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SPECTRAL ANALYSIS OF SET-VALUED MAPPINGS

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ABSTRACT. This note deals with the spectral analysis of a general set-valued mapping A defined from a normed space X into its topological dual space X^*. The following two issues are addressed: (i) identification of set-valued mappings which have only nonnegative eigenvalues; (ii) analysis of recession eigenvalues and recession eigenvectors.

1. Introduction

Eigenvalues and eigenvectors play a fundamental role in the analysis of linear systems. These two notions provide also a valuable information on the structure of more complex systems, such as those described by a set-valued (or multivalued) mapping. Leizarowitz [5], for instance, studies the eigenvalue problem

$$\lambda u \in A(u) , \quad u \neq 0 \tag{1.1}$$

in connection with the asymptotic analysis of the trajectories of the differential inclusion

$$\dot{x}(t) \in A(x(t)) , \quad t \in \mathbf{R}_+ . \tag{1.2}$$

He considers the case in which $A : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ is a convex process, i.e. the graph of A is a convex cone. The question concerning the "controllability" of the differential inclusion (1.2) has to do with the eigenvalue problem

$$\lambda w \in A^*(w) , \quad w \neq 0 ,$$

where $A^* : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$ stands for the adjoint mapping of the convex process $A : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$. This fact is discussed in detail in a remarkable paper by Aubin, Frankowska and Olech [3].

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As shown in the above mentioned works [3, 5], it is possible to extend in a reasonable way some classical results concerning eigenvalues of linear systems to the case of convex processes. For multivalued systems which are not convex processes, the chances of obtaining a bona fide extension are more remote, unless there is some additional structure involved. Aubin, for instance, obtains a Perron-type theorem for the case in which A maps the elements of the standard simplex $\{x \in \mathbf{R}^n_+ : x_1 + \cdots + x_n = 1\}$ into compact convex subsets of R^n_+. He is concerned with the existence of a positive eigenvalue of A. The details can be consulted in [1] or [2, Section 15.9].

In our opinion, the spectral theory of general set-valued mappings is still at an early stage of development. The purpose of this note is to contribute to this area of research by exploring the following two issues

(i) identification of set-valued mappings which have only nonnegative eigenvalues;

(ii) analysis of recession eigenvalues and recession eigenvectors.

These two themes will be treated, respectively, in Sections 3 and 4. Basic definitions and preliminary results will be given in Section 2.

2. Basic definitions and preliminary results

Unless otherwise specified we consider the case of a set-valued mapping from a real normed space $(X, \ ||\cdot||)$ into its topological dual space X^*. The spaces X and X^* are paired in duality by means of the canonical bilinear form

$$\langle y, x\rangle := y(x) \quad \forall x \in X \ , \ y \in X^* \ .$$

The notation $||\cdot||_*$ refers to the dual norm associated to $||\cdot||$, and $I : X \rightrightarrows X^*$ is the duality mapping defined by

$$I(x) := \left\{ y \in X^* : ||y||_*^2 = ||x||^2 =< y, x > \right\} .$$

In this general setting, the concept of eigenvalue is introduced as follows:

Definition 2.1. A real number λ is an *eigenvalue* of $A : X \rightrightarrows X^*$ if there is a nonzero vector $u \in X$ such that

$$0 \in (A - \lambda I)(u) \ .$$

We then call u an *eigenvector* of A associated to the eigenvalue λ.

Remark 2.1. If $(X, ||\cdot||)$ is a Hilbert space, then the symbol $\langle\cdot,\cdot\rangle$ is understood as the inner product in X. The dual space X^* is identified with X, and the duality mapping I is simply the identity mapping over X. In this case the inclusion $0 \in (A - \lambda I)(u)$ takes the more familiar form $\lambda u \in A(u)$.

Remark 2.2. Suppose X is a finite dimensional space equipped with a norm $||\cdot||$ which is not Hilbertian. Even if one identifies X^* with X, the duality mapping I does not coincide with the identity mapping over X. Thus, one has to be careful with the fact that the inclusion $0 \in (A - \lambda I)(u)$ is not equivalent to $\lambda u \in A(u)$. In other words, one has to distinguish between the eigenvalues of A relative to the duality mapping, and the eigenvalues of A relative to the identity mapping.

