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ON COHEN-MACAULAYNESS AND GORENSTEINNESS OF REES ALGEBRAS OF INTERGRALLY CLOSED FILTRATIONS OF HEIGHT TWO EQUIMULTIPLE IDEALS

DUONG QUOC VIET

1. Introduction

Let $(A, \boldsymbol{m})$ be a Noetherian local ring of $\dim A = d > 0$ and I an ideal of A with $\mathrm{ht}(I) = h > 0$. An element x of A is said to be integral over I if there is a positive integer n such that $x^n + c_1x^{n-1} + \cdots + c_n = 0$ for some $c_i \in I^i$ $(1 \leq i \leq n)$. Let $\bar{I}$ be the set of integral elements over I. It is well-known that $\bar{I}$ is an ideal and $\mathcal{F} = \{\overline{I^n}\}_{n\geq 0}$ is a filtration of A. This filtration $\mathcal{F}$ is called the integrally closed filtration of an ideal I. We call the graded rings

$$\overline{R(I)} = \oplus_{n\geq 0}\overline{I^n}t^n \quad \text{ and } \quad \overline{G(I)} = \oplus_{n\geq 0}(\overline{I^n}/\overline{I^{n+1}})$$

the Rees algebra and the associated graded ring of integrally closed filtration of I, respectively. The filtration $\mathcal{F} = \{\overline{I^n}\}_{n\geq 0}$ is called an I-good filtration if $\overline{I^{n+1}} = I\overline{I^n}$ for all large n. If $\mathcal{F}$ is an I-good filtration and I is an equimultiple ideal, we can find elements $x_1, \ldots, x_h$ of I such that $x_1, \ldots, x_h$ is a minimal reduction system of $\mathcal{F}$ [3].

Throughout this paper we assume that the residue field of A is an infinite field, the filtration $\mathcal{F} = \{\overline{I^n}\}_{n\geq 0}$ is an I-good filtration and I is an equimultiple ideal. In this case the number of elements of a minimal reduction system of I and $\mathcal{F}$ is exactly $\mathrm{ht}(I)$ and $\overline{R(I)}$ is a Noetherian ring of dimension $d + 1$ [10]. To determine when the Rees algebra $\overline{R(I)}$ is a Cohen-Macaulay or Gorenstein ring in terms of A and the ideal I is usually a hard problem. This problem is investigated by some authors in the cases A being a Cohen-Macaulay (Gorenstein or regular) ring (see [1], [2], [3], [4]).

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The aim of this paper is to give criteria for the Rees algebra $\overline{R(I)}$ of a height two equimultiple ideal I to be a Cohen-Macaulay or Gorenstein ring in terms of A and the ideal I. From these criteria we obtain interesting information on the structure of A and I. For example, if $\overline{R(I)}$ is a Cohen-Macaulay (Gorenstein) ring, then A is a Cohen-Macaulay (Gorenstein) ring.

Our main result are the following theorems.

Theorem 1.1. *Let I be a height two equimultiple ideal of A and $\mathcal{F}$ the integrally closed filtration of I. Let $r(\mathcal{F})$ be the reduction number of $\mathcal{F}$. Suppose that $\mathcal{F}$ is an I-good filtration. Then $\overline{R(I)}$ is a Cohen-Macaulay ring if and only if the following conditions are satisfied.*

(i) *A is a Cohen-Macaulay ring.*

(ii) *$J \cap \overline{I^n} = J\overline{I^n}$ for all $n \geq 0$, where J is an arbitrary ideal of the principal class of A such that $\mathrm{ht}(J) = d-2$ and (I, J) is an $\boldsymbol{m}$-primary ideal.*

(iii) *$r(\mathcal{F}) \leq 1$.*

Theorem 1.2. *Let I be a height two equimultiple ideal of A and $\mathcal{F}$ the integrally closed filtration of I. Suppose that $\mathcal{F}$ is an I-good filtration. Then $\overline{R(I)}$ is a Gorenstein ring if and only if the following conditions are satisfied.*

(i) *A is a Gorenstein ring.*

(ii) *$\overline{I^n} = I^n$ for all n.*

(iii) *I is a complete intersection ideal of A.*

A satisfactory tool for the proofs of these theorems is the theorem for the Rees algebra of a filtration to be Cohen-Macaulay by the author [8] or Gorenstein given by Trung-Viet-Zarzuela [7].

