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SEMI-TRACES AND PROCESSES OF PETRI NETS

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ABSTRACT. It is well known that the processes of a Petri net represent its true concurrent behaviour [11]. For the safe nets, the processes can be replaced by traces on firing sequences [2, 3]. In the present paper we generalize this result for a special class of nets which is strictly larger than the class of safe nets. Namely, we prove that the processes of such nets can be replaced by some semi-traces on firing sequences. We also show that for a net in which every final process has a global observation, some set of its final semi-traces on firing sequences forms a prime event structure, and hence its induced finitary algebraic domain is equivalent to the one generated by the set of processes.

1. INTRODUCTION

Two representations of behaviours of nets are well-known: firing sequences and processes. Relationship between them has been investigated in [2, 6, 7]. For 1-safe Petri nets, processes are equivalent to traces as presented in [2, 3]. This result is not true for general nets as shown in [2]. E. Ochmanski [10], D. V. Hung and E. Knuth [6] used maximal semi-traces in representing concurrent behaviours of a net. In this paper, we shall investigate relationship between the semi-traces on firing sequences and the processes for some kind of nets.

Some definitions and results of the papers [2, 3, 7, 10] will be recalled in Section 2. In Section 3 some properties of nets are specified. We prove that a net has a global observation or global time for every process if and only it has no firable semi-cycle of transitions. Also for the nets with such properties, some set of semi-traces on firing sequences and the set of processes are equivalent in the sense that they induce the same partially ordered sets of transition occurrences.

In Section 4, we consider the nets whose processes with final event (i.e., having only one maximal event) have a global observation. We show that

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for such a net, some set of final semi-traces on firing sequences determines a prime event structure, and hence, by results of [13], it is equivalent to a finitary algebraic domain corresponding to the *Lin-sets* (sequential observations) of processes. Since Scott domains are closely related to denotational semantics, these sets of final semi-traces on firing sequences of Petri nets can be used as semantics for parallel programming languages.

2. Basic definitions and notions

In this section we recall some necessary terminology and results.

Definition 2.1. (i) A triple (S, T, F) is a net if S and T are disjoint sets, $F \subseteq (S\times T)\cup(T\times S)$ and $T \subset \mathrm{dom}\,(F)\cup\mathrm{cod}\,(F)$. The relation F is also interpreted as the characteristic function of F: $F(x, y) = 1 \Leftrightarrow (x, y) \in F$.

(ii) $\Sigma = (S, T, F, M_0)$ is a system net (or Petri net) if (S, T, F) is a net and $M_0 : S \to \mathbf{N}$ is a marking, where $\mathbf{N}$ is the set of non-negative integers.

(iii) A net $N = (S, T, F)$ is an occurrence net if $\forall s \in S$: $|{}^\bullet s|\leq 1 \wedge |s^\bullet|\leq 1$ and F^+ (the transitive closure of F) is acyclic, where, for any $x \in S \cup T$,

$$^\bullet x = \{y \in S\cup T \mid (y, x) \in F\} \text{ and } x^\bullet = \{y \in S\cup T \mid (x, y) \in F\}.$$

Definition 2.2. Let $N = (B, E, F)$ be an occurrence net.

(i) We define two sets **li, co** $\subseteq (B\cup E) \times (B\cup E)$ by

$$(x, y) \in \mathbf{li} \Leftrightarrow (x < y \vee y < x \vee x = y),$$

$$(x, y) \in \mathbf{co} \Leftrightarrow ((x, y) \notin \mathbf{li}\ ) \vee x = y);$$ where $<$ stands for F^+.

(ii) $\mathbf{l} \subseteq B \cup E$ is a **li-set** iff $\forall x, y \in \mathbf{l}$: $(x, y) \in \mathbf{li}$.

(iii) $\mathbf{c} \subseteq B\cup E$ is a **co-set** iff $\forall x, y \in \mathbf{c}$: $(x, y) \in \mathbf{co}$.

(iv) The interval between two **co-sets** c_1, c_2 is defined by

$$[c_1, c_2] = \{z \in B \cup E \mid \exists x \in c_1\ \exists y \in c_2\colon x \leq z \leq y\}.$$

(v) A net N is discrete with respect to a **co-set c** if

$$\forall x \in B \cup E\ \exists n \in \mathbf{N}\colon \forall \textbf{ li-set } \mathbf{l}\colon |[\mathbf{c}, x]\cap\mathbf{l}\ | \leq n \wedge |[x,\mathbf{c}]\cap \mathbf{l}\ | \leq n.$$

Definition 2.3. Let $\Sigma = (S, T, F)$ be a net, M a marking and $t, t' \in T$.

(i) M enables t if $\forall s \in S$: $F(s, t)\leq M(s)$.

(ii) M' is produced from M by the firing of t if M enables t and $\forall s \in S$: $M'(s) = M(s) - F(s, t) + F(t, s)$. In that case, we write $M[t\rangle M'$.

(iii) M enables concurrently $\{t_i \mid 1 \leq i \leq k\}$ if $\forall s \in S$: $\sum_{i=1}^{k} F(s, t_i) \leq M(s)$.

Definition 2.4. Let $\Sigma = (S, T, F, M_0)$ be a system net.

(i) $\sigma = M_0 t_1 M_1 \ldots t_i M_i \ldots$ is a firing sequence of Σ if $\forall i \geq 1$: $M_{i-1}[t_i\rangle M_i$. Sometime we write: $\sigma = t_1 t_2 \ldots t_i \ldots$ and $M_0[t_1 \ldots t_i\rangle M_i$. The marking M_i is said to be reachable from M_0.

(ii) The set of all finite and infinite firing sequences of Σ is denoted by $\mathcal{F}(\Sigma)$.

(iii) $[M_0\rangle$ denotes the set of all markings of the net Σ reachable from the initial marking M_0.

