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WIDTH OF COMPLEXES OF MODULES

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ABSTRACT. The concept of the width of complex of modules (that is a dual of depth) is introduced and the dual of the generalization of the Auslander-Buchsbaum equality is proved.

INTRODUCTION

The extension of homological algebra from modules to complexes of modules was started already in the last chapter of [CE] and pursued in [H] and [F]. Finiteness of injective and flat dimensions of certain complexes is a key ingredient in [H], while the actual values of these dimensions are studied in [F], where also the concept of *depth* of complexes is introduced, and the formula

$$\operatorname{depth}(X \underset{=}{\otimes} Y) = \operatorname{depth} Y - \sup(k \underset{=}{\otimes} X)$$

is proved for bounded complexes X and Y such that X is of finite flat dimension. Here $\underset{=}{\otimes}$ is the derived of the tensor product functor and $\sup(X)$ is supremum of $\ell \in Z$ such that $H_\ell(X) \neq 0$).

If, in addition, $H_i(X)$ is finite for all $i \in \mathbf{Z}$ then $\sup(k \underset{=}{\otimes} X) = \operatorname{pd} X$, the projective dimension of X. Thus for $Y = R$ (considered as a complex concentrated in degree zero) the above formula reads

$$\operatorname{depth} R = \operatorname{pd} X + \operatorname{depth} X,$$

that is, the Auslander-Buchsbaum equation for complexes of R-modules.

Also in [F], the formula

$$\operatorname{id} Y = \operatorname{depth} X - \inf \underset{=}{\operatorname{Hom}}(X, Y)$$

is proved for bounded complexes X and Y such that $H_i(X)$ and $H_i(Y)$

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are finite for all $i \in Z$ and Y is of finite injective dimension. (Here $\underline{\underline{\mathrm{H}}}\mathrm{om}$ is the derived of the homomorphism functor and $\inf(X)$ is infimum of $\ell \in \mathbf{Z}$ such that $H_\ell(X) \neq 0$). Thus for $X = R$ (consider as a complex concentrated in degree zero) the above formula reads

$$\operatorname{id} Y = \operatorname{depth} R - \inf(Y),$$

that is the Bass's theorem for complexes of R-modules.

In Section 2, for any complex X we introduce width X; this is a dual of $\operatorname{depth} X$. We prove that the dual of the above generalization of the Auslander-Buchsbaum equality, that is, if X is a bounded complex of finite injective dimension then

$$\operatorname{width} \underline{\underline{\mathrm{H}}}\mathrm{om}(Y, X) = \operatorname{depth} Y + \inf(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, X))$$

for all bounded complexes X.

If, in addition, $H_i(X)$ is Artinian for all $i \in \mathbf{Z}$ then $-\inf \underline{\underline{\mathrm{H}}}\mathrm{om}(k, X) = \operatorname{id} X$, the injective dimension of X. Thus for $Y = R$ (considered as a complex concentrated in degree zero) the above formula reads

$$\operatorname{depth} R = \operatorname{id} X + \operatorname{width} X,$$

that is, the dual Auslander-Buchsbaum equation for complexes of R-modules.

Also, for a bounded complex X with finite flat dimension, if $H_i(X)$ is Artinian for all $i \in \mathbf{Z}$ then

$$\operatorname{fd} X = \operatorname{depth} R + \sup(X).$$

That is the dual of Bass's theorem.

Finally, for a bounded complex X with finite injective dimension we show that

$$\operatorname{id} X = \sup_p(\operatorname{depth}_{R_p} R_p - \operatorname{width}_{R_p} X_p).$$

The above equality is proved for modules in [C], and a corresponding formula for finite flat dimension of complexes is proved in [F]. This is actually also a generalized dual version of the Auslander-Buchsbaum equality.

Throughout this paper the ring R is commutative Noetherian with a non-zero identity element.

1. Homological algebra

First we bring some definitions about complexes that we use in the rest of this paper. The reader is referred to [F] for details of the following brief summary of the homological theory of complexes of modules.

