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STRONG CONSISTENCY OF LEAST SQUARES ESTIMATES AND MEAN SQUARE REGRESSION

NGUYEN BAC VAN

Abstract. We prove that the least squares estimates of regression parameters are strongly consistent if and only if the regression is the mean square one.

1. Introduction

Cramér called mean square (msq) regression of a numerical random variable (rv) Y on another rv X a function $g(X)$ which, among all functions belonging to some given class, is one that gives the best possible representation of Y according to the principle of least squares (see [4]). For example, the linear msq regression is the function $\alpha + \beta X$ that minimizes $E\{Y - g(X)\}^2$ in the class of linear functions $g(X)$.

Let $\big(X(1), Y(1)\big), \dots, \big(X(n), Y(n)\big)$ be n i.i.d. versions of the pair (X, Y). In the general linear representation $Y \approx c + dX$ we could estimate the parameter (c, d) by the least square method, i.e. by seeking values of (c, d) so as to minimize the square distance $\sum\limits_{i=1}^{n} \big(Y(i) - c - dX(i)\big)^2$. Such values are called least squares values of (c, d). Under what conditions a least squares value will converge almost surely (a.s.) to the true parameter value as $n \to \infty$? In this paper we will prove that

If EY^2, EX^2 are finite and X is not reduced a.s. to a constant, then a least squares value is strongly consistent when and only when the line $y = c + dx$ is the msq regression line.

We will state this result in a much more general setting. In the same way as in Bac Van (1992, 1994), we consider the r-dimensional polygonal regression model

$$Y' = \sum_{i=1}^{k} b_i'(X) q_i I_{S(i)}(X) + \varepsilon, \tag{1.1}$$

where Y is an $r \times 1$ random vector variable, X is a rv taking values in

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an arbitrary measurable space $(H, \mathcal{A})$, ε is some residual, $S(1), \dots, S(k)$ are specified disjoint sets of $\mathcal{A}$ such that $P\{X \in S(i)\} > 0$, $i = 1, \dots, k$, $k \geq 1$ is fixed, and for each index i, $I_{S(i)}$ is the indicator of $S(i)$, $b_i(\cdot)$ is a known $\ell(i) \times 1$ measurable vector-valued function on $S(i)$ and q_i an unknown $\ell(i) \times r$ matrix parameter. The above linear case corresponds to $r = 1$, $k = 1$, $S(1) = H$, $\ell(1) = 2$, $b_1(X) = (X\ 1)'$, $q_1 = (d\ c)'$. Let

$$\ell = \ell(1) + \cdots + \ell(k),$$
$$q = \big(q_1' \dots q_k'\big)'$$

and Q be the rangle of the parameter q in

$$M_{\ell \times r} = \text{the linear spaces of } \ell \times r \text{ real matrices.}$$

For example, in the above case if $EX = EY = 0$ and $\operatorname{Var} X = \operatorname{Var} Y = 1$, then the msq regression line is $y = \rho x$, the parameter is ρ and its range is the segment $[-1, 1]$. Besides, the constraints imposed on the parameters q_i's in (1.1), if any, can always be expressed through a definite shape of Q. The following definition of generalized least squares (GLS) estimates, already stated in [2], takes into account the arbitrariness of Q and the flexibility to some extent in the choice of metric in the space of response observations. It consists in

(i) Ranking the observations by subset $S(i)$: rearrange the pair of observations $\big(X(t), Y(t)\big)$ $(t = 1, \dots, n)$ on (X, Y) according to the successive entrance of $X(1), \dots, X(n)$ into each domains $S(i)$ by setting

$$\begin{aligned} X_{ij} &= \text{The } j^{th} \text{ element of the sequence } X(1), \dots, X(n) \\ &\qquad \text{falling into } S(i), \\ Y_{ij} &= \text{The } Y\text{-observation paired with } X_{ij}, \\ d(i) &= \#S(i) \cap \{X(1), \dots, X(n)\}. \end{aligned}$$

(ii) Minimizing some distance $d(.,.)$ from the overall response observation matrix

$$U = \big(Y_{11} \dots Y_{1d(1)} \vdots \cdots \vdots Y_{k1} \dots Y_{kd(k)}\big)' \tag{1.2}$$

to the product matrix B_p, where

$$B = \operatorname{diag}(B_1 \dots, B_k)$$

with

$$B_i = \big(b_i(X_{i1}) \dots b_i(X_{id(i)})\big)', \tag{1.3}$$

and where p varies on some affine manifold $\mathcal{M}$ containing the parameter range Q and contained in $M_{\ell\times r}$.