From the Theorem 1.1 and Theorem 1.2 we derive in Section 5 some interesting results in the case A is a domain of dimension 3 or A is an analytically unramified ring.

2. Preliminaries

We shall see that in dealing with problems on the Cohen-Macaulay and Gorenstein Rees algebras of filtrations, one can restrict the investigation to the associated graded rings which are easier to be handled due to the standard graded structure. Let $F = \{I_n\}_{n \geq 0}$ be a filtration of A with dim $A = d > 0$ such that $R(F)$ is a Noetherian ring of dimension $d+1$. Denote by M the maximal graded ideal of $R(F)$. First we mention some results which are used frequently in this paper.

Theorem 2.1 [8]. *Suppose that $R(F)$ is a Noetherian ring of dimension $d+1$. Then $R(F)$ is a Cohen-Macaulay ring if and only if*
(i) $[H^i_M(G(F))]_n = 0$ *for all* $n \neq -1, i = 0, ..., d-1$.
(ii) $[H^d_M(G(F))]_n = 0$ *for* $n \geq 0$.
In this case, $H^i_M(G(F)) \simeq H^i_{\boldsymbol{m}}(A)$ *for* $i = 0, ..., d-1$.

If F is an equimultiple $\boldsymbol{m}$-primary filtrations, then from Theorem 2.1 we already obtained some results on the structure of the ring A and filtration F with $R(F)$ being a Cohen-Macaulay Rees algebra, see [8].

Denote by $K_{G(F)}$ the canonical module of $G(F)$ if $G(F)$ admits a canonical module. Then we have the following result.

Theorem 2.2 [7]. *$R(F)$ is a Gorenstein ring iff the following conditions are satisfied:*
(i) *$R(F)$ is a Cohen-Macaulay ring.*
(ii) $\oplus_{n\geq 2}[K_{G(F)}]_n \simeq G(F)(-2)$.

Next, we shall prove some results on the Cohen-Macaulay property of Rees algebras of equimultiple filtrations. A filtration $F = \{I_n\}_{n\geq 0}$ is called an equimultiple filtration if there exists an equimultiple ideal $I \subseteq I_1$ such that F is an I-good filtration. Let $I' \subseteq I$ be a minimal reduction of an I-good filtration F. The reduction number of F with respect to I' is the number

$$r_{I'}(F) = \min \{r; I_{n+1} = I'I_n \text{ for all } n \geq r\}.$$

The reduction number of F is the number

$$r(F) = \min \{r_{I'}(F); I' \text{ is a minimal reduction of } F\}.$$

An ideal J of the ring A is called a complete intersection ideal of A if $J = 0$ or J is generated by a regular sequence of A.

Then we have the following theorem.

Theorem 2.3. *Let F be an equimultiple filtration of A and J an ideal of principal class of A such that* $\mathrm{ht}\,(J) = \dim A$ - $\mathrm{ht}\,(I_1)$ *and (J, I_1) is an $\boldsymbol{m}$-primary ideal of A. Then $R(F)$ is a Cohen-Macaulay ring if and only if the following conditions are satisfied.*
(i) *J is a complete intersection ideal of A.*
(ii) $J \cap I_n = JI_n$ *for all* $n \geq 0$.
(iii) $\oplus_{n\geq 0}(I_n + J/J)t^n$ *is a Cohen-Macaulay ring.*

Proof. ($\Longrightarrow$) Since F is an equimultiple filtration, one can use the same argument as in [6] to obtain the fact that there exists a system of parameters J^* of $R(F)$ such that J is a subset of J^*. Note that $R(F)$ is

