ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 1, 1998, pp. 65–74

REGULARIZED SOLUTIONS OF A CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN AN IRREGULAR LAYER: A THREE DIMENSIONAL MODEL

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1. Introduction

Consider the problem of finding a function u, harmonic in the domain D defined by

$$D = \{(x, y, z) \ : \ -\infty < x, y < \infty \ , \ 0 < z < \phi(x, y)\}$$

and continuous on $\overline{D}$, given u, u_x, u_y and u_z on the portion of the boundary represented by the surface $z = \phi(x, y)$. Here ϕ is of class C^1.

This is a Cauchy problem for the Laplace equation and is well known as an ill-posed problem, i.e., solutions of the problem do not always exist and, whenever they do exist, there is no continuous dependence on the given data. The reader is referred to [1, 2, 3, 4, 6, 7, 9, 10] on the earlier literature on the Cauchy problem for the Laplace equation.

For numerical computations, ill-posed problems need to be regularized. A regularized solution is a stable approximate solution. An important question arises as to how close a regularized solution is to an exact solution, especially when the measured data is affected with noise. The problem of regularisation of the Cauchy problem for the Laplace equation in a rather general context was considered, e.g., in [5]; using the method of quasi-reversibility, the authors (loc. cit.) stabilized the problem, but no error estimates are given. We shall take the approach followed in [1] by taking the boundary value $v(x, y) = u(x, y, 0)$ as our unknown and we shall show that if the discrepancy between the given values of u, u_x, u_y, u_z on the surface $z = \phi(x, y)$ and their exact values is of the order ε, then, assuming

Received April 4, 1996
1991 Mathematics Subject Classification. 35R30, 35J05
Key words and phrases. Regularized solution Cauchy problem Laplace equation 3-D layer.
The work of the first named author was completed with financial support from the National Program of Basic Research in the Natural Sciences of Viet Nam

the exact solution $v_0(x,y)$ to be smooth (in $H^1(R^2)$), the discrepancy between the regularized solution and the exact solution $v_0(x,y)$ is of the order $(ln\frac{1}{\varepsilon})^{-1}$ as $\varepsilon \to 0$.

2. Integral equation formulation and regularization

First, we set some notations:

$$\begin{aligned} u_x(x,y,\phi(x,y)) &= f(x,y) \\ u_y(x,y,\phi(x,y)) &= g(x,y) \\ u_z(x,y,\phi(x,y)) &= h(x,y) \\ u(x,y,\phi(x,y)) &= u_1(x,y) \end{aligned} \tag{1}$$

These functions, we recall, are given. Let us put

$$\begin{aligned} \Gamma(x,y,z;\xi,\eta,\zeta) &= \frac{1}{4\pi} \cdot \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} , \\ G(x,y,z;\xi,\eta,\zeta) &= \Gamma(x,y,z;\xi,\eta,\zeta) - \Gamma(x,y,z;\xi,\eta,-\zeta) , \end{aligned} \tag{2}$$

where Γ is a fundamental solution of the Laplace equation and G is the Green's function for the Laplacian corresponding to a Dirichlet condition at the boundary $z = 0$.

It is sufficient to determine $u(x,y,0) = v(x,y)$. Once this is done, $u(x,y,z)$ is known. We shall derive an integral equation in v. In order to do this, suppose that

(i) $\dfrac{\partial \phi}{\partial x}(x,y) = \dfrac{\partial \phi}{\partial y}(x,y) = 0$ for large $r = \sqrt{x^2+y^2}$.

(ii) $f(x,y), g(x,y), h(x,y), u_1(x,y)$ tend to 0 sufficiently fast, say as $\dfrac{1}{\sqrt{x^2+y^2}}$ for $\sqrt{x^2+y^2} \to \infty$.

(iii) $\sqrt{1+x^2+y^2}\ .\ v(x,y)$ is in $L^2(R^2)$.

