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FINITE PROPER HOLOMORPHIC SURJECTIONS AND LINEAR TOPOLOGICAL INVARIANTS $(\overline{\Omega})$ AND $(\widetilde{\Omega})$

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Abstract. It is shown that linear topological properties $(\overline{\Omega})$, $(\widetilde{\Omega})$ and isomorphisms of spaces of holomorphic functions are invariant under finite proper holomorphic surjections between Stein spaces.

Introduction

Let E be a Frechet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}$. For each subset B of E we define a semi-norm $\|\cdot\|_B^*$ on E^*, the strongly dual space of E, with values in $[0,+\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}.$$

We write

$$\|\cdot\|_p^* = \|\cdot\|_{U_p}^*, \quad \text{with} \quad U_p = \{x \in E : \|x\|_p \le 1\}.$$

We say that E has the property (P) and write $E \in (P)$ if (P) holds, where (P) is one of the following conditions:

(Ω) $\forall p\ \exists q\ \forall k\ \exists d > 0, c > 0\ s.t.\ \|\cdot\|_q^{*d+1} \le c\|\cdot\|_k^*\ \|\cdot\|_p^{*d}$,

$(\widetilde{\Omega})$ $\forall p\ \exists q, d > 0\ \forall k\ \exists c > 0\ s.t.\ \|\cdot\|_q^{*d+1} \le c\|\cdot\|_k^*\ \|\cdot\|_p^{*d}$,

$(\overline{\Omega})$ $\exists d > 0\ \forall p\ \exists q\ \forall k\ \exists c > 0\ s.t.\ \|\cdot\|_q^{*d+1} \le c\|\cdot\|_k^*\ \|\cdot\|_p^{*d}$,

$(\overline{\overline{\Omega}})$ $\forall p\ \exists q\ \forall k, d > 0\ \exists c > 0\ s.t.\ \|\cdot\|_q^{*d+1} \le c\|\cdot\|_k^*\ \|\cdot\|_p^{*d}$,

$(\underline{DN})$ $\exists p\ \forall q\ \exists k, d > 0, c > 0\ s.t.\ \|\cdot\|_q^{1+d} \le c\|\cdot\|_k\ \|\cdot\|_p^{d}$.

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The above properties were introduced and investigated by D. Vogt (see [8, 9, 10, 11]). For a complex space X, we denote by $H(X)$ the space of holomorphic functions on X equipped with the compact-open topology. The main results of the present paper are the following theorems.

Theorem 1. *Let $\theta : X \to Y$ be a finite proper holomorphic surjection between Stein spaces. Then $H(X) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$) if and only if $H(Y) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$).*

Theorem 2. *Let $\theta : X \to Y$ be a finite proper holomorphic surjection between Stein spaces. Then $H(X) \cong H(D^{\dim X})$ if and only if so is $H(Y)$ where D denotes the open unit disc in C.*

The proofs of theorems 1 and 2 are given Section 1 and Section 2, respectively.

1. Proof of theorem 1

Lemma 1.1. *Let X be a Stein space with $H(X) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$). Then $H^0(X, S) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$) for an arbitrary coherent sheaf S on X.*

Proof. Give a coherent sheaf S on X. Let $\{K_p\}$ be an increasing exhaustion sequence of compact subsets of X. By Cartan Theorem A in [2], for each $x \in X$ there exists a neighbourhood U_x of x and $\sigma_{1x}, \dots, \sigma_{mx} \in H^0(X, S)$ which generate S_y for every $y \in U_x$. Thus, by compactness of K_p, there exists a sequence $\{\sigma_n\} \subset H^0(X, S)$ such that $\{\sigma_{nx}\}$ generates S_x for every $x \in X$. Since $H^0(X, S)$ is Frechet, we can assume that $\{\sigma_n\}$ is bounded in $H^0(X, S)$. Consider the Banach coherent sheaf $\mathcal{H}_X^{\ell_1}$ of germs of holomorphic functions on X with values in ℓ_1 and the morphism $\eta : \mathcal{H}_X^{\ell_1} \to S$ given by

$$\eta(f)(x) = \sum_{n\geq 1} \sigma_n(x) f_n(x),$$

for $f = (f_n) \in \mathcal{H}_X^{\ell_1}$.

By choice of σ_n, we infer that η is surjection. By [6], $\ker\eta$ is a Banach coherent sheaf. Hence

$$H^1(X, \ker\eta) = 0.$$

It follows that the map $\hat{\eta} : H^0(X, \mathcal{H}_X^{\ell_1}) = H(X, \ell_1) \to H^0(X, S)$ is surjective. From the hypothesis, $H(X) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$), it is easy to see that $H(X, \ell_1) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$) which implies $H^0(X, S) \in (\overline{\Omega})$ (resp, $(\widetilde{\Omega})$).