Cohen-Macaulay and $JR(I)$ is an ideal of principal class of $R(F)$. Then $JR(F)$ is a complete intersection ideal of $R(F)$. Thus, J is a complete intersection ideal of A. Since dim $[R(F)/JR(F)] > 0$ and

$$\dim[R(F)/(JR(F) + I_1 \oplus_{n>0} I_n t^n)] = 0,$$

we get

$$JR(F) : (I_1 \oplus_{n>0} I_n t^n)^k = JR(F)$$

for all $k \geq 1$. It follows that $(J : I_1) = J$ and $(JI_n : I_n) \cap I_n = JI_n$ for all $n > 0$. Because

$$JI_n = (JI_n : I_n) \cap I_n \supseteq J \cap I_n \supseteq JI_n$$

for all $n \geq 1$, we get $J \cap I_n = JI_n$ for all $n \geq 0$. Thus,

$$\begin{aligned} R(F)/JR(F) = \oplus_{n\geq 0}(I_n/JI_n)t^n &= \oplus_{n\geq 0}(I_n/J \cap I_n)t^n \\ &\simeq \oplus_{n\geq 0}(I_n + J/J)t^n. \end{aligned}$$

Since $R(F)/JR(F)$ is a Cohen-Macaulay ring, it follows that $\oplus_{n\geq 0}(I_n + J/J)t^n$ is a Cohen-Macaulay ring.

($\Longleftarrow$) Assume that ht $(J) = j$ and $J = (a_1, .., a_j)A$. Since

$$(a_1, ..., a_i, a_{i+1}^k, ..., a_j^k) \cap I_n = (a_1, ..., a_i, a_{i+1}^k, ..., a_j^k)I_n$$

for all $k \geq 1$, it follows that

$$\bigcap_{k\geq 1}[(a_1, ..., a_i, a_{i+1}^k, ..., a_j^k) \cap I_n] = \bigcap_{k\geq 1}[(a_1, ..., a_i, a_{i+1}^k, ..., a_j^k)I_n].$$

Therefore, $(a_1, \ldots, a_i) \cap I_n = (a_1, \ldots, a_i)I_n$ for all $n \geq 0$. Using the equality just obtained and the regular property of the sequence $a_1, \ldots, a_j$ we get

$$\begin{aligned} &[(a_1, \ldots, a_i)I_n : a_{i+1}] \cap I_n = [((a_1, \ldots, a_i) \cap I_n) : a_{i+1}] \cap I_n \\ &= [(a_1, \ldots, a_i) : a_{i+1}] \cap (I_n : a_{i+1}) \cap I_n = (a_1, \ldots, a_i) \cap I_n = (a_1, \ldots, a_i)I_n \end{aligned}$$

for all $n \geq 0, i < j$. Hence

$$(a_1, \ldots, a_i)R(F) : a_{i+1}R(F) = (a_1, \ldots, a_i)R(F)$$

for all $i < j$ and $JR(F)$ is a complete intersection ideal of $R(F)$. Since

$$R(F)/JR(F) \simeq \oplus_{n\geq 0}(I_n + J/J)t^n$$

is a Cohen-Macaulay ring, it follows that $R(F)$ is a Cohen-Macaulay ring.

Proposition 2.4. *Let F be an equimultiple filtration of A such that I_1 is an $\boldsymbol{m}$-primary ideal and $R(F)$ is a Cohen-Macaulay ring. Let $x_1, \ldots, x_d$ be a minimal reduction system of F. Then*

$$I_n \cap (x_1, \ldots, x_i) = (x_1, \ldots, x_i)I_{n-1}$$

for all $n \geq 0$, $i \leq d$.

Proof. Set $J_{i,k} = (x_1, \ldots, x_i, x_{i+1}^k, \ldots, x_d^k)$. By [8, Corollary 2.5], we have

$$J_{i,k} \cap I_n = (x_1, \ldots, x_i)I_{n-1} + (x_{i+1}^k, \ldots, x_d^k)I_{n-k}.$$

Since

$$\bigcap_{k\geq 1}[J_{i,k} \cap I_n] = (x_1, \ldots, x_i) \cap I_n$$

and

$$\bigcap_{k\geq 1}[(x_1, \ldots, x_i)I_{n-1} + (x_{i+1}^k, \ldots, x_d^k)I_{n-k}] = (x_1, \ldots, x_i)I_{n-1},$$

it follows that

$$I_n \cap (x_1, \ldots, x_i) = (x_1, \ldots, x_i)I_{n-1}.$$

Let A be a generalized Cohen-Macaulay ring and $a_1, \ldots, a_d$ a standard system of parameters of A. By [9] $a_1, \ldots, a_d$ is $\big[\sum_{i=1}^{d}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) : a_i\big]$-independent and if J is an ideal of A such that $a_1, \ldots, a_d$ is J-independent, then

$$\Big[\sum_{i=1}^{d}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) : a_i\Big] \subseteq J.$$