(iii) Defining a norm in the range space of U: use

$z(.) =$ an arbitrary $\mathcal{A}$-measurable $r \times r$ positive definite (p.d.) matrix function on H,

set

$$Z_i = \operatorname{diag}\big(z(X_{ij}), j = 1, \dots, d(i)\big), \quad i = 1, \dots, k,$$
$$Z = \operatorname{diag}(Z_1, \dots, Z_k), \tag{1.4}$$

and define the inner product of elements u and v in the range space of U as

$$\big(u, v\big)_Z = [u]'Z[v]$$

by means of the notation

$$[u] = \big(u_1 \; u_2 \; \dots\big)',$$

where $u_1, u_2, \dots$ are the successive rows of the matrix u.

Then the norm of U is

$$\|U\|_Z = \big\{[U]'Z[U]\big\}^{1/2}.$$

This is not the general norm in the U-space, but it tolerates an arbitrary scaling of Y given a value of X.

Thus we define a *GLS value* $\hat{q}$ by

$$\|U - B\hat{q}\|_Z = \min_{p\in\mathcal{M}} \|U - Bp\|_Z. \tag{1.5}$$

A GLS value $\hat{q}$ always exists whenever

$$d(i) > 0 \quad \forall i = 1, \dots, k, \tag{1.6}$$

since $B\hat{q}$ is the orthogonal projection of U on the image $B\mathcal{M} = \{Bp : p \in \mathcal{M}\}$. When there exists a unique GLS value $\hat{q}$, it is called a *GLS estimate* (GLSE) for q. Specifically, for $\mathcal{M} = M_{\ell\times r}$ GLS values $\hat{q}$ are defined by

$$\|U - B\hat{q}\|_Z = \min_{p\in M_{\ell\times r}} \|U - Bp\|_Z. \tag{1.7}$$

The characterization of msq regression by strong consistency of GLS estimates of the regression parameter will be proved for the polygonal model (1.1) in the following section.

2. Results

We now restate and generalize the definition of msq regression applied to Model (1.1). We shall write

$$\|Y\|^2_{z(X)} = Y'z(X)Y.$$

Definition 1. The function $\sum_{i=1}^{k} q_i' b_i(X) I_{S(i)}(X)$ is called the msq regression of Y if $q = \big(q_1' \cdots q_k'\big)'$ is a value that minimizes

$$E\Big\|Y - \sum_{i=1}^{k} p_i' b_i(X) I_{S(i)}(X)\Big\|^2_{z(X)}$$

among all non-random values $p = \big(p_1' \cdots p_k'\big)'$ in $M_{\ell\times r}$.

For any matrix $A = (a_{st})$ and any positive integer m, we shall write $\|A\|^2 = \sum_{s,t} |a_{st}|^2$ and I_m is the $m \times m$ unit matrix. We first state the following lemma.

Lemma. *In Model* (1.1) *suppose that*

$$E\|z^{1/2}(X)Y\|^2 < \infty, \tag{2.1}$$

and that

$$E_{\{X\in S(i)\}}\{\|b_i(X)\|^2 \mathrm{Tr}\ z(X)\} < \infty \quad \forall i = 1, \dots, k. \tag{2.2}$$

Then the function $\sum_{i=1}^{k} q_i' b_i(X) I_{S(i)}(X)$ *is the msq regression of* Y *according to Definition 1 if and only if* q *satisfies the condition:*

$$E_{\{X\in S(i)\}}\left\{(b_i(X)\otimes z(X))\,Y - (b_i(X)b_i'(X)\otimes z(X))\,[q_i]\right\} \tag{2.3}$$

$$\textit{exists and vanishes} \quad \forall i = 1, \dots, k.$$

Given the functions $z(.)$, $b_i(.)$ *and the sets* $S(i)$ $(i = 1, \dots, k)$, *there always exists some value* q *in* $M_{\ell\times r}$ *satisfying* (2.3).