Integrating Green's identity on $D_\varepsilon, \varepsilon > 0$, where $D_\varepsilon = D \setminus D'_\varepsilon$ and D'_ε is the closed ball in D of radius ε centered at (x,y,z), and let $\varepsilon \to 0$, we then have, after some rearrangements

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{zv(\xi,\eta)}{((x-\xi)^2+(y-\eta)^2+z^2)^{3/2}} d\xi d\eta =$$

$$= u(x,y,z) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,z;\xi,\eta,\phi(\xi,\eta))f_1(\xi,\eta)d\xi d\eta$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(x,y,z;\xi,\eta,\phi(\xi,\eta))u_1(\xi,\eta)d\xi d\eta \ , \tag{3}$$

where $-\infty < x, y < \infty, 0 < z < \phi(x,y)$,

$$f_1(\xi,\eta) = h(\xi,\eta) - f(\xi,\eta)\frac{\partial}{\partial\xi}\phi(\xi,\eta) - g(\xi,\eta)\frac{\partial}{\partial\eta}\phi(\xi,\eta)$$

and

$$G_1(x,y,z;\xi,\eta,\phi(\xi,\eta)) = G_\zeta(x,y,z;\xi,\eta,\phi(\xi,\eta))$$

$$- G_\xi(x,y,z;\xi,\eta,\phi(\xi,\eta))\frac{\partial}{\partial\xi}\phi(\xi,\eta) - G_\eta(x,y,z;\xi,\eta,\phi(\xi,\eta))\frac{\partial}{\partial\eta}\phi(\xi,\eta). \tag{4}$$

Letting $z \to \phi(x,y)$ in (3), we have (see [8])

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\phi(x,y)v(\xi,\eta)}{\left((x-\xi)^2+(y-\eta)^2+\phi^2(x,y)\right)^{3/2}}d\xi d\eta =$$

$$= \frac{1}{2}u_1(x,y) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,\phi(x,y);\xi,\eta,\phi(\xi,\eta))f_1(\xi,\eta)d\xi d\eta$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(x,y,\phi(x,y);\xi,\eta,\phi(\xi,\eta))u_1(\xi,\eta)d\xi d\eta \ , \tag{5}$$

which is an integral equation in $v(x,y)$. We shall convert (5) into a convolution equation.

We note that the function

$$H(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{zv(\xi,\eta)}{\left((x-\xi)^2+(y-\eta)^2+z^2\right)^{3/2}}d\xi d\eta$$

is harmonic in the upper half space $z > 0$. The value $H(x, y, \phi(x, y))$ is then the right hand side of (5). Furthermore, we can calculate $\frac{\partial H}{\partial n}(x, y, \phi(x, y))$ as the limit from below of the directional derivative of the right hand side of (3) when $(x, y, z) \to (x, y, \phi(x, y))$, $\vec{n}$ being the inner unit normal to the surface $z = \phi(x, y)$.

Let $\quad \lambda(x, y) = H(x, y, \phi(x, y))\ , \qquad \mu(x, y) = \frac{\partial H}{\partial n}(x, y, \phi(x, y))$

Then $\lambda(x, y)$ and $\mu(x, y)$ are defined on R^2, and depend continuously on $\phi(x, y)$, $\frac{\partial \phi}{\partial x}(x, y)$, $\frac{\partial \phi}{\partial y}(x, y)$, $u_1(x, y)$, $\frac{\partial u_1}{\partial x}(x, y)$, $\frac{\partial u_1}{\partial y}(x, y)$, $f(x, y)$, $g(x, y)$ and $h(x, y)$ in the L^2-sense. Furthermore, $H(x, y, z)$ can be represented as a potential with densities λ, μ on the domain $z > \phi(x, y)$. In fact, integrating Green's identity in the domain

$$D_R = \{(x, y, z)\ :\ x^2 + y^2 < R^2\ ,\ \phi(x, y) < z < R\}$$

and letting $R \to \infty$, we get

$$\begin{aligned} H(x, y, z) = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x, y, z; \xi, \eta, \phi(\xi, \eta))\mu(\xi, \eta)d\xi d\eta \\ & - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma_1(x, y, z; \xi, \eta, \phi(\xi, \eta))\lambda(\xi, \eta)d\xi d\eta \end{aligned} \tag{6}$$

