ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 2, 1998, pp. 247–256

ON F-DISCRETE DISTRIBUTION

PHAM XUAN BINH

Abstract. We investigate a new family of discrete distributions which are called F-discrete distributions. A moment recursion formula and some limit theorems for these distributions are proved. These results are similar to those for the well-known F-distributions in the continuous case.

1. Introduction

Let U be a random variable which has the density

$$f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{\gamma^\beta \Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(x+\gamma)^{-\alpha-\beta} & \text{if } x > 0 \ (\alpha, \beta, \gamma > 0). \end{cases} \tag{1}$$

We note that when putting $\alpha = \dfrac{n_1}{2}$, $\beta = \dfrac{n_2}{2}$, $\gamma = \dfrac{n_2}{n_1}$, we obtain the well-known F-distribution (Fisher-Snedecor distribution) with parameters (n_1, n_2). It is easy to check that the density f in (1) satisfies the differential equation

$$x(x+\gamma)f'(x) = [\gamma(\alpha-1) - (\beta+1)x]f(x). \tag{2}$$

So the distribution of U belongs to Pearson's system (see [4], page 133).

We can write (2) in the following form

$$\int_0^x [\gamma\alpha - (\beta-1)t]f(t)dt = x(x+\gamma)f(x), \quad x \ge 0. \tag{3}$$

This leads us to consider, in the discrete case, the analogous equation

---

Received October 29, 1996; in revised form May 30, 1997.
1991 Mathematics Subject Classification. 60 A 99.
Key words and phrases. F-discrete distribution; F-distribution.

$$(4)\qquad \sum_{j=0}^{k}[r\alpha-(\beta-1)j]p_j=(k+r)(k+\alpha)p_k,\quad k=0,1,2,\dots,$$

where $\alpha,\beta,r>0$.

In this paper we shall show that (4) and the condition $\sum_{k=0}^{\infty}p_k=1$ uniquely define the following family of discrete distributions

$$(5)\qquad p_k=\frac{\Gamma(\alpha+\beta)\Gamma(\beta+r)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(r)}\,\frac{\Gamma(\alpha+k)\Gamma(r+k)}{\Gamma(\alpha+\beta+r+k)k!},\quad k=0,1,2,\dots$$

It is interesting that we can give a probabilistic model which generates these distributions. Such a model can be obtained by reviewing Polya's one (see [4], Chapter 6, and [2], page 120) in another way (see the model after Proposition 2.2).

We shall deliver a recursion formula for the moments of these distributions (Theorem 3.2). We shall also prove some limit theorems (Section 4) which show that Poisson and Pascal distributions can be obtained under certain conditions. These results show that the new distribution is similar to the F-distribution in the continuous case. Therefore we call it the F-discrete distribution.

2. Preliminaries and Definition

Lemma 2.1 (see [1], page 73). *We have*

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

where $Re(c)>Re(b)>0,\ Re(c-a-b)>0.$

Theorem 2.1. *Let $p_k,\ k=0,1,2,...,$ be a sequence of real numbers. Then p_k is given by* (5) *if and only if p_k satisfies* (4) *and the condition* $\sum_{k=0}^{\infty}p_k=1.$

Proof. Suppose that p_k satisfies (4) and the condition $\sum p_k=1$. By (4) we have

$$[r\alpha-(\beta-1)(k+1)]_{p_{k+1}}=(k+1+r)(k+1+\alpha)p_{k+1}-(k+r)(k+\alpha)p_k,$$

$k = 0, 1, 2, ...$ or

$$p_{k+1} = \frac{(k+\alpha)(k+r)}{(k+1)(k+\alpha+\beta+r)} p_k, \quad k = 0, 1, 2, ...$$

Hence we get

$$p_k = \frac{\Gamma(\alpha+\beta+r)\Gamma(\alpha+k)\Gamma(r+k)}{\Gamma(\alpha)\Gamma(r)\Gamma(\alpha+\beta+r+k)k!} p_0, \quad k = 0, 1, 2, ... \tag{6}$$

By Lemma 2.1 we see that

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+\beta+r)\Gamma(\alpha+k)\Gamma(r+k)}{\Gamma(\alpha)\Gamma(r)\Gamma(\alpha+\beta+r+k)k!} p_0 = \frac{\Gamma(\beta)\Gamma(\alpha+\beta+r)}{\Gamma(\alpha+\beta)\Gamma(\beta+r)} p_0.$$

By the condition $\sum_{k=0}^{\infty} p_k = 1$ we obtain

$$p_0 = \frac{\Gamma(\alpha+\beta)\Gamma(\beta+r)}{\Gamma(\beta)\Gamma(\alpha+\beta+r)} . \tag{7}$$

From (6) and (7) we get (5).