Proof. For any rv ξ we have $\xi = \sum_{i=1}^{k} \xi I_{S(i)}(X) + \xi I_{S(0)}(X)$, where $S(0) = H - (S(1) + \cdots + S(k))$. If $E\xi$ exists, $E\xi = \sum_{i=1}^{k} \mathrm{P}\{X \in S(i)\} E_{\{X\in S(i)\}}\xi + E\xi I_{S(0)}(X)$. Let

$$\xi = \|Y - \sum_{i=1}^{k} q_i' b_i(X) I_{S(i)}(X)\|^2_{z(X)} \; . \tag{2.4}$$

Then, to minimize $E\xi$ as $q = \left(q_1' \cdots q_k'\right)'$ varies over $M_{\ell\times r}$ is equivalent to minimize $E_{\{X\in S(i)\}}\xi$ separately as q_i varies over $M_{\ell(i)\times r}$ $(i = 1, \ldots, k)$, because $E\xi I_{S(0)}(X) = E\|Y\|^2_{z(X)} I_{S(0)}(X)$ is independent of q. Thus, our problem is reduced to k problems of minimizing $E_{\{X\in S(i)\}}\|Y - q_i' b_i(X)\|^2_{z(X)}$ as q_i varies over $M_{\ell(i)\times r}$ $(i = 1, \ldots, k)$ or, equivalently, to the only problem of

$$\text{minimizing } E\|Y - q'b\|^2_z \text{ as } q \text{ varies over } M_{\ell\times r} \; . \tag{2.5}$$

Here we write E, q, b, z, ℓ instead of $E_{\{X\in S(i)\}}$, q_i, $b_i(X)$, $z(X)$, $\ell(i)$.

Let us tackle it. Let S_z be the set of all $r\times 1$ random vectors η defined up to an equivalence and such that $E\|\eta\|^2_z = E(\eta' z\eta) = E\|z^{1/2}\eta\|^2 < \infty$. We can check that S_z is a linear space and that the function $\varphi(\eta, \zeta) = E(\eta' z\zeta)$ is an inner product in S_z since z is p.d. Further, q is an $\ell\times r$-matrix and b is an $\ell \times 1$-matrix. The algebraic Propositions 3.2 and 4.3 in [2] successively give

$$q'b = [b'q] = (b' \otimes I_r)[q] \; , \tag{2.6}$$

$$\begin{aligned} E\|z^{1/2}q'b\|^2 &\le E\{\|z^{1/2}(b' \otimes I_r)\|^2\|q\|^2\} = \|q\|^2 E\|(b \otimes I_r)z^{1/2}\|^2 \\ &= \|q\|^2 E\|(b \otimes z^{1/2})\|^2 = \|q\|^2 E\{\mathrm{Tr}\,(b \otimes z^{1/2})(b' \otimes z^{1/2})\} \\ &= \|q\|^2 E\{(\mathrm{Tr}\, bb')(\mathrm{Tr}\, z)\} = \|q\|^2 E\{\|b\|^2 \mathrm{Tr}\, z\} \; . \end{aligned}$$

By Assumption (2.2) it follows that $q'b$ belongs to S_z $\forall q \in M_{\ell\times r}$. In particular, from (2.6) each column vector of $(b' \otimes I_r)$ belongs to S_z. Thus in the space S_z $q'b$ varies over the finite-dimensional subspace G generated by ℓr column vectors of $b' \otimes I_r$. By Assumption (2.1) also $Y \in S_z$. In the Euclidean space S_z the distance $E\|Y - q'b\|^2_z$ is minimized when and only when $q'b$ is the orthogonal projection of Y on G. Since G is a finite-dimensional subspace of the Euclidean space S_z, the orthogonal projector from S_z onto G exists (see the remark on p. 370 in [3]). Thus there exists a unique vector $q'b \in G$ such that $E\|Y - q'b\|_z^2$ is minimized as q varies over $M_{\ell\times r}$ or, equivalently, by virtue of (2.6) there exists some value $q \in M_{\ell\times r}$ such that $Y - q'b \perp G$. This condition is successively written as

$$E\{(Y' - b'q)z(b' \otimes I_r)\} = 0 \ ,$$

$$E\{(b \otimes z)Y - (bb' \otimes z)[q]\} = 0 \ , \tag{2.7}$$

since $(b \otimes I_r)zY = (b \otimes z)Y$ and, by (2.6),

$$(b \otimes I_r)zq'b = (b \otimes z)q'b = (b \otimes z)(b' \otimes I_r)[q] = (bb' \otimes z)[q] \ . \tag{2.8}$$