For subsequent use, it is convenient to denote by

$$\sigma(A) := \big\{\lambda \in \mathbf{R} : \lambda \text{ is an eigenvalue of } A\big\}$$

the *spectrum* of A, and by

$$E_\lambda(A) := \big\{u \in X : 0 \in (A - \lambda I)(u)\big\}$$

the *eigenset* of A associated with the value $\lambda \in \mathbf{R}$. The standard notation

$$\begin{aligned} D(A) &:= \{x \in X : A(x) \neq \emptyset\} && \text{(domain of } A),\\ Gr\ A &:= \{(x,y) \in X \times X^* : y \in A(x)\} && \text{(graph of } A),\\ \text{Im}\ (A) &:= \cup\{A(x) : x \in X\} && \text{(range of } A), \end{aligned}$$

will also be in force.

Basic properties of the eigensets $\{E_\lambda(A) : \lambda \in \mathbf{R}\}$ are derived from the structure of the graph of A. The next three propositions can be proven in a rather easy way, so their proofs are omitted. Recall that a set Q in a linear space is said to be a cone if $\alpha Q \subset Q$ for all $\alpha > 0$.

Proposition 2.1. *Suppose the graph of $A : X \rightrightarrows X^*$ is a cone (resp. contains the origin). Then, for each $\lambda \in \mathbf{R}$, $E_\lambda(A)$ is a cone (resp. contains the origin).*

Proposition 2.2. *Let $(X, ||\cdot||)$ be a Hilbert space or a finite dimensional normed space. If the graph of $A : X \rightrightarrows X$ is closed, then all the eigensets $\{E_\lambda(A) : \lambda \in \mathbf{R}\}$ are closed.*

Proposition 2.3. *Let $(X, ||\cdot||)$ be a Hilbert space. Suppose the graph of $A : X \rightrightarrows X$ is convex. Then all the eigensets $\{E_\lambda(A) : \lambda \in \mathbf{R}\}$ are convex.*

Remark 2.3. If $(X, ||\cdot||)$ is not a Hilbert space, then the graph of the duality mapping $I : X \rightrightarrows X^*$ is not necessarily convex. As a consequence, Proposition 2.3 fails in a non-Hilbertian setting. To see this, consider the space $X = \mathbf{R}^2$ equipped with the norm $||x|| = \text{Max}\,\{|x_1|\ ,\ |x_2|\}$. The dual norm $||\cdot||_*$ on $X^* = \mathbf{R}^2$ is, of course, given by $||y||_* = |y_1| + |y_2|$. If $\lambda = 1$ and $A : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear mapping defined by $A(x_1, x_2) = (x_1, x_2)$, then $(2, 0) \in E_\lambda(A)$ and $(0, 2) \in E_\lambda(A)$. However, the convex combination $(1, 1) = \frac{1}{2}(2, 0) + \frac{1}{2}(0, 2)$ does not belong to $E_\lambda(A)$. So, $E_\lambda(A)$ is not convex, even if the graph of A is linear!

If $A : X \rightrightarrows X$ is a closed convex process over a Hilbert space $(X, ||\cdot||)$, then it is possible to express the eigensets $\{E_\lambda(A) : \lambda \in \mathbf{R}\}$ in terms of the adjoint mapping $A^* : X \rightrightarrows X$. Recall that the adjoint A^* of A is the set-valued mapping defined by

$$p \in A^*(q) \iff \langle p, x\rangle \ \le\ \langle q, y\rangle \quad \text{for all} \quad (x, y) \in Gr\ A\ .$$

The next proposition is a refinement of a result stated in [3, Lemma 1.14].

Proposition 2.4. *Let $(X, ||\cdot||)$ be a Hilbert space. Suppose the graph of $A : X \rightrightarrows X$ is a closed convex cone. Then, for all $\lambda \in \mathbf{R}$, one has*

$$E_\lambda(A) = \big[\ Im\ (A^* - \lambda I)\big]^-\ ,$$

where the notation K^- stands for the negative polar cone of $K \subset X$. In particular,

$$\sigma(A) = \{\lambda \in \mathbf{R} :\ \overline{\text{Im}\ (A^* - \lambda I)} \ne X\}\ ,$$

where the upper bar denotes the closure operation.

