ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 2, 1998, pp. 257–262

NORMAL STRUCTURE AND FIXED POINT PROPERTY IN LINEAR METRIC SPACES

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A linear metric space (X, d) is said to have the fixed point property if for every non-empty weakly compact convex subset K of X, every non-expansive map $T : K \to K$ has a fixed point. In this paper we discuss a class of linear metric spaces which has the fixed point property.

Normal structure is one of the fundamental tools in fixed point theory for non-expansive maps. A central problem in the fixed point theory of non-expansive maps is to determine those spaces which have the fixed point property (f.p.p.). With the appearance of Alspach's example [1], we know that there is a weakly compact convex set in the Banach space $L_1[0, 1]$ which need not have the f.p.p. for non-expansive self maps. On the other hand, Kirk [4] proved that if a reflexive Banach space has weak normal structure then it has the f.p.p. For general Banach spaces it was proved by Dulst and Sims [2]. Here we prove this result in linear metric spaces by generalising the result of Kirk [4] as well as of Dulst and Sims [2]. We start with a few definitions.

Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X, if for all x, y in X and $\alpha \in [0, 1]$ the following condition is satisfied:

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$$

for all $u \in X$. A metric space X with convex structure is called a *convex metric space*.

This notion of convexity in metric spaces was introduced by W. Takahashi [6] in 1970. Clearly, a Banach space or any convex subset of it is a convex metric space with $W(x, y, \alpha) = \alpha x + (1 - \alpha)y$. More generally, if X is a linear space with a translation invariant metric d satisfying $d(\alpha x + (1 - \alpha)y, 0) \leq \alpha d(x, 0) + (1 - \alpha)d(y, 0)$, then X is a convex metric space. There are many convex metric spaces (see Takahashi [6]) which cannot be embedded in any Banach space.

A non-empty set K of a convex metric space X is said to be *convex* if

Received October 16, 1996.

$W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. For linear spaces, convexity of K requires that $\alpha x + (1-\alpha)y \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

Takahashi [6] proved that in a convex metric space X, open balls $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed balls $B[x, r] = \{y \in X : d(x, y) \leq r\}$ are convex and if $\{K_\alpha : \alpha \in \Lambda\}$ is a family of convex sets in X then $\bigcap \{K_\alpha : \alpha \in \Lambda\}$ is convex.

A convex metric space (X, d) is said to satisfy *property (I)* if for all $x, y, p \in X$ and $\alpha \in [0, 1]$,

$$d(W(x, p, \alpha), W(y, p, \alpha)) \leq \alpha d(x, y).$$

Property (I) is always satisfied in any normed linear space. For details we refer to Guay, Singh and Whitfield [3].

Let B be a bounded set in a convex metric space X and let $\delta(B)$ be its diameter. An element $x \in B$ is said to be a *diametral point* of B if $\sup_{y \in B} d(x, y) = \delta(B)$. For closed balls in X, diametral points are precisely the boundary points. A point $x \in B$ is called a *non-diametral point* of B if $\sup_{y \in B} d(x, y) < \delta(B)$.

A convex subset S of a convex metric space X is said to have *normal structure* if every bounded convex subset S_1 of S which contains more than one point has a point that is not a diametral point.

Any compact convex subset of a convex metric space has normal structure (Takahashi [6], Proposition 5).

Let X be a metric space. For subsets H, K of X, H bounded, let

$$\begin{aligned} r_x(H) &= \sup \{d(x, y) : y \in H\}, \quad x \in K, \\ r(H, K) &= \inf \{r_x(H), x \in K\}, \\ C(H, K) &= \{x \in K : r_x(H) = r(H, K)\}. \end{aligned}$$

The set $C(H, K)$ is frequently referred to as the *Chebyshev centre* of H with respect to (w.r.t.) K in X or the set of *best simultaneous approximation* in K to H, and $r(H, K)$ is called the *Chebyshev radius* of H w.r.t. K.

If K is compact convex and H is a bounded subset of a convex metric space X, then $C(H, K)$ is non-empty, closed, convex subset of K (Naimpally, Singh and Whitfield [5], Lemma 3.1).

A locally convex linear metric space (X, d) is said to have *weak normal structure* if every non-trivial weakly compact convex subset of X has normal structure. The space X is said to have the *fixed point property* (f.p.p.) if for every non-empty weakly compact convex subset K of X, every non-expansive map $T : K \to K$ has a fixed point i.e. there exists $x \in K$ such that $Tx = x$.