Corollary 1.2. *Let X be a Stein space. Then $H(X) \in (\Omega)$.*

Proof. By Remmert Theorem, there exists a proper injective holomorphic map γ from X into C^n for some n. By $\mathcal{H}_X$ we denote the structure sheaf of X, and by $\gamma * \mathcal{H}_X$ the direct image of $\mathcal{H}_X$ under γ. Since $H(C^n) \in (\Omega)$ and since $\gamma * \mathcal{H}_X$ is coherent (see [2]), as in the proof of Lemma 1.1, we can show

$$H(X) = H^0(C^n, \gamma * \mathcal{H}_X) \in (\Omega).$$

Lemma 1.3. *Let* $E = \lim \operatorname{proj}(E_n, w_{nm})$ *be a projective limit of Frechet spaces* E_n *such that the canonical maps* $w_{nm} : E_n \to E_m$ *are surjective. Assume that* $E_n \in (\overline{\Omega})$ *(resp.* $(\widetilde{\Omega})$ *or* (Ω)*) for some* $n \geq 1$*. Then* $E \in (\overline{\Omega})$ *(resp.* $(\widetilde{\Omega})$ *or* (Ω)*).*

Proof. Consider only the case $E_n \in (\overline{\Omega})$ for some $n \geq 1$, because the other cases are proved similarly. Given a neighbourhood U of $0 \in E$. Take n such that U is a neighbourhood of $0 \in E_n$. Since $E_n \in (\overline{\Omega})$, there exists a neighbourhood V of $0 \in E_n$ such that for every neighbourhood W of $0 \in E_n$ one can find a number $c > 0$ satisfying

$$\| \cdot \|_V^{*2} \leq c \| \cdot \|_W^* \; \| \cdot \|_U^* \quad \text{on} \quad E_n^*.$$

Since W is also a neighbourhood of $0 \in E$ and since the above inequality obviously holds on $E^* \setminus E_n^*$, we infer that $E \in (\overline{\Omega})$.

Now we can prove Theorem 1 as follows.

Proof of Theorem 1. For each $n \geq 1$ suppose that X_n (resp. Y_n) is the union of irreducible branches of X (resp. Y) of dimension $\leq n$. Then

$$H(X) = \lim \ \operatorname{proj}\,(H(X_n), R_{nm}),$$

and

$$H(Y) = \lim \operatorname{proj}\,(H(Y_n), R_{nm}),$$

where R_{nm} are the restriction maps. By [2], R_{nm} are surjective. Hence, by Lemma 1.3, it suffices to prove the theorem for $\theta : X_n \to Y_n$ which is also surjective. Thus, without loss of generality we may assume that $\dim X < \infty$ and $\dim Y < \infty$. We will only prove the theorem for $(\overline{\Omega})$, because the proof is the same for $(\widetilde{\Omega})$.

a) Let $H(X) \in (\overline{\Omega})$.

(i) First assume that Y is normal. By the integrality lemma in [2], θ is a branched covering map. Let p be the branched order of θ and $f \in H(X)$. Then the form

$$(Pf)(y) = \frac{1}{p} \sum_{\theta(x) = y} m_x f(x)$$

defines a continuous linear map from $H(X)$ onto $H(Y)$, where for every $x \in X$ m_x denotes the branched order of θ at x. Hence $H(Y) \in (\overline{\Omega})$.

(ii) Let $\gamma : \tilde{Y} \to Y$ be the normalization of Y. Put

$$\tilde{J} = \{h \in H(\tilde{Y}) : h\big|_{\gamma^{-1}(SY)} = 0\},$$

where SY denotes the singular locus of Y.

By Cartan Theorem, the sequence

$$0 \to \tilde{J} \to H(\tilde{Y}) \to H(\gamma^{-1}(SY)) \to 0$$

is exact.

Assume that $H(\tilde{Y}) \in (\overline{\Omega})$. By Lemma 1.1, $\tilde{J} \in (\overline{\Omega})$ and by the inductive hypothesis on dimension, we have

$$H(SY) \in (\overline{\Omega}).$$

Consider the coherent sheaf $\mathcal{R}$ on Y given by

$$\mathcal{R}_y = \Big\{\varphi \in \mathcal{H}_{Y,y} : \varphi\big(\gamma * \mathcal{H}_{\tilde{Y}}\big)_y \subseteq \mathcal{H}_{Y,y}\Big\}.$$