(iv) Σ is 1-safe if $\forall M \in [M_0\rangle, \forall s \in S$: $M(s) \leq 1$.

(v) Σ is safe if $\exists n \in \mathbf{N}$ such that $\forall M \in [M_0\rangle, \forall s \in S$: $M(s) \leq n$.

(vi) Σ is self-concurrency free if $\forall t \in T$ and $\forall M \in [M_0\rangle$, M does not enable concurrently $\{t, t\}$.

Definition 2.5. Let $\Sigma = (S, T, F, M_0)$ be a system net, $N = (B, E, F')$ an occurrence net and p a mapping: $B \cup E \rightarrow S \cup T$. The pair (N, p) is a process of Σ if

(i) $p(B) \subseteq S$, $p(E) \subseteq T$.

(ii) $\mathrm{Min}\,(N)$ is a *B-cut* of N, i.e., a maximal **co-set** consisting of elements of B.

(iii) N is discrete with respect to $\mathrm{Min}\,(N)$.

(iv) $\forall e \in E, s \in S$: $F(s, p(e)) = |p^{-1}(s) \cap {}^\bullet e| \wedge F(p(e), s) = |e^\bullet \cap p^{-1}(s)|$.

(v) $\forall s \in S$: $M_0(s) = |p^{-1}(s) \cap \mathrm{Min}\,(N)|$.

From the practical point of view we only consider countable system nets that are degree-finite (i.e., $\forall x \in B \cup E$: $x^\bullet$ and ${}^\bullet x$ are finite sets) and have finite markings. For a process $\pi = (B, E, F, p)$ of a system net, we call $O_\pi = (E, \leq_\pi, p)$ the labelled partial ordering derived from π, where $\leq_\pi = F^*{}_{|E}$. The basic relationship between processes and firing sequences, which is presented in [3], is based on the following construction.

Construction 2.6. Let $\Sigma = (S, T, F, M_0)$ be a system net and $\sigma = M_0 t_1 M_1 \ldots$ be a firing sequence of Σ. A set $\Pi(\sigma)$ of processes is associated to σ as follows.

We construct labelled occurrence nets $(N_i, p_i) = (B_i, E_i, F_i, p_i)$, where $i \in \mathbf{N}$ and p_i: $B_i \cup E_i \rightarrow S \cup T$, by the following recursive procedure:

Define $E_0 = F_0 = \emptyset$, and B_0 consists of $M_0(s)$ distinct conditions b with $p_0(b) = s$ for each $s \in S$. Suppose (N_i, p_i) has already been constructed. For each $s \in {}^\bullet t_{i+1}$ we choose a condition $b = b(s) \in \mathrm{Max}(N_i) \cap p_i^{-1}(s)$. Then we add a new event e and put $p_{i+1}(e) = t_{i+1}$ and $(b, e) \in F_{i+1}$. Also for each $s \in t_{i+1}^\bullet$ we add a new condition $b' = b'(s)$ and put $p_{i+1}(b') = s$ and $(e, b') \in F_{i+1}$. For $x, y \in B_i \cup E_i$ we define $p_{i+1}(x) = p_i(x), (x, y) \in F_{i+1} \Leftrightarrow (x, y) \in F_i$.

For $\sigma = M_0 t_1 \ldots t_n M_n$ the procedure stops at $i = n$, and we put $\pi = (N, p) \in \Pi(\sigma)$ with $N = N_n$ and $p = p_n$. If σ is infinite, we put $\pi = (\cup B_i, \cup E_i, \cup F_i, \cup p_i) \in \Pi(\sigma)$.

The following theorem is taken from [4].

Theorem 2.7. *Let Σ be a degree-finite system net with a finite initial marking. Then*

(i) *For each firing sequence σ of Σ, $\Pi(\sigma)$ is a set of processes of Σ.*

(ii) *For each process π of Σ there exists a firing sequence σ such that $\pi \in \Pi(\sigma)$.*

Definition 2.8. Let $\pi_i = (B_i, E_i, F_i, p_i), i = 1, 2$ be processes of a system net Σ. We define $\pi_1 \approx \pi_2$ if there is a bijection β: $E_1 \to E_2$ such that $\forall e, e_1, e_2 \in E_1$: $((p_1(e) = p_2(\beta(e)) \wedge (e_1 <_1 e_2 \Leftrightarrow \beta(e_1) <_2 \beta(e_2)))$, where $<_i = F_i^+$.

For a system net Σ, let $\mathcal{P}(\Sigma)$ denote the set of processes of Σ.

Theorem 2.9. *Let Σ be a system net, $\pi = (B, E, F, p)$ be a process of Σ and let $\sigma = t_1 t_2 \ldots$ be a sequence of transitions of Σ. Then σ is a firing sequence of Σ and $\pi \in \Pi(\sigma)$ if and only if there exists a bijection β: $E \to \{1, 2, \ldots\}$ such that*

$$\forall e, e_1, e_2 \in E\colon\ ((p(e) = t_\beta(e)) \wedge (e_1 <_\pi e_2 \Rightarrow \beta(e_1) < \beta(e_2))).$$

Proof. It follows immediately from 2.10 and the proof of 2.10 in [2].