So the existing value $q \in M_{\ell\times r}$ satisfying (2.7) is a solution to Problem (2.5). Therefore there always exists some value $q = \left(q_1' \cdots q_k'\right)' \in M_{\ell\times r}$ satisfying the condition (2.3), and (2.3) is necessary and sufficient for q to minimize ξ in (2.4). □

The following theorem gives necessary conditions of strong consistency of GLS values.

Theorem 1. *Assume* (2.2). *If, on the basis of i.i.d. observations* $(X(t), Y(t))$, $t = 1, \ldots, n$, *some GLS value* $\hat{q}$ *defined by* (1.7) *tends a.s. to a value* q *as* $n \to \infty$, *then* q *necessarily satisfies the condition* (2.3).

Since the existence of GLS estimate is not supposed, this theorem is more general than the necessity result stated in [2, Remark 5.2]. Here, we also give a direct proof.

Proof. We first prove that Equation (1.7) is equivalent to a much simpler one. As p varies over $M_{\ell\times r}$, $[Bp] = (B \otimes I_r)[p]$ varies over the vector space, which is denoted by $M\{B \otimes I_r\}$ and generated by the columns of the matrix $B \otimes I_r$. Let us define the inner product $[u']Z[v]$ (see(1.4)) for arbitrary elements $[u]$ and $[v]$ in the range space of the rv $[U]$ (see (1.2)). Then $\|U - Bp\|_Z$ is minimized if and only if $[U - Bp]$ is orthogonal to $M\{B \otimes I_r\}$. Thus Equation (1.7) is equivalent to

$$\left[U - B\hat{q}\right]' Z(B \otimes I_r) = 0. \tag{2.9}$$

We now assume (1.6) to ensure that $\hat{q}$ exists. Replace (X, Y) by the t-th observation $(X(t), Y(t))$ in (1.1). After ranking, it follows that, up to some residual,

$$Y_{ij}' = b_i'(X_{ij})q_i, \quad i = 1, \ldots, k, \ j = 1, \ldots, d(i).$$

Then, using (1.2), (1.3) we can write

$$U = Bq + e,$$

where e is some residual. Hence (2.9) can be rewritten as

$$\big[B(\hat{q} - q)\big]' Z(B \otimes I_r) = [e]' Z(B \otimes I_r). \tag{2.10}$$

Consider block diagonal matrices

$$\begin{aligned} A &= \operatorname{diag}\big(d(i) I_{d(i)r}\big), \\ C &= \operatorname{diag}(C_i), \quad (i = 1, \dots, k), \end{aligned}$$

where (see (1.3) and (1.4))

$$C_i = Z_i^{1/2}(B_i \otimes I_r). \tag{2.11}$$

Then

$$\begin{aligned} A^{-1}C' &= C'A^{-1}, \\ C &= Z^{1/2}(B \otimes I_r). \end{aligned}$$

Hence (2.10) is successively equivalent to

$$\begin{aligned} (B' \otimes I_r) Z (B \otimes I_r)[\hat{q} - q] &= (B' \otimes I_r) Z[e] \ , \\ A^{-1}C'C[\hat{q} - q] &= A^{-1}C'Z^{1/2}[e] \ . \end{aligned}$$

By denoting

$$\begin{aligned} T &= C'A^{-1}C \ , \\ g &= C'A^{-1}Z^{1/2}[e] \ , \end{aligned}$$

it follows that, under (1.6), Equation (1.7) is equivalent to the equation

$$T[\hat{q} - q] = g \ .$$

From the latter we have

$$\|g\|^2 \le \|\hat{q} - q\|^2 (\operatorname{Tr} T)^2 \ , \tag{2.12}$$

since $\|T\|^2 = \operatorname{Tr} TT' = \operatorname{Tr} T^2 \le (\operatorname{Tr} T)^2$.