A complex X of R-modules is a sequence of R-linear homomorphisms $\{\partial_n : X_n \to X_{n-1}\}_{n\in\mathbf{Z}}$ such that $\partial_n\partial_{n+1} = 0$ for all n. (We only use subscripts and all differentials have degree -1). We set

$$\inf(X) = \inf\{n \in \mathbf{Z} \big| H_n(X) \neq 0\} \quad \text{and}$$
$$\sup(X) = \sup\{n \in \mathbf{Z} \big| H_n(X) \neq 0\}.$$

We identify any module M with a complex of R-modules, which has M in degree zero and is trivial elsewhere.

A homology isomorphism is a morphism $\alpha : X \to Y$ such that $H(\alpha)$ is an isomorphism; homology isomorphisms are marked by the sign $\simeq$, while $\cong$ is used for isomorphisms. The equivalence relation generated by the homology isomorphisms is also denoted by $\simeq$.

The *derived category* of the category of modules over R, cf. [H], is denoted by $\mathcal{C}$.

The full subcategory of $\mathcal{C}$ consisting of complexes with finite homology modules is denoted by $\mathcal{C}^f$, and we write $\mathcal{C}_+$, $\mathcal{C}_-$, $\mathcal{C}_b$, $\mathcal{C}_0$, for the full subcategories defined by $H_n(X) = 0$ for, respectively, $n \ll 0$, $n \gg 0$, $|n| \gg 0$, $n \neq 0$.

The left derived functor of the tensor product functor of R-complexes is denoted by $-\underset{=R}{\otimes} -$, and the right derived functor of the homomorphism functor of complexes of the R-modules is denoted by $\underset{=}{\mathrm{H}}\mathrm{om}_R(-,-)$. Thus, for arbitrary $X, Y \in \mathcal{C}$ there are complexes $X \underset{=R}{\otimes} Y$ and $\underset{=}{\mathrm{H}}\mathrm{om}_R(X, Y)$ which are defined uniquely up to isomorphism in $\mathcal{C}$, and possess the expected functorial properties.

Familiar invariants of R-modules have been extended to complexes in several non-equivalent ways. We use the notions introduced in [F].

The *(Krull) dimension* of an R-complex is defined in terms of the (Krull) dimensions of its homology modules by the formula:

$$\dim_R(X) = \sup\{\dim_R(H_i(X) - i \big| i \in \mathbf{Z}\},$$

with the convention that the dimension of the zero module is equal to $-\infty$. The *depth* of an R-complex X is defined by the formula

$$\mathrm{depth}_R(X) = -\sup \underset{=}{\mathrm{H}}\mathrm{om}_R(k, X),$$

hence $-\infty \leq \operatorname{depth}_R X \leq \infty$. In case X is an R-module the notions of dimension and depth coincide with the standard ones.

For any $X \in \mathcal{C}_b$ we set:

$$\operatorname{id}_R X = \sup\Big\{ -\inf\big(\underline{\underline{\operatorname{H}}}\mathrm{om}_R(M,X)\big)\big| M \in \mathcal{C}_0^f \Big\},$$
$$\operatorname{fd}_R X = \sup\Big\{ \sup\big(X \underline{\underline{\otimes}}_R M)\big)\big| M \in \mathcal{C}_0^f \Big\}.$$

We call $\operatorname{id}_R X$ (resp. $\operatorname{fd}_R X$) the injective (resp. flat) dimension of X over R.

2. Width of complexes of modules

In this section we introduce the notion of width of complexes. This is a dual notion to the depth of complexes.

Definition 2.1. If $X \in \mathcal{C}_+$ the *width* of X is defined by

$$\operatorname{width} X = \inf\big(k \underline{\underline{\otimes}} X\big).$$

Here and always when the word "width" is mentioned in the future, the ring $(R, \mathbf{m})$ is supposed to be local and $k = R/\mathbf{m}$.

We use the notation $\mathcal{C}^{art}$ for the class of all complexes X such that $H_\ell(X)$ is an Artinian R-module for all $\ell \in \mathbf{Z}$.