Then $\mathcal{R}_y \neq 0$ for $y \in Y$. By Cartan Theorem A (see [2]), $H^0(Y, \mathcal{R}) \neq 0$. Moreover, there exists a $\varphi \in \mathcal{R}$ such that $\varphi \neq 0$ on every irreducible branch of Y. Indeed, let $Y = \bigcup\limits_{i \geq 1} Y_i$ be the irreducible decomposition of Y. For each $i \geq 1$, put

$$\begin{aligned} G_i &= \Big\{\varphi \in H^0(Y, \mathcal{R}) : \varphi\big|_{Y_i} \neq 0\Big\} \\ &= H^0(Y, \mathcal{R}) \setminus \Big\{\varphi \in H^0(Y, \mathcal{R}) : \varphi\big|_{Y_i} = 0\Big\}. \end{aligned}$$

Thus G_i is open. We prove that G_i is dense in $H^0(Y, \mathcal{R})$ for $i \geq 1$. For each $i \geq 1$, take $y_i \in \mathcal{R}(Y_i)$, the regular locus of Y_i. Since $1_{y_i} \in \mathcal{R}_{y_i}$, by Cartan Theorem A, there exist $g_1, \dots, g_m \in H^0(Y, \mathcal{R})$ and $\sigma_1, \dots, \sigma_m \in \mathcal{H}_{Y,y_i}$ such that

$$\sum_{1 \leq j \leq m} g_{j,y_i} \sigma_{j,y_i} = 1_{y_i}.$$

Hence, there exists j_0 such that $g_{j_0} \in G_i$. Thus, $G_i \neq \emptyset$ for $i \geq 1$, which implies that G_i is dense in $H^0(Y, \mathcal{R})$ for $i \geq 1$. By Baire Theorem, there exists $\varphi \in \bigcap_{i\geq 1} G_i$. Thus we can choose $\varphi \in H(Y)$, $\varphi \neq 0$ on every irreducible branch of Y such that

$$\varphi H(\tilde{Y}) \subseteq H(Y).$$

Then

$$\varphi H(\tilde{Y}) \cong H(\tilde{Y}) \in (\overline{\Omega}).$$

Now consider the exact sequence

$$0 \to \varphi H(\tilde{Y}) \to H(Y) \to H(Y)/\varphi H(\tilde{Y}) \to 0.$$

Observe that

$$H(Y)/\varphi H(\tilde{Y}) \cong H^0(Y, \mathcal{H}_Y/\varphi\gamma * \mathcal{H}_{\tilde{Y}}),$$

and

$$\operatorname{supp}\Big(\mathcal{H}_Y/\varphi\gamma * \mathcal{H}_{\tilde{Y}}\Big) \subseteq Z(\varphi) := \Big\{y \in Y : \varphi(y) = 0\Big\}.$$

Hence $\mathcal{H}_Y/\varphi\gamma * \mathcal{H}_{\tilde{Y}}$ can be considered as a coherent $\mathcal{H}_{Z(\varphi)}$ sheaf, By Lemma 1.1 we have

$$H(Y)/\varphi H(\tilde{Y}) \in (\overline{\Omega}).$$

Then by D. Vogt [9], we get $H(Y) \in (\overline{\Omega})$.

(iii) Finally consider the following commutative diagram of finite proper holomorphic surjection of Stein spaces

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{\theta}} & \tilde{Y} \\ {\scriptstyle\tilde{\gamma}}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{\theta} & Y \end{array}$$

where $Z = X \times_Y \tilde{Y}$ is the fiber product of X and $\tilde{Y}$, and $\tilde{\theta}$, $\tilde{\gamma}$ are the canonical projections. Since $H(X) \in (\overline{\Omega})$ by Lemma 1.1,

$$H(Z) = H^0(X, \gamma * \mathcal{H}_Z) \in (\widetilde{\Omega}).$$

By (i) $H(\tilde{Y}) \in (\overline{\Omega})$. Hence, by (ii) we have

$$H(Y) \in (\overline{\Omega}).$$

b) Now we assume that $H(Y) \in (\overline{\Omega})$. By Lemma 1.1,

$$H(X) = H^0(Y, \theta * \mathcal{H}_X) \in (\overline{\Omega}).$$

Corollary 1.4. *Let X be a Stein space. Then $H(X) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$) if and only if $H(Z) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$) for every irreducible branch Z of X.*

Proof. By Cartan Theorem B it suffices to prove the sufficiency of the corollary. Consider the normalization $\gamma : \tilde{X} \to X$ of X. By Theorem 1, $H(\tilde{Z}) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$) for every irreducible branch $\tilde{Z}$ of $\tilde{X}$. Since

$$H(\tilde{X}) = \prod \Big\{ H(\tilde{Z}) : \tilde{Z} \text{ is irreducible in } \tilde{X} \Big\},$$

we have $H(\tilde{X})$ and hence, $H(X) \in (\overline{\Omega})$ (resp. $(\widetilde{\Omega})$.