In [5] D. Rees showed that if A is a quasi-unmixed ring and $a_1, \ldots, a_d$ is a system of parameters of A then $a_1, \ldots, a_d$ is $\overline{Q}$-independent with $Q = (a_1, \ldots, a_d)A$. Note that if A is a generalized Cohen-Macaulay ring, then A is a quasi-unmixed ring. By the results just mentioned, we get the following lemma.

Lemma 2.5. *Let A be a generalized Cohen-Macaulay ring, $a_1, \dots, a_d$ a standard system of parameters of A, $Q = (a_1, \dots, a_d)A$. Then*

$$\Big[\sum_{i=1}^{d}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d) : a_i\Big] \subseteq \overline{Q}.$$

3. Criterion for Cohen-Macaulayness of Rees algebras $\overline{R(I)}$

First we have the following proposition in the case dim $A = 2$.

Proposition 3.1. *Let* dim $A = 2$, *I an $\boldsymbol{m}$-primary equimultiple ideal of A and $\mathcal{F}$ the integrally closed filtration of I. Suppose that $\mathcal{F}$ is an I-good filtration. Then $\overline{R(I)}$ is a Cohen-Macaulay ring iff A is a Cohen-Macaulay ring and $r(\mathcal{F}) \leq 1$.*

Proof. Set $\overline{R(I)} = R$, $\overline{G(I)} = G$.

($\Longrightarrow$) Let a, b be a minimal reduction system of $\mathcal{F}$ and x, y their images in G, respectively. Since R is a Cohen-Macaulay ring, we obtain that a, b (resp. x, y) is a standard system of parameters of A (resp. G) by Lemma 2.4 in [8]. From [8, Theorem 2.1] we get $[H^0_M(G)]_0 \simeq H^0_{\boldsymbol{m}}(A)$ and $[H^0_M(G)]_0 = 0$. Thus, depth$(A) > 0$ and depth$(G) > 0$. Since x, y is a standard system of parameters of G, it follows that the element x is a non-zero-divisor in G. Hence, there is the exact sequence

$$0 \longrightarrow G \xrightarrow{x} G \longrightarrow G/xG \longrightarrow 0. \tag{1}$$

From this exact sequence we get for all $n \neq -1$,

$$[H^0_M(G/xG)]_n \simeq [H^1_M(G)]_{n-1} \tag{2}$$

and the exact sequence

(3)
$$0 \longrightarrow [H^1_M(G)]_n \longrightarrow [H^1_M(G/xG)]_n \longrightarrow [H^2_M(G)]_{n-1} \longrightarrow [H^2_M(G)]_n \longrightarrow 0.$$

Consider the following exact sequences:

$$0 \longrightarrow xG : y/xG \longrightarrow G/xG \xrightarrow{y} (x, y)G/xG \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow (x,y)G/xG \longrightarrow G/xG \longrightarrow G/(x,y)G \longrightarrow 0. \tag{5}$$

Since x, y is a standart system of parameters of G, $H^1_M(xG : y/xG) = 0$ and $yH^0_M(G/xG) = 0$. Hence from (4) we get

$$[H^1_M(G/xG)]_n \simeq [H^1_M((x,y)/xG)]_{n+1}. \tag{6}$$

From (5), it follows that

$$0 \longrightarrow [H^0_M(G/xG)]_n \longrightarrow [H^0_M(G/(x,y)G)]_n \longrightarrow [H^1_M((x,y)G/xG)]_n \longrightarrow 0. \tag{7}$$