Now, suppose that p_k is given by (5). By Lemma 2.1 we see that the condition $\sum_{k=0}^{\infty} p_k = 1$ is satisfied. We shall prove that p_k satisfies (4) by induction. It is easy to verify that (4) holds when $k = 0$. Assume that (4) is true when $k = n$. By noting that

$$p_n = \frac{(n+1)(n+\alpha+\beta+r)}{(n+r)(n+\alpha)} p_{n+1} ,$$

we have

$$\begin{aligned}
&\sum_{k=0}^{n+1} [r\alpha - (\beta-1)k] p_k = \\
&\quad = [r\alpha - (\beta-1)(n+1)] p_{n+1} + \sum_{k=0}^{n} [r\alpha - (\beta-1)k] p_k \\
&\quad = [r\alpha - (\beta-1)(n+1)] p_{n+1} + (n+\alpha)(n+r) p_n \\
&\quad = [r\alpha - (\beta-1)(n+1)] p_{n+1} + (n+1)(n+\alpha+\beta+r) p_{n+1} \\
&\quad = [n^2 + (\alpha+r+2)n + (r\alpha+r+\alpha+1)] p_{n+1} \\
&\quad = (n+1+\alpha)(n+1+r) p_{n+1}. \ \Box
\end{aligned}$$

By Theorem 2.1 we can give the following definition.

Definition 2.1. A random variable X is said to have F-discrete distribution (F-d.d.) with parameters (α, β, r) if

$$P(X = k) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\beta+r)}{\Gamma(r)} \frac{\Gamma(\alpha+k)\Gamma(r+k)}{\Gamma(\alpha+\beta+r+k)k!}, \quad k = 0, 1, 2, ...,$$

where α, β, r are positive real numbers.

Proposition 2.2. *Let U be a random variable which has the density* (1). *Suppose that for a random variable X we have*

$$P(X = k \mid U = x) = \frac{\Gamma(r+k)}{\Gamma(r)k!} \Big(\frac{\gamma}{x+\gamma}\Big)^r \Big(\frac{x}{x+\gamma}\Big)^k, \ k = 0, 1, 2, ..., \ r > 0.$$

Then X has F-d.d. with parameters (α, β, r).

Proof. For $k = 0, 1, 2, ...,$ we have

$$\begin{aligned} P(X = k) &= \frac{\gamma^{\beta+r}\Gamma(\alpha+\beta)\Gamma(r+k)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(r)k!} \int_0^{+\infty} x^{\alpha+k-1}(x+\gamma)^{-(\alpha+\beta+r+k)} dx \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(r+k)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(r)k!} \int_0^1 t^{\alpha+k-1}(1-t)^{\beta+r-1} dt \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\beta+r)}{\Gamma(r)} \frac{\Gamma(\alpha+k)\Gamma(r+k)}{\Gamma(\alpha+\beta+r+k)k!}. \ \square \end{aligned}$$

When α, β, r are integers, F-d.d. can be described as follows. Suppose that there is an urn which contains α white and β red balls. A ball is drawn at random. It is replaced and, moreover, one ball of the color drawn is added. A new random drawing is made from the urn (now containing $\alpha+\beta+1$ balls), and this procedure is repeated until r red balls are drawn. Let Y be the random variable which represents for the number of drawing. Then the random variable $X = Y - r$ has F-d.d. with parameters (α, β, r).

This can be proved by induction. It is easy to check that if $r = 1$ then X has F-d.d. with parameters $(\alpha, \beta, 1)$.

Now suppose that X has F-d.d. with parameters $(\alpha, \beta, n-1)$ when $r = n-1$ and we find the probability of the event A that the procedure terminates at the $(n+k)$-th step, i.e., the n-th red ball is drawn at $(n+k)$-th step, $n \geq 2$.