Lemma 2.2. *Let $(R, \mathbf{m})$ be local and let $-^\vee = \operatorname{Hom}(-, E(R/\mathbf{m}))$ be the Matlis duality. Then for $X \in \mathcal{C}_+$ the following holds:*

$$\operatorname{depth} X^\vee = \operatorname{width} X.$$

Proof. We use the definition of depth and width

$$\begin{aligned}
\operatorname{depth} X^\vee &= -\sup\big(\underline{\underline{\operatorname{H}}}\mathrm{om}(k, X^\vee)\big) \\
&= -\sup\big(k \underline{\underline{\otimes}} X\big)^\vee \qquad [F, 5.2] \\
&= \inf(k \underline{\underline{\otimes}} X) \\
&= \operatorname{width} X. \qquad \square
\end{aligned}$$

Lemma 2.3. *If $(R, \mathbf{m})$ is a local ring, then the following are equivalent for $X \in \mathcal{C}_b$.*

(i) $\mathbf{m} \in \operatorname{supp}(X)$, *in other words* $X \underline{\underline{\otimes}} k \not\simeq 0$.

(ii) depth $X < \infty$.

(iii) width $X < \infty$.

Proof. It follows by [F, 6.3]. □

The first part of the next result is more general than [F, 6.5].

Theorem 2.4. *For* $X \in \mathcal{C}_+$ *we have*

(a) $\operatorname{depth} \underline{\underline{\mathrm{H}}}\mathrm{om}(X,Y) = \operatorname{width} X + \operatorname{depth} Y$ *for* $Y \in \mathcal{C}_-$.

(b) $\operatorname{width}\big(X \underline{\underline{\otimes}} Y\big) = \operatorname{width} X + \operatorname{width} Y$ *for* $Y \in \mathcal{C}_+$.

Proof. (a): We have

$$\begin{aligned}\operatorname{depth} \underline{\underline{\mathrm{H}}}\mathrm{om}(X,Y) &= -\sup\big(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, \underline{\underline{\mathrm{H}}}\mathrm{om}(X,Y))\big)\\ &= -\sup\big(\underline{\underline{\mathrm{H}}}\mathrm{om}(k,Y)\big) + \inf(k \underline{\underline{\otimes}} X) \quad [F, 5.9]\\ &= \operatorname{depth} Y + \operatorname{width} X.\end{aligned}$$

(b): We have

$$\begin{aligned}\operatorname{width}\big(X \underline{\underline{\otimes}} Y\big) &= \operatorname{depth}(X \underline{\underline{\otimes}} Y)^{\vee} && (2.2)\\ &= \operatorname{depth}\big(\underline{\underline{\mathrm{H}}}\mathrm{om}(X, Y^{\vee})\big) && [F, 5.2]\\ &= \operatorname{depth} Y^{\vee} + \operatorname{width} X && (a)\\ &= \operatorname{width} Y + \operatorname{width} X. && (2.2) \quad \square\end{aligned}$$

Lemma 2.5. *If* $X \in \mathcal{C}_+$ *is non-trivial then*

$$\operatorname{width} X \geq \inf(X)$$

and the equality holds if and only if $\mathbf{m} \in \operatorname{Coass} \mathrm{H}_{-\inf(x)}(X)$.

Proof. We have

$$\begin{aligned}\operatorname{width} X &= \operatorname{depth} X^{\vee}\\ &\geq -\sup(X^{\vee})\\ &= \inf(X).\end{aligned}$$

Since $\mathrm{H}_{\sup(X^{\vee})}(X^{\vee}) \cong \big(\mathrm{H}_{\inf(X)}(X)\big)^{\vee}$, we have the equality

$\operatorname{Ass} \mathrm{H}_{\sup(X^\vee)}(X^\vee) = \operatorname{Coass} \mathrm{H}_{\inf(X)}(X)$ (see [Y, 1.7]. Now the assertion follows by [F, 6.6]. □

For $X \in C_b$ with $\operatorname{fd} X < \infty$ we have

$$\operatorname{depth}(X \underset{=}{\otimes} Y) = -\sup(k \underset{=}{\otimes} X) + \operatorname{depth} Y \quad \text{for } Y \in C_b \quad \text{[F, 6.37]}.$$

As mentioned in the introduction this is a generalization of the Auslander-Buchsbaum equality for complexes of modules.

Now we provide the dual of the above results.