Definition 2.10. Let π be a process of a system net Σ. We denote the *Lin-set* of π by

$$\mathrm{Lin}(\pi) = \{\sigma \mid \sigma \text{ is a firing sequence of } \Sigma \text{ and } \pi \in \Pi(\sigma)\}.$$

A semi-commutative system $SC = \langle A, R \rangle$ is a semi-Thue system, where A is a finite alphabet and R is a set of rules of the form $ab \to ba$ with $a, b \in A$ and $a \neq b$. If all the rules in R are symmetrical, then we say $\langle A, R \rangle$ is a commutative system. For a semi-commutative system $SC = \langle A, R \rangle$ let R_S denote the set of symmetrical rules of R, i.e.

$$R_S = \big\{ab \to ba \in R \big| ba \to ab \in R\big\}.$$

We say $\langle A, R_S \rangle$ is the commutative system derived from $\langle A, R \rangle$. We write $x \to_R y$ for $x, y \in A^*$ if $x = x_1 ab x_2$, $y = x_1 ba x_2$ and $ab \to ba \in R$. The reflexive and transitive closure of $\to_R$ is denoted by $\to_R^*$. A semi-trace generated by a string x will be defined as a set of all words derived from x by rules of R and denoted by $[x\rangle_R$, i.e., $[x\rangle_R = \{y \in A^* \mid x \to_R^* y\}$.

We define an equivalence relation associated with a commutative system $\langle A, R_S\rangle$: $x \equiv y$ iff $x \to^*_{R_S} y$. The equivalence class of x is denoted by $[x]_{R_S}$ and called a trace.

Let us fix some semi-commutative system $\langle A, R\rangle$. Unless there is a confusion, we omit the subscripts R, R_S for semi-traces and traces. A semi-commutative monoid over $\langle A, R\rangle$ is a triple $(M, \circ, \{\epsilon\})$, where $M = \{[x\rangle \mid x \in A^*\}$, $[x\rangle \circ [y\rangle = [x.y\rangle$. We denote a free partial commutative monoid over $\langle A, R_S\rangle$ by $(M_S, \circ, \{\epsilon\})$, where $M_S = \{[x] \mid x \in A^*\}$ and $[x] \circ [y] = [x.y]$.

The following lemma is obvious from the above mentioned notations (see [6, 10]).

Lemma 2.11. $[u\rangle = [v\rangle \iff [u] = [v]$.

Definition 2.12. Let $w = a_1 a_2 \dots a_n \in A^*$. Set

$$O(w) = \{(a,k) \mid a \in \mathrm{alph}(w), k \in \mathbf{N},\ 1 \le k \le |w|_a\}.$$

We define an ordering $\le_w$ on $O(w)$ as follows: $(a,k) \le_0 (a',m)$ if $\exists ua'$: $w \in ua'A^*$ with $|ua'|_{a'} = m$ and $\exists va$: $ua' \in vaA^*$ with $|va|_a = k$ and $aa' \to a'a \notin R$. For $\le_\omega = (\le_0)^*$, the labelled partial ordering derived from ω is denoted by $O_\omega = (O(\omega), \le_\omega, lb)$, where $lb((a,n)) = a$, for all $a \in A$, and all $n \in \mathbf{N}$.

Let us consider the following orderings on M and M_S between semi-traces and traces of the same length:

$$[\alpha\rangle \le [\beta\rangle \quad \text{iff} \quad [\alpha\rangle \subseteq [\beta\rangle, \text{ i.e. } \ \beta \to^* \alpha.$$
$$[\alpha] \le [\beta] \quad \text{iff} \quad [\alpha\rangle \le [\beta\rangle, \text{ i.e. } \ \beta \to^* \alpha.$$

Lemma 2.13 (see [6]). *Let $w, w' \in A^*$. Then*

$$[w'\rangle \le [w\rangle \iff (O(w) = O(w') \wedge \le_w \subseteq \le_{w'}).$$

We now extend the definition of finite semi-traces and traces to infinite ones. Let $SC = \langle A, R\rangle$ be a semi-commutative system and $A^\omega = A^* \cup A^\infty$.

Definition 2.14. Let $x, y \in A^\omega$. Then $x \to^\omega y$ if $\forall$ prefix u of y $\exists$ prefix v of x and $w \in A^*$ such that: $v \to^* uw$. We define and denote semi-traces and traces, respectively, by

$$[x\rangle = \{y \in A^\omega \mid x \to^\omega y\}, \ \ [x] = \{y \in A^\omega \mid x \to^\omega y \wedge y \to^\omega x\},$$

and the labelled partial ordering $(O(x), \le_x, lb)$ associated with $x \in A^\omega$ by

$$O(x) = \cup O(u) \ \text{ and } \ \le_x = \cup \le_u \text{ for all prefices } u \text{ of } x.$$

From Definition 2.14 and Lemma 2.13 it immediately implies

Lemma 2.15. *Let $x, y \in A^\omega$. Then*
(i) $x \rightarrow^\omega y$ *iff* $O(y) \subseteq O(x)$ *and* $\leq_x \subseteq \leq_y$,
(ii) $[x\rangle = [y\rangle$ *iff* $[x] = [y]$.

Definition 2.16. Let $\Sigma = (S, T, F, M_0)$ be a system net. The semi-commutative system derived from Σ is $SC(\Sigma) = \langle T, R\rangle$, where

$$R = \{tt' \rightarrow t't \mid t^\bullet \cap {}^\bullet t' = \emptyset \wedge t \neq t'\},$$
$$R_S = \{tt' \rightarrow t't \mid t^\bullet \cap {}^\bullet t' = t'^\bullet \cap {}^\bullet t = \emptyset \wedge t \neq t'\}.$$

For a given system net Σ we are always dealing with labelled partial ordering $(O_\sigma, \leq_\sigma, lb)$ induced by the firing sequence σ, and with semi-traces and traces on firing sequences in the corresponding semi-commutative system $SC(\Sigma)$.

The following notations will be used in the sequel:
$\mathcal{S}(\Sigma)$ is the set of all semi-traces on firing sequences of Σ.
$\mathcal{T}(\Sigma)$ is the set of all traces on firing sequences of Σ.
$M\mathcal{S}(\Sigma)$ is the set of all maximal semi-traces on firing sequences of Σ.
$M\mathcal{T}(\Sigma)$ is the set of all maximal traces on firing sequences of Σ.