The following result deals with the continuity of the function r_x defined above:

Lemma 1. *The function $r_x : K \to \mathbf{R}$ defined above is uniformly continuous.*

Proof. Let $x, y \in K$. Then $d(x, z) \leq d(x, y) + d(y, z)$ for all $z \in H$. Therefore $\sup_{z\in H} d(x, z) \leq d(x, y) + \sup_{z\in H} d(y, z)$ and so $r_x(H) \leq d(x, y) + r_y(H)$ or $r_x(H) - r_y(H) \leq d(x, y)$. Interchanging x and y, we get $r_y(H) - r_x(H) \leq d(y, x)$. So $|r_x(H) - r_y(H)| \leq d(x, y)$ for all $x, y \in K$ and hence the result follows.

The following result deals with the convexity of the function r_x:

Lemma 2. *If K is a convex subset of a convex metric space (X, d) then the function $r_x : K \to \mathbf{R}$ defined above is a convex function i.e. $r_{W(x_1,x_2,t)}(H) \leq tr_{x_1}(H) + (1-t)r_{x_2}(H)$ for all $x_1, x_2 \in K$ and $0 \leq t \leq 1$.*

Proof. Let $x_1, x_2 \in K$ and $0 \leq t \leq 1$. Since K is convex, $W(x_1, x_2, t) \in K$. We have

$$\begin{aligned} r_{W(x_1,x_2,t)}(H) &= \sup_{y\in H} d(W(x_1, x_2, t), y) \\ &\leq \sup_{y\in H} \big[td(x_1, y) + (1-t)d(x_2, y)\big] \\ &\leq t \sup_{y\in H} d(x_1, y) + (1-t) \sup_{y\in H} d(x_2, y) \\ &= tr_{x_1}(H) + (1-t)r_{x_2}(H) \end{aligned}$$

and so r_x is convex.

Remark 1. The result is true if K is a convex subset of a linear metric space (X, d) satisfying $d(tx + (1-t)y, 0) \leq td(x, 0) + (1-t)d(y, 0)$ for all $x, y \in X$ and $t \in [0, 1]$.

It is well known that if K is a weakly compact convex subset of a Banach space X then $C(H, K)$ is non-empty, weakly compact and convex. In linear metric spaces we have:

Lemma 3. *If K is a weakly compact convex subset of a locally convex linear metric space (X, d) having convex structure and with property (I) then $C(H, K)$ is non-empty, weakly compact and convex.*

Proof. Since K is a weakly compact and convex set in the locally convex space X, the function $r_x : K \to \mathbf{R}$, being continuous and convex (by Lemmas 1 and 2), is weakly lower semi-continuous. Let $C(H,K) = \{x \in K : r_x(H) = r(H,K)\} = \{x \in K : r_x(H) \leq r(H,K)\}$ be the Chebyshev centre of H w.r.t. K. Suppose $C(H,K) = \emptyset$. Then $r_x(H) > r(H,K)$ for all $x \in K$. Let x be any point of K. Since $\frac{r(H,K) + r_x(H)}{2} < r_x(H)$ and r_x is weakly lower semi-continuous, there exists a weak neighbourhood W_x of x in K such that $\frac{r(H,K) + r_x(H)}{2} \leq r_y(H)$ for all $y \in W_x$. Now $\{W_x : x \in K\}$ is a covering of K by weakly open sets and K is weakly compact, there exists a finite subcovering of K, say $\{W_{x_1}, W_{x_2}, \ldots, W_{x_n}\}$. Let $a = \min_{1 \leq j \leq n} r_{x_j}(H)$. Then $a > r(H,K)$ as each $r_{x_j}(H) > r(H,K)$. Let y be any element of K. Then $y \in W_{x_j}$ for some j, $1 \leq j \leq n$. Hence $\frac{r(H,K) + r_{x_j}(H)}{2} \leq r_y(H)$. So $\frac{r(H,K) + a}{2} \leq \frac{r(H,K) + r_{x_j}(H)}{2} \leq r_y(H) \cdot$ Therefore $\frac{r(H,K) + a}{2} \leq \inf_{y \in K} r_y(H) = r(H,K)$ i.e. $a \leq r(H,K)$, a contradiction. Hence $C(H,K) \neq \emptyset$.

We now show that $C(H,K)$ is convex. Let x_1, $x_2 \in C(H,K)$ and $0 \leq t \leq 1$. Then $r_{tx_1 + (1-t)x_2}(H) \leq t r_{x_1}(H) + (1-t) r_{x_2}(H) = t r(H,K) + (1-t) r(H,K) = r(H,K)$. Therefore $tx_1 + (1-t)x_2 \in C(H,K)$ and so $C(H,K)$ is convex.