Lemma 2.6. *If X, $Y \in C_b$ with $\operatorname{id} Y < \infty$, then the following hold:*
(a) width $\underset{=}{\mathrm{H}}\mathrm{om}(X,Y) = \operatorname{depth} X + \inf(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y))$,
(b) width $Y = \operatorname{depth} R + \inf(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y))$.
(c) width $\underset{=}{\mathrm{H}}\mathrm{om}(X,Y) = \operatorname{width} Y + \operatorname{depth} X - \operatorname{depth} R$.

Proof. (a): We have

$$\begin{aligned}\operatorname{width} \underset{=}{\mathrm{H}}\mathrm{om}(X,Y) &= \inf(k \underset{=}{\otimes} \underset{=}{\mathrm{H}}\mathrm{om}(X,Y)) \\ &= -\sup\big(\underset{=}{\mathrm{H}}\mathrm{om}(k,X)\big) + \inf\big(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y)\big) \quad \text{[F, 5.11]} \\ &= \operatorname{depth} X + \inf\big(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y)\big).\end{aligned}$$

(b) is (a) with $X = R$. Also (c) is (a) combined with (b) □

Corollary 2.7. *If X, $Y \in C_b$ and $\operatorname{id} Y < \infty$, then*

$$\operatorname{width} \underset{=}{\mathrm{H}}\mathrm{om}(X,Y) = \operatorname{depth}\big(X \underset{=}{\otimes} Y^\vee\big).$$

Proof. We have

$$\text{(2.6a)} \qquad \operatorname{width} \underset{=}{\mathrm{H}}\mathrm{om}(X,Y) = \operatorname{depth} X + \inf\big(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y)\big).$$

Since $\operatorname{fd} Y^\vee$ is finite, we have

$$\operatorname{depth}(X \underset{=}{\otimes} Y^\vee) = \operatorname{depth} X - \sup\big(k \underset{=}{\otimes} Y^\vee\big) \quad \text{[F, 6.37]}.$$

The assertion follows from the equality

$$\inf\big(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y)\big) = -\sup\big(\underset{=}{\mathrm{H}}\mathrm{om}(k,Y)\big)^\vee = -\sup\big(k \underset{=}{\otimes} Y^\vee\big). \qquad \square$$

Theorem 2.8. *If $Y \in \mathcal{C}_b^{art}$ has finite injective dimension, then*

$$\operatorname{width} \underset{=}{\mathrm{H}}\mathrm{om}(X, Y) = \operatorname{depth} X - \operatorname{id} Y.$$

In particular, $\operatorname{depth} R = \operatorname{id} Y + \operatorname{width} Y$.

Proof. Since $Y \in \mathcal{C}_b^{art}$, we have $\operatorname{Supp} Y = \{\mathbf{m}\}$ by [Y, 2.10]. Since

$$\operatorname{id} Y = \sup \Big\{ -i \big| \mu^i(\mathbf{p}, Y) \neq 0 \quad \text{for some } \mathbf{p} \in \operatorname{Supp} Y \Big\}$$

by [F, 6.22], we have that $\operatorname{id} Y = -\inf \big(\underset{=}{\mathrm{H}}\mathrm{om}(k, Y)\big)$. Now the assertion follows from (2.6). □

For $X \in \mathcal{C}_b^f$ with $\operatorname{id} X < \infty$ we have

$$\operatorname{depth} R - \inf (X) = \operatorname{id} X$$

by [F, 6.29]. This is the Bass Theorem, cf. [B, 3.3], for complexes of modules. Now we give the dual of this result.

Theorem 2.9. *For $X \in \mathcal{C}_b^{art}$ with $\operatorname{fd} X < \infty$ we have*

$$\operatorname{depth} R + \sup(X) = \operatorname{fd} X.$$

Proof. It follows from [Y, 2.10] that $\operatorname{Supp} X = \{\mathbf{m}\}$ so we have $\operatorname{depth} X = -\sup(X)$ by [F, 6.6]. Since

$$\operatorname{fd} X = \sup \Big\{ n \big| \beta_n^{A_p}(X_p) \neq 0 \text{ for some } \mathbf{p} \in \operatorname{Supp} X \Big\}$$

by [F, 6.34], we have that $\operatorname{fd} X = \sup(k \underset{=}{\otimes} X)$. Now the assertion follows from $\operatorname{depth} X = \operatorname{depth} A - \sup(k \underset{=}{\otimes} X)$, which is Auslander and Buchsbaum equality, cf. [F, 6.37]. □