3. Globally observable nets

Now we want to find conditions for a system net, in which the behaviour can be represented by its semi-traces.

Definition 3.1. A process π of a system net Σ is globally observable if there exists a firing sequence $\sigma \in \mathcal{F}(\Sigma)$ such that: $\mathrm{Lin}(\pi) = [\sigma\rangle$.

In this case σ is said to be a global observation of the process π. A net Σ is globally observable iff all of its processes are globally observable.

Definition 3.2. A process $\pi = (B, E, F', p)$ of a net Σ has a global time if there exists a bijection tm: $E \rightarrow \{1, 2, \ldots, |E|\}$ such that, $\forall e_1, e_2 \in E$:

$$((e_1 \leq e_2 \Rightarrow tm(e_1) \leq tm(e_2)) \wedge (e_1 \mathbf{co} e_2 \wedge p(e_2)^\bullet \cap {}^\bullet p(e_1) \neq \emptyset) \Rightarrow \\ \Rightarrow (tm(e_1) < tm(e_2)))$$

We say a net Σ has a global time if all of its processes have their global time.

Definition 3.3. A system net $\Sigma = (S, T, F, M_0)$ is said to have a firable semi-cycle at a marking $M \in [M_0\rangle$ if there exist $t'_i, t_i \in T$ and $u_i \in t'_i T^* \cap T^* t_i$ for $i = 1, \ldots, k$ such that: $k \geq 2, M = \sum_{i=1}^{k} M_i, M_i[u_i\rangle M'_i, t'_i \leq_{u_i} t_i, {t'}^\bullet_{i+1} \cap {}^\bullet t_i \neq \emptyset$ with $i = 1, \ldots, k, t'_{i+1} \neq t_i$ and $t'_{k+1} = t'_1$.

In the case $u_i = t'_i = t_i$ for $i = 1, \dots, k$ we say the net Σ has a firable cycle at the marking M.

Example 3.4

(i) Consider the system net given in Fig. 1

Fig. 1

(ii) The process π of Σ, given in Fig. 2, is not globally observable, because it has two maximal semi-traces of $\mathrm{Lin}(\pi)$, which are not $\equiv$-equivalent:

Fig. 2

$[t_1t_3t_2t_4\rangle$ and $[t_2t_4t_1t_3\rangle \in MS(\mathrm{Lin}(\pi))$, but $t_1t_3t_2t_4 \not\equiv t_2t_4t_1t_3$. Obviously, Σ has a firable semi-cycle at M_0: $M_0 = {}^\bullet t_1 + {}^\bullet t_2$ with $u_1 = t_1t_3, u_2 = t_2t_4$.

Theorem 3.5. *Let Σ be a system net. The following three conditions are equivalent:*

(i) Σ *is a globally observable,*

(ii) Σ *has a global time,*

(iii) Σ *has no firable semi-cycle at any reachable marking.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let $\pi = (B, E, F', p) \in \mathcal{P}(\Sigma)$ and σ be the global observation of π. Take $tm = \mathrm{pos}_\sigma$. We show that tm is a global time of π.

Assume for the contrary that tm is not a global time. By property of pos_σ in Theorem 2.9 and Definition 3.2 this may happen only in the following case:

$\exists e_1, e_2 \in E, e_1 \ \mathbf{co} \ e_2, p(e_2)^\bullet \cap {}^\bullet p(e_1) \neq \emptyset$, but $tm(e_2) < tm(e_1)$.

From "$e_1 \ \mathbf{co} \ e_2$" we have an observation

$$\sigma' = \alpha p(e_1) p(e_2) \beta \in Lin(\pi) = [\sigma\rangle.$$

So $\sigma \rightarrow^* \sigma'$. Because of

$$\mathrm{pos}_\sigma(e_2) = tm(e_2) < tm(e_1) = pos_\sigma(e_1),$$

$\sigma = \gamma_1 p(e_2) \gamma_2 p(e_1) \gamma_3$ for some $\gamma_1, \gamma_2, \gamma_3 \in T^*$. Hence, by definition,

$$\sigma \rightarrow^* \alpha p(e_2) p(e_1) \beta \text{ and } p(e_2) p(e_1) \rightarrow p(e_1) p(e_2).$$

It follows that

$$p(e_2)^\bullet \cap {}^\bullet p(e_1) = \emptyset.$$

This contradicts the assumption

$$p(e_2)^\bullet \cap {}^\bullet p(e_1) \neq \emptyset.$$

Hence tm is a global time of π.

$(ii) \Rightarrow (iii)$. Let every process of Σ have a global time. Assume that Σ has a firable semi-cycle as in Definition 3.3. Then we can find a process π of Σ as in Fig. 3:

Fig. 3

Let $p(e'_i) = t'_i, p(e_i) = t_i$ and tm a global time of π. By the construction of process π we have:

$$e'_i \leq_\pi e_i, \ e_i \, \mathbf{co} \, e'_{i+1} \quad \text{and} \quad p({e'}^\bullet_{i+1}) \cap {}^\bullet p(e_i) \neq \emptyset$$

for $i = 1, \dots,$ and $e'_{k+1} = e'_1$. So from definition of global time tm it follows:

$$tm(e'_1) < tm(e_1) < tm(e'_2) < \dots < tm(e'_k) < tm(e_k) < tm(e'_1).$$

This is a contradiction. So Σ has no firable semi-cycle.

(iii)$\Rightarrow$ (i). The proof is by induction on the number of events of processes of Σ. Assume that every process of Σ with number of events less than n has been proved to be globally observable. Let $\pi = (B, E, F', p)$ be a process with n events. Now we prove that process π is a globally observable.