2. Proof of Theorem 2

Lemma 2.1. *Let X be a complex space. Then $H(X) \in (\underline{DN})$ if X has a finite number of irreducible branches.*

Proof. Consider the Hironaka singular resolution $\gamma : \hat{X} \to X$ of X. By the hypothesis, $\hat{X}$ has a finite number of connected components. By [11], $H(\hat{X})$ and hence $H(X)$, have the property $(\underline{DN})$.

Corollary 2.2. *Let X be an irreducible complex space. Then $H(X) \in (\overline{\overline{\Omega}})$ if and only if* dim $H(X) = 1$.

Proof. Sufficiency is trivial. Assume now that $H(X) \in (\overline{\overline{\Omega}})$. By Lemma 2.1, $H(X) \in (\underline{DN})$. Hence, it is isomorphic to a subspace of the space $\Lambda_1(\alpha)$ for some exponent sequence α (see [9]). Then, by applying Vogt's result in [10] to the embedding $H(X) \to \Lambda_1(\alpha)$, we get dim $H(X) < \infty$. Since X is irreducible, dim $H(X) = 1$.

Now we can prove Theorem 2 as follows.

Proof of Theorem 2. Assume that $H(X)$ is isomorphic to $H(D^d)$, where $d = \dim X$. Since $H(D^d) \in (\overline{\Omega})$, by Theorem 1, $H(Y) \in (\overline{\Omega})$. Since $H(Y)$ is contained in $H(X)$ as a subspace of $H(X)$, as in [9], we have

$$H(Y) \cong H(D^d).$$

Conversely, assume that $H(Y) \cong H(D^d) \in (\overline{\Omega})$, $d = \dim Y$. By Theorem 1, $H(X) \in (\overline{\Omega})$. On the other hand, by Lemma 2.1, $H(X) \in (\underline{DN})$ and hence, it is isomorphic to a subspace of $\Lambda_1(n^{1/d}) \cong H(D^d)$. Hence, by [9], we get $H(X) \cong H(D^d)$.

Remark. It is shown in [1] and [4] that the space $H(X)$ of holomorphic functions on a Stein manifold X is isomorphic to the space $H(D^{\dim X})$

if and only if $H(X) \in (\overline{\Omega})$ or equivalently, X is hyperconvex, where D denotes the open unit disc in C. Thus, in the non-singular case Theorem 2 and the invariance of $(\overline{\Omega})$ is an immediate consequence of the invariance of hyperconvexity under finite proper holomorphic surjection.

Moreover, by the same argument as in [13], we can prove that if given X a Stein space such that $H(X)$ is isomorphic to $H(D^d)$, then X is hyperconvex. We do not know whether the conserve statement is true.

By considering the normalization of a complex space and by Theorem 1 and 2, we get a partial answer to the above mentioned question.

Corollary 2.3. *Let X be a Stein space of dimension 1. Then the following are equivalent.*

(i) $H(X) \cong H(D)$,

(ii) $H(X) \in (\overline{\Omega})$,

(iii) X *is hyperconvex.*

References

1. A. Aytuna, *On Stein manifolds M for which $O(M)$ is isomorphic to $O(\Delta^n)$ as Frechet space*, Manuscripta Math. **62** (1988), 297-315.
2. G. Fischer, *Complex Analytic Geometry*, Lecture Notes in Math. **538**, Springer-Verlag, 1976.
3. E. Fornaess and R. Narasimhan, *The Levi problem on complex spaces with singularities*, Math. Ann. **248** (1980), 47-72.
4. N. V. Khue and L. V. Thanh, *On invariance of q-convexity*, Trans. Amer. Math. Soc. **302** (1987), 45-54.
5. M. Klimek, *Pluripotential Theory*, Oxford, 1991.
6. J. Leiterer, *Banach coherent Frechet sheaves*, Math. Narchr. **85** (1978), 91-108.
7. N. Sibony, *Prolongement des fonctions holomorphes bornées et metrique de Caratheodory*, Invent. Math. **29** (1975), 205-230.
8. D. Vogt, *On two classes of (F)-spaces*, Arch. Math. **45** (1985), 255-266.
9. D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. **37** (1982), 269-301.
10. D. Vogt, *Frécheträume zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. **345** (1983), 182-200.
11. D. Vogt, *Charakterisierung der Unteräume eines nuklearen stabilen Potenzreihenräumes won endlichem Typ*, Studia Math. **71** (1982), 251-270.
12. D. Vogt, *On the functor $Ext^1(E,F)$*, Studia Math. **85** (1987), 163-197.
13. V. D. Zaharjuta, *Isomorphism of spaces of analytic functions*, Dokl. Akad. Nauk SSSR **155** (1980), 11-14 (in Russian).

14. A. Zeriahi, *Fonctions de Green pluricomplex a pole a l'infinite sur un espace de Stein parabolique et applications*, Scand. Math. **69** (1990), 98-126.

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