Next we show that $C(H,K)$ is weakly compact. Let x belong to the weak closure of K in H and $\varepsilon > 0$ be given. Then there exists a weak neighbourhood W of x in K such that $r_x(H) - \varepsilon \leq r_y(H)$ for all $y \in W$. As x belongs to the weak closure of $C(H,K)$ in K, there exists a $y \in C(H,K) \cap W$. For this y, $r_H(y) = r(H,K)$. So $r_x(H) - \varepsilon \leq r(H,K)$ or $r_x(H) \leq r(H,K) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $r_x(H) \leq r(H,K)$ and so $x \in C(H,K)$. Thus $C(H,K)$ is a weakly closed subset of the weakly compact set K and hence $C(H,K)$ is weakly compact.

The following result gives a relation between the diameters of $C(K,K)$ and K:

Lemma 4. *Let K be a weakly compact convex subset of a locally convex linear metric space (X,d) having convex structure and with property (I) and $C(K,K)$ be the Chebyshev centre of K w.r.t. itself. If K has normal structure then* $\operatorname{diam} C(K,K) < \operatorname{diam} K$.

Proof. Since K has normal structure, there exists $x \in K$ such that $\sup \{d(x,y) : y \in K\} < \operatorname{diam} K$ or $r_x(K) < \operatorname{diam} K$. Let x_1, x_2 be any two points of $C(K,K)$. Then $d(x_1,x_2) \leq r_{x_1}(K) = r(K,K)$. So $\operatorname{diam} C(K,K) \leq r(K,K) \leq r_x(K) < \operatorname{diam} K$.

Alspach [1] proved that the Banach space $L_1[0,1]$ does not have the f.p.p.. Kirk [4] proved that if a reflexive Banach space has weak normal structure then it has the f.p.p.. The following theorem shows that there are certain linear metric spaces which have the f.p.p.:

Theorem. *Let (X,d) be a locally convex linear metric space having convex structure and with property (I). Then X has the f.p.p. if X has weak normal structure.*

Proof. Let K be a non-empty weakly compact convex subset of X and $T: K \to K$ a non-expansive map. Let $\mathcal{F}$ be the family of all non-empty weakly compact convex subsets of K which are invariant under T. $\mathcal{F}$ is non-empty as $K \in \mathcal{F}$. For K_1, $K_2 \in \mathcal{F}$ let $K_1 \leq K_2$ if $K_1 \supseteq K_2$. This is a partial ordering in $\mathcal{F}$. Let $\{K_\alpha\}$ be any chain in $\mathcal{F}$. Obviously, $\{K_\alpha\}$ has the finite intersection property. Since K is weakly compact, $\{K_\alpha\}$ must have a non-empty intersection, say C_1. Then C_1 is a non-empty weakly compact convex subset of K which is invariant under T. Obviously, C_1 is an upper bound of $\{K_\alpha\}$ in $\mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ must have a maximal element, say M. If M is a singleton, then obviously T has a fixed point in K. So assume that M has more than one point. Since X has weak normal structure, M has normal structure. By Lemma 4, $\operatorname{diam} C(M,M) < \operatorname{diam} M$ and so $C(M,M) \subseteq M$ but $C(M,M) \neq M$. Also by Lemma 3, $C(M,M)$ is a non-empty weakly compact and convex set. We now show that $C(M,M)$ is invariant under T. Let $x \in C(M,M)$. We have $d(Tx,Ty) \leq d(x,y)$ for all $y \in M$. So $d(Tx,Ty) \leq r_x(M) = r(M,M)$ for all $y \in M$. Let B be the closed ball in X with centre Tx and radius $r(M,M)$. Then $Ty \in B$ for all $y \in M$, i.e. $T(M) \subseteq B$. Consequently, $T(B \cap M) \subseteq T(M) \subseteq M \cap B$. Since $M \cap B$ is a non-empty weakly compact convex subset of K, $B \cap M \in \mathcal{F}$. By the maximality of M, $M \cap B = M$ or $M \subseteq B$. Hence for all $y \in M$, $d(Tx,y) \leq r(M,M)$ or $r_{Tx}(M) \leq r(M,M)$ and so $Tx \in C(M,M)$. Therefore $C(M,M) \in \mathcal{F}$ and so by the maximality of M we must have $C(M,M) = M$, a contradiction. Hence M must be a singleton proving thereby that T has a fixed point in K.

Remark 2. Since Banach spaces are locally convex linear metric spaces having convex structure and with property (I), the above theorem generalizes the corresponding result of Kirk [4] as well as of Dulst and Sims [2].

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