Using (2), (6), (7) and $[H^1_M(G)]_n = 0$ for all $n \neq -1$, we get

$$[G/(x,y)G]_n = [H^0_M(G/(x,y)G)]_n = 0$$

for all $n \geq 2$. Therefore $\overline{I^{n+1}} = (a,b)\overline{I^n}$ for all $n \geq 1$. Thus, $r(\mathcal{F}) \leq 1$. Note that $[H^0_M(G/xG)]_n \simeq [H^1_M(G)]_{n-1}$ and $[H^1_M(G)]_n = 0$ for $n \neq -1$. Further, from $(a) \cap \overline{I^n} = (a)\overline{I^{n-1}}$ for all $n \geq 1$ by Proposition 2.4 and $r(\mathcal{F}) \leq 1$, we get

$$\begin{aligned}
[H^0_M(G/xG)]_1 &= [xG : y/xG]_1 \\
&= [(a\bar{I}, \bar{I^3}) : b] \cap \bar{I}/(a, \bar{I^2}) \\
&= [(aA \cap \bar{I^2} : b) \cap \bar{I} + \bar{I^2}]/(a, \bar{I^2}) \\
&= [(aA : b) \cap (\bar{I^2} : b) \cap \bar{I} + \bar{I^2}]/(a, \bar{I^2}) \\
&= [(aA : b) \cap \bar{I} + \bar{I^2}]/(a, \bar{I^2}) \\
&= [(aA : b) \cap \bar{I}]/[a, (a : b) \cap \bar{I^2}] \\
&= [(aA : b) \cap \bar{I}]/[aA + (aA : b) \cap (a,b) \cap \bar{I^2}] \\
&= [(aA : b) \cap \bar{I}]/[aA + aA \cap I^2] \\
&= [aA : b] \cap \bar{I}/aA.
\end{aligned}$$

Note that $[H^0_M(G/xG)]_1 = 0$. Hence, we have $[(a) : b] \cap \bar{I} = (a)$. Since A is a generalized Cohen-Macaulay ring and a, b is a standard system of parameters of A, it follows that $(a) : b \subseteq \overline{(a,b)A} \subseteq \bar{I}$. Thus, $(a) : b = [(a) : b] \cap \bar{I} = (a)$. Since the element a is a non-zero-divisor in A, $(a) : b = (a)$ and $\dim A = 2$, it follows that A is a Cohen-Macaulay ring.

($\Longleftarrow$) Since $r(\mathcal{F}) \leq 1$, it follows that there exists a minimal reduction system a, b of $\mathcal{F}$ such that $\overline{I^{n+1}} = (a,b)\overline{I^n}$ for all $n \geq 1$. Thus,

$\overline{I^{n+1}} \cap (a,b) = (a,b)\overline{I^n}$ for all $n \geq 0$. Hence the filtration $\mathcal{F}$ satisfies the conditions (i) and (ii) of the Theorem 2.3 in [8]. From the above equalities together with A being a Cohen-Macaulay ring, it follows that $\overline{R(I)}$ is a Cohen-Macaulay ring, by [8, Theorem 2.3].

One can replace the condition $(a) : b \subset \bar{I}$ by $(a) : b \subset I_1$ and use the same argument as in the proof of Proposition 3.1 to prove the following proposition.

Proposition 3.2. *Let* $\dim A = 2$, F *an equimultiple filtration of* A *such that* I_1 *is an* $\boldsymbol{m}$*-primary ideal. Suppose that there is a minimal reduction* $J = (a,b)$ *of* I_1 *such that* $(a) : b \subset I_1$*. Then* $R(F)$ *is a Cohen-Macaulay ring if and only if* A *is a Cohen-Macaulay ring and the reduction number of* F *with respect to* J *is smaller than 2.*

Proof of Theorem 1.1.
($\Longrightarrow$) Let $J = (a_3, ..., a_d)$ be an ideal of principal class of A such that $\mathrm{ht}(I) = d-2$ and $\overline{(I,J)}$ is an $\boldsymbol{m}$-primary ideal. Set $J_k = (a_3^k, ..., a_d^k)$ for all $k \geq 1$, $\overline{R(I)} = R$, $\overline{G(I)} = G$. Since R is a Cohen-Macaulay ring, it follows that $\oplus_{n\geq 0}(\overline{I^n} + J_k/J_k)t^n$ is a Cohen-Macaulay ring and J_k is a complete intersection ideal of A for all $k \geq 1$, by Theorem 2.3. Let a, b be a minimal reduction system of $\mathcal{F}$ and x, y their images in A/J_k. Since $(a) : b \subseteq \bar{I}$ we get $(x) : y \subseteq (\bar{I} + J_k/J_k)$. Note that $\{\overline{I^n} + J_k/J_k\}_{n\geq 0}$ is an equimultiple filtration of the ring A/J_k and x, y is a minimal reduction system, it follows that A/J_k is a Cohen-Macaulay ring and if r^* is a reduction number of the filtration $\{\overline{I^n} + J_k/J_k)\}_{n\geq 0}$ then $r^* \leq 1$, by Proposition 3.2. Using the results just obtained and by Proposition 3.2 it follows that