For an R-module M with finite flat dimension

$$\operatorname{fd} M = \sup_{\mathbf{p} \in \operatorname{Spec} R} \big(\operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{depth}_{R_\mathbf{p}} M_\mathbf{p}\big) \quad [\text{C}, 1.2]$$

and for an R-module M with finite injective dimension

$$\operatorname{id} M = \sup_{\mathbf{p} \in \operatorname{Spec} R} \big(\operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{width}_{R_\mathbf{p}} M_\mathbf{p}\big) \quad [\text{C}, 3.1].$$

Also for $X \in \mathcal{C}_b$ with $\operatorname{fd} X < \infty$ we have

$$\operatorname{fd} X = \sup_{\mathbf{p} \in \operatorname{Spec} R} \big(\operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{depth}_{R_\mathbf{p}} X_\mathbf{p}\big) \quad [F, 6.39].$$

Now we provide the dual of this result.

Theorem 2.10. *If $X \in \mathcal{C}_b$ has $\operatorname{id} X < \infty$ then*

$$\operatorname{id} X = \sup_{\mathbf{p} \in \operatorname{Spec} R} \big(\operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{width}_{R_\mathbf{p}} X_\mathbf{p}\big).$$

Proof. "$\geq$" We have

$$\begin{aligned} \text{(2.6)} \qquad \operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{width}_{R_\mathbf{p}} X_\mathbf{p} &= -\inf\big(\underline{\underline{\mathrm{H}}}\mathrm{om}(k(\mathbf{p}), X_\mathbf{p})\big) \\ &\leq \operatorname{id}_{R_\mathbf{p}} X_\mathbf{p} \qquad [F, 6.22] \\ &\leq \operatorname{id} X \qquad [F, 6.23] \end{aligned}$$

"$\leq$" If $\ell \leq \operatorname{id} X$ then there exists $n \in \mathbf{Z}$ and $\mathbf{p} \in \operatorname{Spec} R$ such that $\ell \leq n$ and $\mu^n(\mathbf{p}, X) \neq 0$ by [F, 6.22]. Thus we have

$$\begin{aligned} n &\leq -\inf\big(\underline{\underline{\mathrm{H}}}\mathrm{om}(k(\mathbf{p}), X_\mathbf{p})\big) \qquad [F, 3.16] \\ \text{(2.6)} \qquad &= \operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{width}_{R_\mathbf{p}} X_\mathbf{p}. \end{aligned}$$

Therefore $\ell \leq n \leq \operatorname{depth}_{R_\mathbf{p}} R_\mathbf{p} - \operatorname{width}_{R_\mathbf{p}} X_\mathbf{p}$. □

If the ring R is local and has a dualizing complex D, cf. [F, 8.1], then for $X \in \mathcal{C}_+^f$ Foxby introduced the dual with respect to D by $X^\dagger = \underline{\underline{\mathrm{H}}}\mathrm{om}(X, D)$, and he proved some results showing the duality between X and $X^\dagger$. Now we use this notion to find relations between the depth and the width of X and $X^\dagger$.

Theorem 2.11. *Let $\dim R = d$ and $X \in \mathcal{C}_b$, then*
(a) width $X^\dagger =$ depth $X - d$.
(b) depth $X^\dagger =$ width $X + d$.

Proof. (a): We have

$$\begin{aligned} \operatorname{width} X^\dagger &= \inf(k \underline{\underline{\otimes}} X^\dagger) \\ &= -\sup\big(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, X) + \inf(k^\dagger) \quad [F, 5.11] \\ &= \operatorname{depth} X - d \qquad [F, 8.12]. \end{aligned}$$

(b): We have

$$\begin{aligned}
\operatorname{depth} X^\dagger &= -\sup\left(\underline{\underline{\mathrm{H}}}\mathrm{om}(k, X^\dagger)\right) \\
&= -\sup(k^\dagger) + \inf(k \underset{=}{\otimes} X) \quad [\mathrm{F}, 5.9] \\
&= d + \operatorname{width} X \quad [\mathrm{F}, 8.12]. \quad \square
\end{aligned}$$

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