Now, we can write $T = \operatorname{diag}(T_i\,,\ i = 1, \dots, k)$, where

$$\begin{aligned} T_i &= d^{-1}(i) C_i' C_i = d^{-1}(i)(B_i' \otimes I_r) Z_i (B_i \otimes I_r) \\ &= d^{-1}(i) \sum_{j=1}^{d(i)} b_i(X_{ij}) b_i'(X_{ij}) \otimes z(X_{ij}) \end{aligned}$$

by using (1.3) and (1.4). We have $\operatorname{Tr} T = \sum_i \operatorname{Tr} T_i$ with

$$\operatorname{Tr} T_i = d^{-1}(i) \sum_{j=1}^{d(i)} \|b_i(X_{ij})\|^2 \operatorname{Tr} z(X_{ij}).$$

On the basis of the infinite sequence of i.i.d. rvs $X(1), X(2), \dots$ the ranked one $\{X_{i1}, X_{i2}, \dots\}$ is a sequence of a.s. defined, i.i.d. rvs (see [1, Theorem 2]). By Kolmogorov strong law of large numbers, $\operatorname{Tr} T_i$ tends a.s. to $E\|b_i(X_{i1})\|^2 \operatorname{Tr} z(X_{i1})$ as $d(i) \uparrow \infty$ in a non-random manner. Hence $\operatorname{Tr} T_i$ tends a.s. to the same limit as $n \to \infty$, because $d(i) \to \infty$ a.s. as $n \to \infty$. From [1, Theorem 2] this limit equals $E_{\{X \in S(i)\}} \|b_i(X)\|^2 \operatorname{Tr} z(X)$ which is finite by Assumption (2.2). Hence $\operatorname{Tr} T$ tends a.s. to a finite limits as $n \to \infty$.

In the basic probability space, let us now consider the almost sure event

$$\Omega_0 = \big\{ d(i) \underset{n \to \infty}{\longrightarrow} \infty \ \forall i = 1, \dots, k \big\}.$$

At each element ω of Ω_0, as soon as n is sufficiently large we have $d(i) > 0$ $\forall i$. Then every GLS value $\hat{q}$ defined by Equation (1.7) will satisfy (2.12). Thus,

If some GLS value defined by (1.7) *tends a.s. to the parameter value* q *as* $n \to \infty$*, the* $g \ \underset{n \to \infty}{\overset{a.s.}{\longrightarrow}} \ 0$.

Write $U_i = \big(Y_{i1} \dots Y_{id(i)}\big)'$. From (1.2) and (1.3) we have

$$e = U - Bq = \big(e_1' \dots e_k'\big)',$$

where $e_i = U_i - B_i q_i$. Then

$$\begin{aligned} [e_i] &= \big(\dots (Y_{ij}' - b_i'(X_{ij}) q_i) \dots \big)', \quad (j = 1, \dots, d(i)), \\ g = A^{-1} C' Z^{1/2} [e] &= \big\{ \operatorname{diag}(d^{-1}(i) C_i' Z_i^{1/2}) \big\} \big([e_1]' \dots [e_k]' \big)'. \end{aligned}$$

By (2.11), $g = \big(g_1', \dots, g_k'\big)'$, where

$$g_i = d^{-1}(i)(B_i' \otimes I_r)Z_i[e_i], \quad i = 1, \dots, k.$$

Besides, from (1.3) and (1.4)

$$\begin{aligned}(B_i' \otimes I_r)Z_i &= \big(\dots (b_i(X_{ij}) \otimes I_r)z(X_{ij}) \dots\big) \\ &= \big(\dots (b_i(X_{ij}) \otimes z(X_{ij})) \dots\big) \quad (j = 1, \dots, d(i)).\end{aligned}$$

Hence

$$g_i = d^{-1}(i) \sum_{j=1}^{d(i)} \big(b_i(X_{ij}) \otimes z(X_{ij})\big)\big(Y_{ij} - q_i' b_i(X_{ij})\big), \quad (i = 1, \dots, k).$$