$$(\overline{I^{n+1}}, J_k)/J_k = [(a,b)\overline{I^n}, J_k]/J_k$$

for all $n \geq 1$, $k \geq 1$. Thus,

$$\overline{I^{n+1}} + J_k = (a,b)\overline{I^n} + J_k.$$

From this it follows that

$$\cap_{k\geq 1}[\overline{I^{n+1}} + J_k] = \cap_{k\geq 1}[(a,b)\overline{I^n} + J_k].$$

Thus, $\overline{I^{n+1}} = (a,b)\overline{I^n}$ for all $n \geq 1$. Hence $r(\mathcal{F}) \leq 1$. Since A/J is a Cohen-Macaulay ring and J is a complete intersection ideal of A, it follows that A is a Cohen-Macaulay ring. The condition (ii) of Theorem 1.1 follows by the condition (ii) of Theorem 2.3.

($\Longleftarrow$) Let J be an ideal of principal class of A such that $\mathrm{ht}(J) = d-2$ and (I, J) an $\boldsymbol{m}$-primary ideal of A such that $J\overline{I^n} = J \cap \overline{I^n}$ for all $n \geq 0$. Consequently,

$$R/JR \simeq \oplus_{n\geq 0}(\overline{I^n} + J/J)t^n.$$

Since $r(\mathcal{F}) \leq 1$, it follows that there exists an ideal $I" = (a', b') \subseteq (I+J/J)$ such that if $\bar{r}$ is the reduction number of the filtration $\{\overline{I^n} + J/J\}$ of the ring A/J with respect to $I"$, then $\bar{r} \leq 1$. Since A is a Cohen-Macaulay ring and J is an ideal of principal class of A, it follows that A/J is a Cohen-Macaulay ring. Hence, by Proposition 3.2 it follows that $\oplus_{n\geq 0}(\overline{I^n}+J/J)t^n$ is a Cohen-Macaulay ring. By the results just mentioned and by Theorem 2.3, it follows that R is a Cohen-Macaulay ring.

4. Criterion for Gorensteiness of Rees algebras $\overline{R(I)}$

First we shall prove Theorem 1.2.

Proof of Theorem 1.2.
($\Longrightarrow$) Set $\overline{R(I)} = R$, $\overline{G(I)} = G$. Since R is a Gorenstein ring, it follows that A is a Cohen-Macaulay ring and $\mathrm{grade}(I) = 2$ by Theorem 1.1. Hence A, G are Gorenstein rings and $a(G) = -2$ by [7, Corollary 3.5]. Let a, b be a minimal reduction system of filtration $\mathcal{F}$ and x, y their images in G. Then a, b and x, y are regular sequences of A and G, respectively. Hence we get an exact sequence

$$0 \longrightarrow G \xrightarrow{x} G \longrightarrow G/xG \longrightarrow 0.$$

Using this exact sequence we have the exact sequence

$$0 \longrightarrow [H^1_M(G/xG)]_n \longrightarrow [H^2_M(G)]_{n-1}.$$

Since $[H^2_M(G)]_n = 0$ for $d > 0$ and $[H^2_M(G)]_n = 0$ for all $n \geq -1$ if $d = 2$, $[H^1_M(G/xG)]_n = 0$ for $n \geq 0$. Since the element y is a non-zero-divisor in G/xG, we get the exact sequence

$$0\longrightarrow G/xG \xrightarrow{y} G/xG \longrightarrow G/(x, y)G \longrightarrow 0.$$