For any subset $E' \subset E$ we define a subprocess $\pi' = (B', E', F'', p')$ with

$$B' = \{b \in B \mid \exists e' \in E' : (e', b) \in F' \vee (b, e') \in F'\},$$
$$F'' = F'_{|(B' \times E') \cup (E' \times B')}, p' = p_{|E'}.$$

The constructed process id denoted by $\pi_{|E'}$.

Let $O_\pi = (E, \leq_\pi, p)$ be the labelled partial ordering derived from π.

If $\mathrm{Max}(O_\pi) = \{e\}$ with $p(e) = t$, i.e. , there is only one maximal event e of π. Then, by induction process $\pi' = \pi_{|\{e\}}$ is globally observable. Let $\mathrm{Lin}(\pi') = [\alpha\rangle$ for some $\alpha \in \mathcal{F}(\Sigma)$. Since e is the unique maximal event, $\mathrm{Lin}(\pi) = [\alpha t\rangle$, so π is a globally observable.

Suppose $\mathrm{Max}(O_\pi) = \{e_1, \dots, e_k\}$ with $k \geq 2$. For $i = 1, \dots, k$ we denote:

$$p(e_i) = t_i, E_i = \{e \in E \mid e \leq_\pi e_i\}.$$

Assume that there exists a maximal event $e_{i_0} \in \mathrm{Max}(O_\pi)$ such that

$$\forall e \in E \setminus E_{i_0} : p(e)^\bullet \cap {}^\bullet p(e_{i_0}) = \emptyset.$$

Let e_{i_0} be such an event and $\pi' = \pi_{|E \setminus \{e_{i_0}\}}$. By induction, π' is a globally observable, so $\mathrm{Lin}(\pi') = [\alpha\rangle$ for some $\alpha \in \mathcal{F}(\Sigma)$. Clearly, $\alpha t_{i_0} \in \mathrm{Lin}(\pi)$. We show $\mathrm{Lin}(\pi) = [\alpha t_{i_0}\rangle$, i.e. , for all $\sigma t_{i_0} \sigma' \in \mathrm{Lin}(\pi)$, $\alpha t_{i_0} \rightarrow^* \sigma t_{i_0} \sigma'$. Obviously, $\sigma\sigma' \in \mathrm{Lin}(\pi')$, thus $\alpha \rightarrow^* \sigma\sigma'$. For all $t \in \mathrm{alph}(\sigma')$, $t \notin E_{i_0}$, i.e., $t \in E \setminus E_{i_0}$. By the assumption this implies $t^\bullet \cap {}^\bullet t_{i_0} = \emptyset$, so $(tt_{i_0} \rightarrow t_{i_0} t)$. Hence,

$$\alpha t_{i_0} \rightarrow^* \sigma\sigma' t_{i_0} \rightarrow^* \sigma t_{i_0} \sigma'.$$

Now we will show that if there is no such maximal event $e_{i_0} \in \mathrm{Max}(O_\pi)$, i.e.

$$\forall e_i \in \mathrm{Max}(O_\pi), \exists e''_i \in E \setminus E_i : p(e''_i)^\bullet \cap {}^\bullet p(e_i) \neq \emptyset \quad \text{with} \quad i = 1, \dots, k,$$

then we can find a firable semi-cycle by the following algorithm:

Step 0. Take $e_{r_0} = e_k \in \mathrm{Max}(O_\pi)$. Note that maximal event e_{r_0} has been marked. Go to *Step i.*

Step i. For $e_{r_i} \in \mathrm{Max}(O_\pi)$ let

$$E''_{r_i} = \{e \in E \setminus E_{r_i} : p(e)^\bullet \cap {}^\bullet p(e_{r_i}) \neq \emptyset.$$

By the assumption $E''_{r_i} \neq \emptyset$. Between all maximal events of E''_{r_i} we can choose the event e''_{r_i} such that for all other maximal event $e \in E''_{r_i}$ we have $p(e)^\bullet \cap {}^\bullet p(e''_{r_i}) = \emptyset$. Otherwise, we get a firable semi-cycle, a contradiction.

We denote the set of indexes of maximal events of π dominating e''_{r_i} by:

$$J_i = \{j \mid j \leq k \wedge e''_{r_i} \in E_j\}.$$

If there exists a marked maximal event in $\{e_j \mid j \in J_i\}$, then the algorithm stops. Otherwise, let $E^{(i)} = \bigcup E_j$ for $j \in J_i$, thus $E^{(i)} \subseteq E \setminus \{e_{r_i}\}$. By induction, $\pi^{(i)} = \pi_{|E^{(i)}}$ is globally observable and $\mathrm{Lin}(\pi^{(i)}) = [\alpha t''_{r_i}\beta\rangle$ for some $\alpha, \beta \in T^*$ and $t''_{r_i} = p(e''_{r_i})$.

Let $t''_{r_i}\beta \in T^* t_{j_0}$ and $e_{j_0} = p^{-1}(t_{j_0})$.

By the assumption, for the maximal event e_{j_0} there exists $e''_{j_0} \in E \setminus E_{j_0}$ such that $p(e''_{j_0})^\bullet \cap {}^\bullet p(e_{j_0}) \neq \emptyset$. We show $e''_{j_0} \in E \setminus E^{(i)}$. Assume for the contrary that $e''_{j_0} \in E^{(i)}$. Let $\gamma = \alpha t''_{r_i}\beta$ and $tm = \mathrm{pos}_\gamma$. By induction tm is a global time of $\pi^{(i)}$. Then $tm(e_{j_0}) < tm(e''_{j_0})$, i.e., by the definition of tm: $\mathrm{pos}_\gamma(e_{j_0}) < \mathrm{pos}_\gamma(e''_{j_0})$. This contradicts the fact that $t_{j_0} = p(e_{j_0})$ is the last transition in $\alpha t''_{r_i}\beta$.