By [1, Theorem 2], $\{(X_{ij}, Y_{ij}),\ j = 1, 2, \dots\}$ is a sequence of i.i.d. rvs. Thus, by Kolmogorov strong law of large numbers $g_i \overset{a.s.}{\to} 0$ as $d(i) \to \infty$ in a non-random manner if and only if

$$E\big\{\big(b_i(X_{i1}) \otimes z(X_{i1})\big)\big(Y_{i1} - q_i' b_i(X_{i1})\big)\big\} \text{ exists and vanishes.} \tag{2.13}$$

Now as $n \uparrow \infty$, $d(i)$ tends a.s. to infinity in increasing by the unit, hence $g_i \overset{a.s.}{\to} 0$ as $d(i)$ goes non-randomly to infinity if and only if $g_i \overset{a.s.}{\to}{}_{n\to\infty} 0$ (see also [2, Proposition 5.6]. By [1, Theorem 2(ii)] and by (2.8), Condition (2.13) is expressed equivalently as follows:

$$E_{\{X \in S(i)\}}\left\{(b_i(X) \otimes z(X))\, Y - (b_i(X) b_i'(X) \otimes z(X))\, [q_i]\right\} \tag{2.14}$$

exists and vanishes.

Thus (2.14) is equivalent to $g_i \xrightarrow[n\to\infty]{a.s.} 0$, hence

$$\text{(2.3) is equivalent to } g \xrightarrow[n\to\infty]{a.s.} 0. \tag{2.15}$$

From the strong consistency of any GLS value defined by Equation (1.7), it follows that (2.3) holds.

From Theorem 1 and Lemma we get

Corollary 1. *Assume* (2.1) *and* (2.2). *If on the basis of i.i.d. observations* $(X(t), Y(t))$ *on* (X, Y), $t = 1, \dots, n$, *some GLS value* $\hat{q}$ *defined by*

Equation (1.7) *tends a.s. to* $q = \left(q_1' \dots q_k'\right)'$ *as* $n \to \infty$, *then the function* $\sum_{i=1}^{k} q_i' b_i(X) I_{S(i)}(X)$ *formed with these values* q *is the msq regression of* Y *according to Definition 1.*

Sufficient conditions for strong consistency of GLSE are stated in the following theorem:

Theorem 2. *Assume* (2.2) *and that*

$$\text{the } P_{\{X \in S(i)\}} - \text{distribution of } b_i(X) \text{ is not concentrated} \tag{2.16}$$

$$\text{is any proper subspace of } R^{\ell(i)},\ i = 1, \dots, k.$$

Then, in the representation (1.1) *and on the basis of i.i.d. observations* $(X(t), Y(t))$ *on* (X, Y), *we have*
(i) *in the basic probability spaces* $\Omega = \{\omega\}$,

$P\left\{\omega : \exists n_0(\omega),\ \forall n \geq n_0(\omega),\ \text{there is a unique solution } \hat{q} \text{ to } (1.5)\right\} = 1,$
(ii) *if the parameter value* q *satisfies* (2.3), *then holds*

$$P\left\{\sup_{\mathcal{M}} \|\hat{q} - q\| \to 0 \text{ as } n \to \infty\right\} = 1,$$

where $\sup_{\mathcal{M}}$ *is taken over the set of all affine manifolds* $\mathcal{M}$ *containing the parameter range* Q *and contained in* $M_{\ell \times r}$.

Proof. Part (i) follows from Assumption (2.16) (see [2, Theorem 5.1]). Part (ii) follows from (2.15) and [2, Theorem 5.2]. □

Corollary 2. *Assume* (2.1), (2.2) *and* (2.16). *On the basis of i.i.d. observations* $(X(t), Y(t))$ *on* (X, Y), $t = 1, \dots, n$, *the GLSE* $\hat{q}$ *defined by Equation* (1.7) *for sufficiently large* n *tends a.s. to the parameter value* $q = \left(q_1' \dots q_k'\right)'$ *as* $n \to \infty$ *if and only if the function* $\sum_{i=1}^{k} q_i' b_i(X) I_{S(i)}(X)$ *is the msq regression of* Y *according to Definition 1.*

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Hochiminhcity National University
227 Nguyen Van Cu, District 5, Ho Chi Minh City, Vietnam