have no common nonlinear factors. In [4], Yang asked the following question: *Is it possible to construct three transcendental prime functions* f_1*,* f_2 *and* f_3 *such that* $(f_1, f_2) = (f_1, f_3) = z$ *but* $f_1 \not\mid f_2(f_3)$ *?*

The following result give an affirmative answer to Yang's question.

Theorem 3. *There exist three transcendental prime functions* f_1*,* f_2 *and* f_3 *such that* $(f_1, f_2) = (f_1, f_3) = z$ *but* $f_1 \not\mid f_2(f_3)$.

2. Proof of Theorem 2

We will follow the proof of Yang [4, Theorem 2]. Suppose that F and G are not relatively prime. It follows from Definition 2 that F and G have a common right factor $h(z)$ (say) which is nonlinear. Then

$$F = f(h), \quad G = g(h) \tag{3}$$

for some meromorphic functions f and g, in which case $FG = (fg) \circ h$. Thus, according to the assumption that FG is prime, there exists a fractional linear function $M(z)$ such that

$$fg = M. \tag{4}$$

This implies that either both f and g are transcendental or both f and g are rational. Next we consider four cases as follows.

1) f and g are entire. Then $M(z)$ is linear. Since F and G are nonconstant, it follows from (4) that both f and g are transcendental. Otherwise, either f or g is constant, a contradiction. Using (4) again we obtain (i) or (ii).

2) f is entire but g has poles. Since G is entire, if f and g are transcendental, it follows from (3) that h is entire and g has only pole z_0 (say) such that

$$h = z_0 + e^{\beta}$$

for some entire function $\beta(z)$. Combining this with (3) and (4) we obtain

$$FG = M \circ (z_0 + e^{\beta}) = M \circ (w^2 + z_0) \circ e^{\beta/2}.$$

This means that FG is not prime, a contradiction. If f and g are not transcendental, then f is a polynomial and g is rational. Since G is entire, from (3) we see that g has at most two poles. Moreover, if g has two poles z_1 and z_2, then there exists an entire function $\beta(z)$ such that

$$L_1 \circ h = e^{\beta}, \quad L_1(w) = \frac{w - z_1}{w - z_2} \cdot$$

Combining this with (3) and (4) we obtain

$$FG = M \circ h = M \circ L_1^{-1} \circ e^{\beta} = M \circ L_1^{-1} \circ w^2 \circ e^{\beta/2}.$$

Thus FG is not prime, a contradiction. Therefore g has only pole which is a Picard exceptional value of h. Let z_0 be the pole of g. Then there exists an entire function $h_0(z)$ such that

$$h(z) = z_0 + \frac{1}{h_0} \cdot$$

Substituting this into (3) we get $F = f(z_0 + 1/h_0)$. Since F is entire and f is a polynomial, h_0 has no zeros, which implies that $h_0 = e^{\alpha}$ for some entire function α. We thus have

$$FG = M \circ h = M \circ (z_0 + w^2) \circ e^{-\alpha/2},$$

a contradiction.

3) f has poles but g is entire. Similarly as in the case 2), we can deduce a contradiction.

4) Both f and g have poles. If f and g have a common pole z_0, then fg can not satisfy (4). If f and g have different poles z_1 and z_2, respectively, then by the same method as in 2) we deduce that FG is not prime, a contradiction.

The proof of Theorem 2 is now complete.

3. Proof of Theorem 3

Let

$$f_1 = z + e^z, \quad f_2 = ze^z, \quad f_3 = ze^{2z}.$$

Then f_j $(j = 1, 2, 3)$ are prime and $(f_1, f_2) = (f_1, f_3) = z$. If

$$f_1 | f_2 \circ f_3 = ze^{z(2+e^{2z})},$$

then there exists a meromorphic function $f(z)$ such that

$$f \circ f_1 = ze^{z(2+e^{2z})}. \tag{5}$$

Since the right hand side is entire and f_1 has no Picard exceptional value, f should be entire. Furthermore, note that the right hand side of (5) has only one zero, thus

$$f(w) = we^{\beta(w)}$$

for some transcendental entire function $\beta(w)$. From this and (5) we obtain

$$(z+e^z)\exp(\beta(z+e^z)) = ze^{z(2+e^{2z})}.$$

This is impossible since there are infinitely many zeros in the left hand side.

The proof of Theorem 3 is now complete.

References

1. C. T. Chuang and C. C. Yang, *Fix-points and Factorization of Meromorphic Functions*, World Scientific Publishing Co., 1990.

2. F. Gross, *Factorization of Meromorphic Functions*, U.S. Goverment Printing Office, Washington, D. C., 1972.

3. C. C. Yang, *Factorization Theory of Meromorphic Functions*, Lecture Notes in Pure and Applied Math. (edited by C. C. Yang), Marcel Dekker Inc., 1982.

4. C. C. Yang, *Great common right factor and least common right multiplier of entire functions*, Preprint.

The Weather Bureau of Anhui Province
Hefei, Anhui Province, P. R. China

Institute of Mathematics, Nanjing University
Nanjiing 210093, P. R. China
E-mail: postmath@netra.nju.edu.cn