for $-\infty < x, y < \infty$, $z > \phi(x, y)$, where

$$\begin{aligned} \Gamma_1(x, y, z; \xi, \eta, \phi(\xi, \eta)) &= \Gamma_\xi(x, y, z; \xi, \eta, \phi(\xi, \eta))\frac{\partial \phi}{\partial \xi}(\xi, \eta) \\ &+ \Gamma_\eta(x, y, z; \xi, \eta, \phi(\xi, \eta))\frac{\partial \phi}{\partial \eta}(\xi, \eta) - \Gamma_\zeta(x, y, z; \xi, \eta, \phi(\xi, \eta)). \end{aligned}$$

Note that as $R \to \infty$, the integral on

$$\begin{aligned} C_R = & \{(x, y, z)\ :\ x^2 + y^2 = R^2,\ \phi(x, y) < z < R\} \\ & \cup \{(x, y, R)\ :\ x^2 + y^2 < R^2\} \end{aligned}$$

tends to 0 as a consequence of our assumption on v (i.e., $\sqrt{1 + x^2 + y^2}\ .$ $v(x, y)$ is in $L^2(R^2)$).

Evaluating $H(x,y,z)$ at (x,y,k) where k is a fixed number greater than $\phi(x,y)$ for all (x,y) in R^2, we have by (6)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{kv(\xi,\eta)}{((x-\xi)^2+(y-\eta)^2+k^2)^{3/2}}d\xi d\eta =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(x,y,k;\xi,\eta,\phi(\xi,\eta))\mu(\xi,\eta)d\xi d\eta$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_1(x,y,k;\xi,\eta,\phi(\xi,\eta))\lambda(\xi,\eta)d\xi d\eta\ ,$$

Let

$$F(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(x,y,k;\xi,\eta,\phi(\xi,\eta))v(\xi,\eta)d\xi d\eta$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_1(x,y,k;\xi,\eta,\phi(\xi,\eta))\lambda(\xi,\eta)d\xi d\eta\ .$$

Then we have a convolution integral equation in $v(\xi,\eta)$

$$\text{(7)}\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{kv(\xi,\eta)}{((x-\xi)^2+(y-\eta)^2+k^2)^{3/2}} = F(x,y), \quad \forall (x,y)\in R^2,$$

which is an integral equation of first kind, and we know that this problem is ill-posed. We shall construct a family (v_β), $\beta>0$, of regularized solutions (see [11]), and we pick a regularized solution that is "close" to the exact solution. We recall that, by regularized solution we mean a function that is stable with respect to variations in the right hand side of (7).

We now state and prove our main result.

Theorem. *Suppose the exact solution* v_0 *of* (7) *in the right hand side is in* $H^1(R^2)$ *and let*

$$|F_0 - F|_2 < \varepsilon\ ,\quad |\ .\ |_2 = L^2(R^2) - norm.$$

Then there exists a regularized solution v_ε of (7) *such that*

$$|v_\varepsilon - v_0|_2 \leq K\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-1} \qquad for \quad \varepsilon \to 0,$$

where K is a constant depending only on the H^1-norm of v_0.

Proof. Letting

$$G(x,y) = \frac{k}{\left(x^2+y^2+k^2\right)^{3/2}} \ ,$$

we have

$$\hat{G}(s,t) = \exp\left(-k\sqrt{s^2+t^2}\right) \ ,$$

where

$$\hat{G}(s,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} G(x,y)e^{-i(xs+yt)}dxdy.$$