We denote by H_j the events that the $(n-1)$-th red ball is drawn at $(j+n-1)$-th step, $j=\overline{0,k}$. Then

$$\begin{aligned}
P(A) &= \sum_{j=0}^{k} P(H_j)P(A \mid H_j) \\
&= \sum_{j=0}^{k} \frac{\Gamma(\alpha+\beta)\Gamma(\beta+n-1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-1)} \frac{\Gamma(\alpha+j)\Gamma(n+j-1)}{\Gamma(\alpha+\beta+n+j-1)j!} \times \\
&\times \frac{\Gamma(\alpha+\beta+n+j-1)\Gamma(\alpha+k)(\beta+n-1)}{\Gamma(\alpha+j)\Gamma(\alpha+\beta+r+k)} \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\beta+n)\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+r+k)} \sum_{j=0}^{k} \frac{\Gamma(j+n-1)}{\Gamma(n-1)j!} \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\beta+n)\Gamma(\alpha+k)\Gamma(n+k)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+n+k)k!} \ .
\end{aligned}$$

3. Moment Properties

From now on, we denote by X a random variable which has F-d.d. with parameters (α, β, r) and we will use the notation $p_k = P(X = k)$, $k = 0, 1, 2, ...$

By the investigations of [1] (page 70), and [3] (page 280), we see that the s-th moment of F-d.d. with parameters (α, β, r) is finite if and only if $s < \beta$.

Proposition 3.1. *For $\beta - 2 > s \geq 0$, we have*

$$E[(X+1)^s(X+\alpha)(X+r)] = E[X^{s+1}(X+\alpha+\beta+r-1)].$$

Proof. Note that

$$(k+\alpha)(k+r)p_k = (k+1)(k+\alpha+\beta+r)p_{k+1}.$$

Multiplying the two sides of this equality by $(k+1)^s$ and summing up we obtain

$$\sum_{k=0}^{\infty}(k+1)^s(k+\alpha)(k+r)p_k = \sum_{k=0}^{\infty}(k+1)^{s+1}(k+\alpha+\beta+r)p_{k+1}.$$

Hence we get

$$\begin{aligned} E[(X+1)^s(X+\alpha)(X+r)] &= \sum_{k=0}^{\infty}(k+1)^s(k+\alpha)(k+r)p_k \\ &= \sum_{k=0}^{\infty}(k+1)^{s+1}(k+\alpha+\beta+r)p_{k+1} \\ &= \sum_{n=0}^{\infty} n^{s+1}(n+\alpha+\beta+r-1)p_n \\ &= E[X^{s+1}(X+\alpha+\beta+r-1)]. \ \square \end{aligned}$$

It is easy to check the validity of the following lemma.

Lemma 3.1. *For $\beta - 1 > m \geq 0$, we have*

$$\lim_{n\to\infty} n^{m+2}.p_n = 0.$$

Theorem 3.2. *For $\beta - 1 > m$, $m = 0, 1, 2, ...$, we have*

$$\begin{aligned} (\beta-1-m)EX^{m+1} &= \sum_{j=0}^{m-2} C_m^j EX^{j+2} + (\alpha+r)\sum_{j=0}^{m-1} C_m^j EX^{j+1} \\ &+ \alpha r \sum_{j=0}^{m} C_m^j EX^j. \end{aligned} \tag{8}$$

Proof. By analogous arguments as in the proof of Proposition 3.1 we have

$$\sum_{k=0}^{n}(k+1)^m(k+\alpha)(k+r)p_k = \sum_{k=0}^{n}(k+1)^{m+1}(k+\alpha+\beta+r)p_{k+1}. \tag{9}$$