Now we take $e_{r_{i+1}} = e_{j_0}$ and $u_{i+1} = t''_{r_i}\beta$. We assume that all maximal events $\{e_j \mid j \in J_i\}$ have been marked. Let $i = i + 1$ and return to the beginning of *Step i.*

Algorithm terminates after at most k steps. As a result, we get a set of words $\{u_1, u_2, \ldots, u_q\}$. To prove that it is a firable semi-cycle, according to the algorithm we need only to show in *Step i* that for every event $e \in E^{(i)}$ with $p(e) \in \mathrm{alph}(\beta)$: e **co** e_{r_i}. By the choice of e''_{r_i} from E''_{r_i}, it follows that for all $e \in E^{(i)}$,

$$p(e)^\bullet \cap {}^\bullet p(e_{r_i}) \neq \emptyset \text{ implies } e < e''_{r_i} \text{ or } e \textbf{ co } e''_{r_i} \wedge p(e)^\bullet \cap {}^\bullet p(e''_{r_i}) = \emptyset.$$

So we can take a global observation $\gamma = \alpha t''_{r_i}\beta$ of $\pi^{(i)}$ such that, for those events e, as mentioned above, all transitions $\{p(e)\}$ have no occurrence in β. Then, $\forall t \in \mathrm{alph}(\beta)$: $t^\bullet \cap {}^\bullet p(e_{r_i}) = \emptyset$. Hence $p^{-1}(t)$ **co** e_{r_i}. We get a firable semi-cycle $\{u_1, \ldots, u_q\}$, a contradiction. So there is a required maximal event e_{i_0}. Hence, every process of Σ with n events is globally observable. The proof of Theorem 3.5 is complete.

Corollary 3.6. *If a system net $\Sigma = (S, T, F, M_0)$ is a globally observable* *and $M \in [M_0\rangle$, then so is the system net $\Sigma' = (S, T, F, M)$.*

Definition 3.7. Let Σ be a system net and σ a firing sequence of Σ. A semi-trace $[\sigma\rangle$ is strict if there is a process π such that: $\mathrm{Lin}(\pi) = [\sigma\rangle$. We denote the set of all strict semi-traces on firing sequences of Σ by $SS(\Sigma)$.

Lemma 3.8. *Let Σ be a self-concurrency free system net. Let*

$$\pi = (B, E, F', p)$$

be a process and σ a firing sequence of Σ. Then

$$\mathrm{Lin}(\pi) = [\sigma\rangle \quad \text{iff} \quad (O(\sigma), \leq_\sigma) = (E, \leq_\pi, p).$$

Proof. ($\Leftarrow$) Clearly.

($\Rightarrow$) Since Σ is self-concurrency free, for the process $\pi \in \Pi(\sigma)$ there is a unique bijection $\beta : E \rightarrow O(\sigma)$ such that

$$\forall e \in E : p(e) = \beta(e)_{|T} \wedge e_1 F'^* e_2 \rightarrow \beta(e_1) \leq_\sigma \beta(e_2).$$

By Theorem 3.5, pos_σ is a global time for the process π. It is obvious that $<_\pi \subseteq <_\sigma$. Let $t^\bullet \cap {}^\bullet t' \neq \emptyset$ and $m < k$ so $(t, m) <_\sigma (t', k)$. We should prove $\beta^{-1}(t, m) <_\pi \beta^{-1}(t', k)$. We may assume $t \neq t'$. Suppose $\beta^{-1}(t, m) \not<_\pi \beta^{-1}(t', k)$. Then $(t, m)\,\mathbf{co}\,(t', k)$ and $t^\bullet \cap {}^\bullet t' \neq \emptyset$. Since pos_σ is a global time for π, $\mathrm{pos}_\sigma(t', k) < \mathrm{pos}_\sigma(t, m)$, i.e., $k < m$, a contradiction.

Corollary 3.9. *Let Σ be a self-concurrency free system net. Then two globally observable processes of Σ with common global observation are $\approx$-equivalent.*

Combining all above presented results we get

Theorem 3.10. *Let Σ be a self-concurrency free globally observable system net. Then the mappings π_0 and Lin are order-preserving bijections between the set $SS(\Sigma)$ of strict semi-traces on firing sequences of Σ and the set of $\approx$-equivalence classes on processes of Σ:*

$$\mathcal{SS}(\Sigma) \cong \mathcal{P}(\Sigma)/_\approx$$

where $\pi_0([\sigma\rangle) = \{\pi \mid \pi \in \Pi(\sigma) \wedge Lin(\pi) = [\sigma\rangle\}$ and $(O(\sigma), \leq_\sigma) = (E, \leq_\pi, p)$ for any process $\pi = (B, E, F, p) \in \pi_0(\sigma)$.

By Theorem 3.10, semi-traces can replace processes for self-concurrency free globally observable nets.

4. Locally observable nets

Now we consider a class of the so called locally observable nets, for which there may be no global observation for all processes in general, but there is a global observation for every of its final subprocesses. A finite process π has a finite set of maximal events, which determines its final subprocesses with maximal events of π as their final events. Global observations of those final subprocesses are final strict semi-traces. So each process π of a locally observable net Σ corresponds to some set of final semi-traces which is denoted by $\text{Loc}(\pi)$. We will show that the set of $\{\text{Loc}(\pi) \mid \pi \in \mathcal{P}(\Sigma)\}$ with some partial order forms a prime algebraic domain where final strict semi-traces are complete primes. So from [13] the set of processes and the set of final strict semi-traces are equivalent in the sense that the second can be used to represent the first.

The following two definitions are taken from [9, 13]:

Definition 4.1. Let $(D, \sqsubseteq)$ be a partial order. A subset X of D is said to be compatible if $\exists p \in D$, $\forall x \in X$: $x \sqsubseteq p$, and to be finitely compatible if every its finite subset is compatible. D is said to be consistently complete if all finitely compatible subsets $X \subseteq D$ have least upper bounds $\sqcup X$.