This implies exact sequence

$$0\longrightarrow [H^0_M(G/(x, y)G]_n \longrightarrow [H^1_M(G/xG)]_{n-1}.$$

Since $[H^1_M(G/xG)]_n = 0$ for all $n \geq 0$, it follows that $[H^0_M(G/(x, y)G)]_n = 0$ for all $n \geq 1$. Note that $J = (a_3, \ldots, a_d)$ is an ideal of the principal class such that (I, J) is an $\boldsymbol{m}$-primary ideal and $J_k = (a_3^k, \dots, a_d^k)$. We shall denote by $(J'_k, x, y)/(x, y)$ the image of J_k in G. Then $(J'_k, x, y)/(x, y)$ is a complete intersection ideal of the ring $G/(x, y)G$, because $G/(x, y)G$ is a Cohen-Macaulay ring. On the other hand, $(J'_k, x, y)/(x, y)$ is a homogeneous ideal of degree 0 of the ring $G/(x, y)G$ for all $k \geq 1$. Using the results just obtained, we get $[H^0_M(G/(J'_k, x, y)G)]_n = 0$ for all $n \geq 1$. Since

$$\begin{aligned}[G/(J'_k, x, y)G]_n &= [H^0_M(G/(J'_k, x, y)G]_n \\ &= [\overline{I^n}/(J_k\overline{I^n} + (a, b)\overline{I^{n-1}} + \overline{I^{n+1}}] = 0\end{aligned}$$

for all $n \geq 1$, we get

$$\overline{I} = J_k\overline{I} + (a, b) + \overline{I^2} = J_k\overline{I} + (a, b) + \overline{I^3} = \cdots = J_k\overline{I} + (a, b) + \overline{I^n} = \cdots$$

for all $k \geq 1$. Thus,

$$\overline{I} = \cap_{n\geq 1}[J_k\overline{I} + (a, b) + \overline{I^n}] = J_k\overline{I} + (a, b)$$

for all $k \geq 1$. Hence, we get

$$\overline{I} = \cap_{k\geq 1}[J_k\overline{I} + (a, b)] = (a, b).$$

Now, from the relations $\overline{I^{n+1}} = (a, b)\overline{I^n}$ for all $n \leq 1$ it follows that $\overline{I^n} = (a, b)^n = I^n$ for all $n \geq 1$ and that I is a complete intersection ideal of A.

($\Longleftarrow$) Since I is a complete intersection ideal of A, there exist regular elements a, b of A such that $I = (a, b)$. Since A is a Cohen-Macaulay ring and $\overline{I^n} = I^n = (a, b)^n$, it follows that

$$\overline{G(I)} = G(I) = (A/I)[X, Y],$$

by [10]. Since A is a Gorenstein ring, A/I and $(A/I)[X, Y]$ are Gorenstein rings. Thus, $\overline{G(I)}$ is a Gorenstein ring. Since $G = (A/I)[X, Y]$, we get the following exact sequences

$$0 \longrightarrow G \xrightarrow{X} G \longrightarrow (A/I)[Y] \longrightarrow 0,$$

$$0 \longrightarrow (A/I)[Y] \xrightarrow{Y} (A/I)[Y] \longrightarrow A/I \longrightarrow 0.$$

Using these exact sequences we get exact sequences
(8)
$$0\longrightarrow[H_M^{d-2}(A/I)]_n\longrightarrow[H_M^{d-1}((A/I)[X])]_{n-1}\longrightarrow[H_M^{d-1}((A/I)[X])]_n\longrightarrow 0,$$

$$0\longrightarrow[H_M^{d-1}((A/I)[X])]_n\longrightarrow[H_M^{d}(G)]_{n-1}\longrightarrow[H_M^{d}(G)]_n\longrightarrow 0. \tag{9}$$

From (8) we get

$$[H_M^{d-1}((A/I)[X])]_{n-1}\simeq[H_M^{d-1}((A/I)[X])]_n \tag{10}$$

for all $n\geq 1$. Since $[H_M^{d-1}((A/I)[X])]_n=0$ for all large n, it follows that

$$[H_M^{d-1}((A/I)[X])]_n=0 \quad \text{for all } n\geq 0.$$

From this and (9) we get $[H_M^d(G)]_n\simeq[H_M^d(G)]_{n-1}$ for all $n\geq 0$. Thus, $[H_M^d(G)]_n=0$ for all $n\geq -1$. It is a plain fact that $a(G)=-2$. Hence the conditions of Corollary 3.5 in [7] are satisfied. Thus, $\overline{R(I)}$ is Gorenstein ring by Corollary 3.5 of [7].