But

$$\begin{aligned} \sum_{k=0}^{n}(k+1)^m(k+\alpha)(k+r)p_k &= \sum_{k=0}^{n}\sum_{j=0}^{m} C_m^j k^j[k^2+(\alpha+r)k+\alpha\, r]p_k \\ = \sum_{j=0}^{m} C_m^j \sum_{k=0}^{n} k^{j+2}p_k &+ (\alpha+r)\sum_{j=0}^{m} C_m^j \sum_{k=0}^{n} k^{j+1}p_k + \alpha\, r\sum_{j=0}^{m} C_m^j \sum_{k=0}^{n} p_k \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{m-2} C_m^j \sum_{k=0}^{n} k^{j+2} p_k + (\alpha + r) \sum_{j=0}^{m-1} C_m^j \sum_{k=0}^{n} k^{j+1} p_k \\
&\quad + \alpha\, r \sum_{j=0}^{m} C_m^j \sum_{k=0}^{n} p_k + \sum_{k=0}^{n} k^{m+2} p_k + m \sum_{k=0}^{n} k^{m+1} p_k \\
&\quad + (\alpha + r) \sum_{k=0}^{n} k^{m+1} p_k, \qquad (10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{n} (k+1)^{m+1} (k + \alpha + \beta + r) p_{k+1} \\
&= \sum_{k=0}^{n} (k+1)^{m+2} p_{k+1} + (\alpha + \beta + r - 1) \sum_{k=0}^{n} (k+1)^{m+1} p_{k+1} \\
&= \sum_{k=0}^{n+1} k^{m+2} p_k + (\alpha + \beta + r - 1) \sum_{k=0}^{n+1} k^{m+1} p_k \\
&= \sum_{k=0}^{n} k^{m+2} p_k + (\alpha + \beta + r - 1) \sum_{k=0}^{n} k^{m+1} p_k + \\
&\quad + (n+1)^{m+2} p_{n+1} + (\alpha + \beta + r - 1)(n+1)^{m+1} p_{n+1}. \qquad (11)
\end{aligned}
$$

From (9), (10), (11), we get

$$
\begin{aligned}
&(\beta - 1 - m) \sum_{k=0}^{n} k^{m+1} p_k + (n+1)^{m+2} p_{n+1} \\
&\quad + (\alpha + \beta + r - 1)(n+1)^{m+1} p_{n+1} = \sum_{j=0}^{m-2} C_m^j \sum_{k=0}^{n} k^{j+2} p_k \\
&\quad + (\alpha + r) \sum_{j=0}^{m-1} C_m^j \sum_{k=0}^{n} k^{j+1} p_k + \alpha r \sum_{j=0}^{m} C_m^j \sum_{k=0}^{n} p_k. \qquad (12)
\end{aligned}
$$

By Lemma 3.1 we get (8) when taking the limit as $n \to \infty$ in (12). $\square$

Remark.

(i) If $\beta > 1$ then $EX = \dfrac{r\alpha}{\beta - 1}$

(ii) If $\beta > 2$ then $EX^2 = \dfrac{r\alpha(r\alpha + \alpha + \beta + r - 1)}{(\beta - 1)(\beta - 2)}$ and

$$\mathrm{Var}(X) = \frac{r\alpha[r\alpha + (\beta - 1)(\alpha + \beta + r - 1)]}{(\beta - 1)^2(\beta - 2)}.$$

4. Some Limit Theorems

Theorem 4.1. *For $c > 0$, we have*

$$\lim_{\substack{\beta, r \to \infty \\ \frac{\beta}{r} \to c}} P(X = k) = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha) k!} p^\alpha q^k, \ \ k = 0, 1, 2, ...,$$

where $p = \dfrac{c}{1 + c}$ and $q = 1 - p$.

Proof. When β, r are large, $\beta = cr + \varepsilon(\beta, r)$, where

$$\lim_{\substack{\beta, r \to \infty \\ \frac{\beta}{r} \to c}} \varepsilon(\beta, r) = 0.$$

Then

$$P(X = k) = \frac{\Gamma(\alpha + k)\Gamma(\alpha + \beta)\Gamma(r + k)\Gamma[(1 + c)r + \varepsilon(\beta, r)]}{k!\Gamma(\alpha)\Gamma(\beta)\Gamma(r)\Gamma[(1 + c)r + \alpha + k + \varepsilon(\beta, r)]} \ .$$