Definition 4.2. Let $\mathcal{D} = (D, \sqsubseteq)$ be a consistently compatible partial order. A complete prime of D is an element $p \in D$ such that $p \sqsubseteq \sqcup X \Rightarrow \exists x \in X; p \sqsubseteq x$ for any compatible set X. D is a prime algebraic domain if $x = \sqcup\{p \sqsubseteq x : p \text{ is a complete prime}\}$ for all $x \in X$. D is a finitary domain if every complete prime p dominates only a finite number of elements, i.e. $\{d \in D : d \sqsubseteq p\}$ is finite.

Definition 4.3. Let Σ be a system net. A finite process π of Σ is final if there is only one maximal event on it.

Σ is called locally observable if every final process of Σ is globally observable.

A semi-trace $[\sigma\rangle$ of Σ is final if there is only one maximal element of $(O(\sigma), \leq_\sigma)$. The set of all final semi-traces on firing sequences of Σ is denoted by $\mathcal{FS}(\Sigma)$.

Definition 4.4. Let $\Sigma = (S, T, F, M_0)$ be a system net. A reachable marking $M \in [M_0\rangle$ is an observable if there exists a final process π of Σ such that M corresponds to some **co-set** of conditions of π.

Proposition 4.5. *Let $\Sigma = (S, T, F, M_0)$ be a system net. The following conditions are equivalent:*

(i) Σ *is locally observable,*
(ii) *Every final process is globally observable,*
(iii) *Every final process has global time,*
(iv) *There is no any firable semi-cycle at a reachable marking such that, after firing this semi-cycle it produces an observable marking of* Σ.

Proof. It is similar to the proof of Theorem 3.5.

Example 4.6. The system net Σ given in Example 3.4 is locally observable because the marking we get after firing semi-cycle is not observable.

Let Σ be a self-concurrency free system net and π a process of Σ. Then π has a finite set of maximal events each of them corresponds to a final subprocess of π. If those final subprocesses are globally observable, each process π of Σ associates with the finite set of final strict semi-traces, which are called local observations of π. We denote the set of local observations of a process π by $\mathrm{Loc}(\pi)$. So $\mathrm{Loc}(\pi) = \{l_i \mid i = 1, \dots, n\}$, where l_i is a global observation of final subprocess π_i with $n = |\mathrm{Max}(O_\pi)| \leq |T|$, and $\pi = \cup \pi_i$ for $i = 1, \dots, n$.

Lemma 4.7. *Let* Σ *be a self-concurrency free system net and* π, π' *processes of* Σ. *Then*

$$\mathrm{Loc}(\pi) = \mathrm{Loc}(\pi') \text{ iff } \pi \approx \pi'.$$

Proof. Let $\mathrm{Loc}(\pi) = \mathrm{Loc}(\pi') = \{l_i \mid i = 1, \dots, n\}$. It is clear that if process π is associated with a set of final subprocesses π_i, then $\leq_\pi = \cup \leq_{\pi_i}$. Each final subprocess π_i is globally observable with l_i as its global observation. By Lemma 3.8, $(E_i, \leq_{\pi_i}, p_i) = (O(l_i), \leq_{l_i})$. Hence $\leq_\pi = \cup \leq_{l_i} = \leq_{\pi'}$. The converse is obvious.

Let us denote $\mathcal{LC}(\Sigma) = \{\mathrm{Loc}(\pi) \mid \pi \in \mathcal{P}(\Sigma)\}$. On the set $\mathcal{LC}(\Sigma)$ we define the following ordering:

$$\gamma_1, \gamma_2 \in \mathcal{LC}(\Sigma) : \gamma_1 \preceq \gamma_2 \Leftrightarrow (\forall l_1 \in \gamma_1, \exists l_2 \in \gamma_2 \wedge \gamma \in T^* : l_1 \circ [\gamma\rangle = l_2).$$

Proposition 4.8. $(\mathcal{LC}(\Sigma), \preceq)$ *is a finitary algebraic domain, where final strict semi-traces are its complete primes.*

Proof. (i) $(\mathcal{LC}(\Sigma), \preceq)$ is a consistently complete partial order. We have to prove that if $\gamma = \{l_i \mid i = 1, \dots, n\}$ is a finite consistent set of final semi-traces, then $\mathrm{Max}(\gamma) = \mathrm{Loc}(\pi)$ for some process $\pi \in \mathcal{P}(\Sigma)$. We may assume that $\gamma = \mathrm{Max}(\gamma)$. Then $\exists \gamma' = \{l'_k \mid k = 1, \dots, p\} \in \mathcal{LC}(\Sigma) : \forall l_i \in \gamma$, there is $l'_{k_i} \in \gamma'$ such that $l_i \preceq l'_{k_i}$ for $i = 1, \dots, n$. Let π' be the process of Σ

with $\gamma' = \text{Loc}(\pi')$. Since $\pi' = \cup\pi'_k$, where π'_k is the final subprocess with final semi-trace l'_k as its global observation, $\text{Lin}(\pi'_k) = l'_k$. In the process π'_{k_i} there is the **co-set** with corresponding marking M_i, where $M_0[l_i\rangle M_i$.

Let π_i be the subprocess of π'_{k_i} with the **co-set** corresponding marking M_i as its maximal conditions. The process π_i is final globally observable and l_i is its global observation. We take the process π as the union of all processes π_i: $\pi = \cup\pi_i$. So $\gamma = \text{Loc}(\pi)$.

(ii) Now we prove that a final semi-trace $[\sigma\rangle$ is a complete prime. We have to show that if $[\sigma\rangle \preceq \{l_i \mid i = 1, \ldots, n\}$, then there exists $k : 0{\le}k{\le}n$ such that $[\sigma\rangle \preceq l_k$. But that is obvious from the definition of the partial ordering $\preceq$ on $\mathcal{LC}(\Sigma$.