For v in $L^2(R^2)$, we then have by (7)

$$\hat{G}(s,t)\hat{v}(s,t) = \hat{F}(s,t).$$

Now let $v_0 \in H^1(R^2)$ be the exact solution of the equation

$$\hat{G}(s,t)\hat{v_0}(s,t) = \hat{F_0}(s,t) \ , \quad \forall (s,t) \in R^2, \tag{8}$$

with F and F_0 in $L^2(R^2)$ such that

$$|F - F_0|_2 < \varepsilon. \tag{9}$$

For every $\beta > 0$, the function

$$\psi(s,t) = \frac{\hat{G}(s,t)\hat{F}(s,t)}{\beta + \hat{G}^2(s,t)} \tag{10}$$

is in $L^2(R^2)$. Let

$$v_\beta(x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \psi(s,t)e^{i(xs+yt)}dsdt.$$

Then $v_\beta \in L^2(R^2)$ and, by (10), v_β satisfies the equation

$$\beta \hat{v_\beta}(s,t) + \hat{G}^2(s,t)\hat{v_\beta}(s,t) = \hat{G}(s,t)\hat{F}(s,t), \quad \forall\ (s,t) \in R^2, \tag{11}$$

and depends continuously on $F(s,t)$.

We now derive error estimates. From (8) and (11), we have

$$\begin{aligned} &\beta(\hat{v_\beta}(s,t) - \hat{v_0}(s,t)) + \hat{G}^2(s,t)(\hat{v_\beta}(s,t) - \hat{v_0}(s,t)) = \\ &\quad -\beta \hat{v_0}(s,t) + \hat{G}(s,t)(\hat{F}(s,t) - \hat{F}_0(s,t)), \quad \forall (s,t) \in R^2. \end{aligned} \tag{12}$$

We multiply both sides of (12) by $\overline{\hat{v_\beta}(s,t) - \hat{v_0}(s,t)}$ and then integrate on R^2. Then we have

$$\begin{aligned} &\beta|\hat{v_\beta} - \hat{v_0}|_2^2 + |\hat{G}\,(\hat{v_\beta} - \hat{v_0})\,|_2^2 \\ &\quad = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta\hat{v_0}(s,t)\left(\overline{\hat{v_\beta}(s,t) - \hat{v_0}(s,t)}\right) dsdt \\ &\qquad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{G}(s,t)(\hat{F}(s,t) - \hat{F}_0(s,t))\left(\overline{\hat{v_\beta}(s,t) - \hat{v_0}(s,t)}\right) dsdt \\ &\qquad\quad \le \beta|\hat{v_0}|_2|\hat{v_\beta} - \hat{v_0}|_2 + |\hat{F} - \hat{F}_0|_2|\hat{v_\beta} - \hat{v_0}|_2. \end{aligned} \tag{13}$$

Let $\beta = \varepsilon$ and note that $|\hat{F} - \hat{F}_0|_2 = |F - F_0|_2 < \varepsilon$, we have

$$\varepsilon|\hat{v_\varepsilon} - \hat{v_0}|_2^2 + |\hat{G}\,(\hat{v_\varepsilon} - \hat{v_0})\,|_2^2 \le \varepsilon\,(|\hat{v_0}|_2 + 1)\,|\hat{v_\varepsilon} - \hat{v_0}|_2. \tag{14}$$

In particular

$$|\hat{v_\varepsilon} - \hat{v_0}|_2 \le (|\hat{v_0}|_2 + 1)\,. \tag{15}$$

Similarly, letting $\beta = \varepsilon$ in (12) and multiplying both sides by $(s^2 + t^2)\left(\overline{\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)}\right)$ and then integrating over R^2, we have

$$
\begin{aligned}
&\varepsilon|\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2^2+|\hat{G}\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2^2\\
&=-\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\varepsilon\hat{v_0}(s,t)\left(s^2+t^2\right)\left(\overline{\hat{v_\varepsilon}(s,t)-\hat{v_0}(s,t)}\right)dsdt\\
&\quad+\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\sqrt{s^2+t^2}\,.\,\hat{G}(s,t)(\hat{F}(s,t)\\
&\quad-\hat{F}_0(s,t))\sqrt{s^2+t^2}\left(\overline{\hat{v_\varepsilon}(s,t)-\hat{v_0}(s,t)}\right)dsdt\\
&\le\varepsilon|Dv_0|_2|\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2\\
&\quad+\frac{1}{ke}|\hat{F}-\hat{F}_0|_2|\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2\\
&\le\varepsilon\left(|Dv_0|_2+\frac{1}{ke}\right)|\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2, \qquad (16)
\end{aligned}
$$

where

$$|Dv_0|_2^2\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s^2+t^2)\hat{v_0}^2(s,t)dsdt.$$