By the continuity of Gamma function and the well-known limit $\lim\limits_{x \to +\infty} \dfrac{\Gamma(x + a)}{x^a \Gamma(x)} = 1$, we have

$$\begin{aligned}
\lim_{\substack{\beta, r \to \infty \\ \frac{\beta}{r} \to c}} P(X = k) &= \lim_{\substack{\beta, r \to \infty \\ \frac{\beta}{r} \to c}} \frac{\Gamma(\alpha + k)\beta^\alpha r^k}{k!\Gamma(\alpha)(1 + c)^\alpha r^\alpha (1 + c)^k r^k} \\
&= \frac{\Gamma(\alpha + k)}{\Gamma(\alpha) k!} \Big(\frac{c}{1 + c}\Big)^\alpha \Big(\frac{1}{1 + c}\Big)^k \\
&= \frac{\Gamma(\alpha + k)}{\Gamma(\alpha) k!} p^\alpha q^k.
\end{aligned}$$

Remark. Let f as in (1). For $c > 0$ we have

$$\lim_{\substack{\beta,\gamma\to\infty \\ \frac{\beta}{\gamma}\to c}} f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0, \\ \dfrac{c^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-cx} & \text{if} \quad x > 0. \end{cases}$$

The next result is analogous to Theorem 4.1

Theorem 4.2. *For $c > 0$, we have*

$$\lim_{\substack{\alpha,\beta\to\infty \\ \frac{\beta}{\alpha}\to c}} P(X=0) = \frac{\Gamma(r+k)}{\Gamma(r)k!} p^r q^k, \quad k = 0, 1, 2, ...,$$

where $p = \dfrac{c}{1+c}$ *and* $q = 1 - p$.

Theorem 4.3. *For $\lambda > 0$, we have*

$$\lim_{\substack{\alpha,\beta,r\to\infty \\ \frac{r\alpha}{\beta}\to\lambda}} P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, ... \tag{13}$$

Proof. For $k = 0, 1, 2, ...,$ we have

$$P(X=k) = \frac{\Gamma(\alpha+\beta)\Gamma(\beta+r)\Gamma(\alpha+k)\Gamma(r+k)}{\Gamma(\beta)\Gamma(\alpha+\beta+r)\alpha^k\Gamma(\alpha)r^k\Gamma(r)} \times$$
$$\times \left(\frac{\alpha r}{\alpha+\beta+r}\right)^k \frac{(\alpha+\beta+r)^k\Gamma(\alpha+\beta+r)}{\Gamma(\alpha+\beta+r+k)k!} \,. \tag{14}$$

By using Stirling formula, we get

$$\lim_{\substack{\alpha,\beta,r\to\infty \\ \frac{r\alpha}{\beta}\to\lambda}} \frac{\Gamma(\alpha+\beta)\Gamma(\beta+r)}{\Gamma(\beta)\Gamma(\alpha+\beta+r)} = e^{-\lambda}. \tag{15}$$

On the other hand

$$\lim_{\substack{\alpha,\beta,r\to\infty \\ \frac{\alpha r}{\beta}\to\lambda}} \frac{\Gamma(\alpha+k)\Gamma(r+k)}{\alpha^k\Gamma(\alpha)r^k\Gamma(r)} = 1, \tag{16}$$

$$\lim_{\substack{\alpha,\beta,r\to\infty \\ \frac{\alpha r}{\beta}\to\lambda}} \frac{(\alpha+\beta+r)^k\Gamma(\alpha+\beta+r)}{\Gamma(\alpha+\beta+r+k)} = 1, \tag{17}$$

$$\lim_{\substack{\alpha,\beta,r\to\infty \\ \frac{\alpha r}{\beta}\to\lambda}} \left(\frac{r\alpha}{\alpha+\beta+r}\right)^k = \lambda^k. \tag{18}$$

From (14), (15), (16), (17), (18) we get (13). □.

Acknowledgements

The author wish to thank the referee for his suggestions on the arrangement of the paper.

References

1. A. Erdelyi et. al., *Higher Transcendental Functions*, Vol.2, McGraw-Hill, New York-Toronto-London, 1953 (in Russian, Nauka, Moscow, 1965).
2. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol.I, 3rd ed., John Wiley and Sons, New York, 1970.
3. G. M. Fiktengols, *Course of Differential and Integral Calculus*, Vol.II, Nauka, Moscow, 1969 (in Russian).
4. V. S. Koroliuk, N. I. Portenko, A. V. Skorokhod and A. F. Turbin, *Handbook of Probability Theory and Mathematical Statistics*, Nauka, Moscow, 1985 (in Russian).

Department of Mathematics
Qui Nhon Pedagogical Institute