(iii) Next, we show that $(\mathcal{LC}(\Sigma), \preceq)$ is finitary and prime algebraic. Let π be a process of Σ. Then $\text{Loc}(\pi) = \{l_i \mid i = 1, \ldots, n\}$, where l_i is a final semi-trace corresponding to final subprocess π_i of process π. So $\text{Loc}(\pi)$ is also the least upper bound of the set $\{l \mid l \in \mathcal{FS}(\Sigma) \wedge l \preceq l_i \wedge 1{\le}i{\le}n\}$. Hence the result follows.

Furthermore, we show that for locally observable nets the set of processes is equivalent to the set of final semi-traces with corresponding consistency relation on its subsets. The following two definitions come from [13].

Definition 4.9. A prime event structure $\mathcal{E} = (E, \text{Con}, \sqsubseteq)$ consisting of a set E of events which are partially ordered by $\sqsubseteq$ is called *the causal dependency relation* and a predicate $\text{Con} \subseteq 2^E$ is called **the consistency relation** if $\{e' \mid e' \sqsubseteq e\}$ is finite, $e \in \text{Con}$, $Y \subseteq X \in \text{Con} {\Rightarrow} Y \in \text{Con}$, $X \in \text{Con} \wedge \exists e' \in X,\ e \sqsubseteq e' {\Rightarrow} X{\cup}e \in \text{Con}$, for all $e \in E$ and all finite subsets X, Y of E.

Define its consistent left-closed subset $\mathcal{L}(\mathcal{E})$ to consist of those subsets $x \subseteq E$ which are *consistent*: $\forall$ finite $X \subseteq x, X \in \text{Con}$ and X is *left-closed*, $\forall e, e' :\quad e' \sqsubseteq e \in x {\Rightarrow} e' \in x$. In particular, define the history of e as $h(e) = \{e' \in E \mid e' \sqsubseteq e\}$.

In [13] the following result has been proved:

Proposition 4.10. *Let $\mathcal{E}$ be a prime event structure. Then $(\mathcal{L}(\mathcal{E}), \subseteq)$ is a finitary and prime algebraic domain in which the complete primes are those elements that are of the form $h(e)$ for $e \in E$.*

Let $\mathcal{D} = (D, \sqsubseteq)$ be a finitary and prime algebraic domain. Define

$$\mathcal{P}r(\mathcal{D}) = (P, \text{Con}, \sqsubseteq_P),$$

where P consists of the complete primes of $\mathcal{D}$: $\sqsubseteq_P = \sqsubseteq_{|P}$ and $X \in \text{Con}$ if X has an upper bound for a finite subset X of P. Then, $\mathcal{P}r(\mathcal{D})$ is a prime event structure (see [13]).

Let $SS_f(\Sigma)$ be a set of all final strict semi-traces on firing sequences of Σ. By Lemma 4.7, Theorems 4.8, 4.10 and Theorem 1.3.7 in [13] we get the following result:

Theorem 4.11. *Let Σ be a self-concurrency free locally observable system net. Then $SS_f(\Sigma)$ is a prime event structure with ϕ: $\mathcal{L}C(\Sigma) \cong \mathcal{L}(SS_f(\Sigma))$, giving an isomorphism of partial orders, where $\phi(\mathrm{Loc}(\pi)) = \{h(l_i) \mid l_i \in \mathrm{Loc}(\pi)\}$ with inverse θ: $\mathcal{L}(SS_f(\Sigma)) \to \mathcal{L}C(\Sigma)$ given by $\theta(x) = \sqcup x$.*

Thus we see that for locally observable nets the set of final semi-traces on firing sequences can replace the set of processes.

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References

1. I. J. Aalbersberg and G. Rozenberg, *Theory of Traces*, Theor. Comp. Sci. **60** (1988), 1-82.
2. E. Best, *Concurrent behaviour: sequences, processes and axioms*, LNCS **197** (1984), 221-245.
3. E. Best and C. Fernandez, *Non-sequential Processes*, A Petri Net View, Springer-Verlag 1988.
4. E. Best and C. Fernandez, *Notation and terminology on Petri net theory*, Arbeitspapiere der GMD **195** (1987).
5. U. Goltz and W. Reisig, *The non-sequential behaviour of Petri nets*, Information and Control **57** (1983), 125-147.
6. D. V. Hung and E. Knuth, *Semi-commutations and Petri nets*, Theor. Comp. Sci. **64** (1989), 67-81.
7. D. V. Hung and T. V. Dung, *On the relation between firing sequences and processes of Petri nets*, LNCS **710** (1993), 309-318.
8. A. Mazurkiewics, *Trace theory*, Advanced Course of Petri nets 86, LNCS **225** (1987), 279-324.
9. M. Nielsen, G. Plotkin and G. Winskel, *Petri nets, Event structures and Domains, 1*, Theor. Comp. Sci. **13** (1981), 85-108.
10. E. Ochmanski, *Semi-commutation and Petri nets*, In: V. Diekert, editor, Proceeding of the ASMICS worshop Free Partial Commutative Monoid, Kochel am See 1989, Report TUM-19002, Technical University of Munich (1990), 151-166.
11. C. A. Petri, *Non-Sequential processes*, GMD-ISF Report **77** (1977).

12. P. S. Thiagarajan, *Some behaviour aspects of net theory*, Theor. Comp. Sci. **71** (1990), 133-153.

13. G. Winskel, *Event structures*, Advanced Course of Petri nets 86, LNCS **225** (1987), 325-392.

14. W. Zielonka, *Notes on finite asynchronous automata*, R.A.I.R.O. Inform. Theor. Appl. **21** (1987), 99-135.

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