In particular,

$$|\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2\le\left(|Dv_0|_2+\frac{1}{ke}\right). \qquad (17)$$

Since

$$|v_\varepsilon-v_0|_2=|\hat{v_\varepsilon}-\hat{v_0}|_2 \qquad (18)$$

and

$$|Dv_\varepsilon-Dv_0|_2=|\sqrt{s^2+t^2}\,(\hat{v_\varepsilon}-\hat{v_0})\,|_2,$$

from (15) and (17) we get

$$\|v_\varepsilon-v_0\|_{H^1(R^2)}=|v_\varepsilon-v_0|_2+|Dv_\varepsilon-Dv_0|_2\le K_1, \qquad (19)$$

where

$$
\begin{aligned}
K_1&=|v_0|_2+|Dv_0|_2+1+\frac{1}{ke}\\
&=\|v_0\|_{H^1(R^2)}+1+\frac{1}{ke}\cdot
\end{aligned}
$$

Now, for any $t_\varepsilon > 0$,

$$\iint\limits_{s^2+t^2\le t_\varepsilon^2} |\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)|^2 dsdt$$
$$\le \iint\limits_{s^2+t^2\le t_\varepsilon^2} e^{2kt_\varepsilon}\hat{G}^2(s,t)|\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)|^2 dsdt$$
$$\le e^{2kt_\varepsilon}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \hat{G}^2(s,t)|\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)|^2 dsdt$$
$$= e^{2kt_\varepsilon}|\hat{G}\left(\hat{v_\varepsilon} - \hat{v_0}\right)|_2^2$$
$$\le e^{2kt_\varepsilon}K_1\varepsilon\left(|\hat{v_0}|_2 + 1\right)$$
$$\equiv K_2\varepsilon e^{2kt_\varepsilon}, \tag{20}$$

$$\iint\limits_{s^2+t^2> t_\varepsilon^2} |\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)|^2 dsdt$$
$$\le \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left(s^2+t^2\right)t_\varepsilon^{-2}|\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)|^2 dsdt$$
$$= t_\varepsilon^{-2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |\sqrt{s^2+t^2}\left(\hat{v_\varepsilon}(s,t) - \hat{v_0}(s,t)\right)|^2 dsdt$$
$$\le K_1 t_\varepsilon^{-2}$$
$$\le K_2 t_\varepsilon^{-2}, \tag{21}$$

where

$$K_2 = K_1\left(|\hat{v_0}|_2 + 1\right).$$

Next consider the equation

$$y^2 e^{2ky} = \frac{1}{\varepsilon}\ \cdot \tag{22}$$

The function $h(y) = y^2 e^{2ky}$ is strictly increasing for $y > 0$ and $h(R^+) = R^+$. Then the equation (22) has a unique solution t_ε and $t_\varepsilon \to \infty$ as $\varepsilon \to 0$. Hence, we have

$$2(1+k)t_\varepsilon \ge 2\ln t_\varepsilon + 2kt_\varepsilon = \ln\frac{1}{\varepsilon}\ \cdot$$

Letting $\varepsilon < 1$, we have

$$t_\varepsilon^{-1} \leq 2(1+k)\left(\ln \frac{1}{\varepsilon}\right)^{-1} . \tag{23}$$

By (20), (21) and (23) we have

$$|v_\varepsilon - v_0|_2^2 \leq 2K_2 t_\varepsilon^{-2} \leq K^2 \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-2} ,$$

where

$$K^2 = 8(1+k)^2 K_2,$$

as desired. This completes the proof of the theorem.

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