Corollary 4.1. *Let* $\dim A=2$*,* I *a height two equimultiple ideal of* A *and* $\mathcal{F}$ *an* I*-good integrally closed filtration of* I*. Then* $\overline{R(I)}$ *is a Gorenstein ring if and only if* A *is a Gorenstein ring and* $\overline{G(I)}\simeq(A/I)[X,Y]$*.*

Proof. ($\Longrightarrow$) From Theorem 1.2 it follows that A is a Gorenstein ring and

$$\overline{G(I)}=G(I)\simeq(A/I)[X,Y].$$

($\Longleftarrow$) Since $\overline{G(I)}\simeq(A/I)[X,Y]$ and A is a Gorenstein ring, it follows that I is a complete intersection ideal of A and $\overline{I^n}=I^n$ for all n. From this it follows that $\overline{R(I)}$ is a Gorenstein ring, by Theorem 1.2.

5. Some applications

First, we are interested in the case A being a normal integral domain with dim $A=3$.

Proposition 5.1. *Let* A *be a Noetherian normal integral domain of dimension 3,* I *a height two equimultiple ideal of* A *and* $\mathcal{F}$ *the integrally closed filtration of* I*. Let* $r(\mathcal{F})$ *be the reduction number of* $\mathcal{F}$*. Then* $\overline{R(I)}$ *is a Cohen-Macaulay ring if and only if* A *is a Cohen-Macaulay ring and* $r(\mathcal{F})\leq 1$*.*

Proof. We need only to show that if A is a Cohen-Macaulay normal integral domain of dimension 3, I a height two equimultiple ideal of A and $\mathcal{F}$ an I-good integrally closed filtration of I, c is an element of A such that (I, c) is an $\boldsymbol{m}$-primary ideal, then $(c) \cap \overline{I^n} = c\overline{I^n}$ for all $n \geq 1$. Assume that a, b is a minimal reduction system of $\mathcal{F}$ and $x \in (c) \cap \overline{I^n}$. It follows that there is an element $y \in A$ such that $x = cy \in \overline{I^n}$ and $c^N y^N \in \overline{I^k}(a, b)^{nN-k}$ for all large N, k. From this, we get $c^N y^N \in (a, b)^{nN-k}$. Thus,

$$y^N \in (a, b)^{nN-k} : c^N = (a, b)^{nN-k},$$

because A is a Cohen-Macaulay ring. Now, let V be a discrete valuation of A. We have

$$NV(y) \geq NV[(a, b)^n] - kV[(a, b)]$$

for all large N. It follows that

$$V(y) \geq V[(a, b)^n] - (k/N)V[(a, b)]$$

for all large N. Therefore, we have $V(y) \geq V[(a, b)^n]$ for all discrete valuations V of A and $y \in \overline{(a, b)^n} \subseteq \overline{I^n}$. Thus, $x \in (c)\overline{I^n}$ and

$$(c) \cap \overline{I^n} = (c)\overline{I^n}$$

for all $n \geq 1$. From this and Theorem 1.1 we get Proposition 5.1.

Proposition 5.2. *Let $(A, \boldsymbol{m})$ be a Noetherian local analytically unmarified ring of $\dim A \geq 2$, I a height two equimultiple ideal of A, $\mathcal{F} = \{\overline{I^n}\}_{n \geq 0}$. Then $\overline{R(I)}$ is a Gorenstein ring if and only if A is a Gorenstein ring, I is a complete intersection ideal of A and $\overline{I^n} = I^n$*

Proof. Since A is analytically unmarified, it follows that the filtration $\mathcal{F}$ is an I-good filtration. Hence this proposition follows from Theorem 1.2.

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Department of Mathematics,
National Polytechnic University, Hanoi, Vietnam.