Proof. Take any $\lambda \in \mathbf{R}$. Since $Gr\ A = \big\{(x, y) \in X.X : (-y, x) \in \big[Gr\ A^*\big]^{-1}\big\}$, the condition $\lambda u \in A(u)$ can be written in the form

$$\langle \lambda u, q\rangle \ \ge\ \langle u, p\rangle \quad \text{for all} \quad (q, p) \in Gr\ A^*\ ,$$

or equivalently,

$$\langle p - \lambda q, u\rangle\ \le 0 \quad \text{for all} \quad q \in D(A^*)\ ,\ p \in A^*(q)\ .$$

In other words,

$$\langle w, u\rangle\ \le 0 \quad \text{for all} \quad w \in\ \text{Im}\ (A^* - \lambda I)\ .$$

In this way, one has shown that $u \in E_\lambda(A)$ if and only if u belongs to the negative polar cone of Im $(A^* - \lambda I)$. In particular, $\lambda \in \sigma(A)$ if and only if $[\,\mathrm{Im}\,(A^* - \lambda I)]^-$ contains a nonzero vector. The latter condition amounts to saying that the closure of Im $(A^* - \lambda I)$ is not the whole space X. □

3. Set-valued mappings with nonnegative eigenvalues

This section addresses the first topic in our agenda, namely the identification of set-valued mappings which have only nonnegative eigenvalues. In connection with this question, the notion of positive semidefinite set-valued mapping emerges in a natural way.

Definition 3.1. $A : X \rightrightarrows X^*$ is said to be *positive semidefinite* if

$$\langle y, x\rangle \;\geq 0 \quad \text{for all} \quad (x,y) \in Gr\ A\ . \tag{3.1}$$

If one uses the notation $\langle A(x), x\rangle := \{\langle y, x\rangle : y \in A(x)\}$, then (3.1) takes the form

$$\langle A(x), x\rangle \;\subset \mathbf{R}_+ \quad \text{for all} \quad x \in D(A)\ . \tag{3.2}$$

Proposition 3.1. *Let $A : X \rightrightarrows X^*$ be positive semidefinite. Then, A has only nonnegative eigenvalues.*

Proof. Take any $\lambda \in \sigma(A)$. From the very definition of an eigenvalue, there are vectors $u \neq 0$ and $v \in I(u)$ such that $\lambda v \in A(u)$. Since A is positive semidefinite, one has

$$0 \leq \;\langle \lambda v, u\rangle \;= \lambda\langle v, u\rangle \;= \lambda||u||^2\ .$$

This shows that $\lambda \geq 0$. □

Remark 3.1. The converse of Proposition 3.1 is true for the particular case $X = X^* = \mathbf{R}$, but it is not true in general. Consider the space $X = \mathbf{R}^2$ equipped with the usual Euclidean norm, and the mapping $A : \mathbf{R}^2 \rightrightarrows \mathbf{R}^2$ given by

$$A(x) := \begin{cases} \{x\} & \text{if} \quad x \neq (1,1)\ , \\ \{(1,1)\ ,\ (-1,0)\} & \text{if} \quad x = (1,1)\ . \end{cases}$$

It can be shown that A has $\lambda = 1$ as unique eigenvalue, but A is not positive semidefinite.

A minor modification of the proof of Proposition 3.1 yields directly:

Proposition 3.2. *Let $A : X \rightrightarrows X^*$ be positive definite in the sense that*

$$(x, y) \in Gr\ A\ , \quad x \neq 0 \Longrightarrow \quad \langle y, x \rangle\ > 0\ .$$

Then, A has only positive eigenvalues.

In what follows we identify two important classes of positive semidefinite set-valued mappings. Recall that $A : X \rightrightarrows X^*$ is called *monotone* (in the sense of Minty [9]) if

$$\langle y_1, x_2 - x_1 \rangle + \langle y_2, x_1 - x_2 \rangle \leq 0 \ \text{ for all } \ (x_1, y_1) \in Gr\ A,\ (x_2, y_2) \in Gr\ A\ .$$

Contrarily to the case of linear systems, monotonicity by itself is not enough to obtain the positive semidefinite property. For this reason one needs to invoke the following stability condition:

Definition 3.2. $A : X \rightrightarrows X^*$ is said to be *stable* if the set $Gr\ A$ is stable, i.e.

$$(3.3) \qquad (x, y) \in Gr\ A \Longrightarrow \exists \alpha \neq 1 \quad \text{such that} \quad \alpha(x, y) \in Gr\ A\ .$$

As a simple example of stable mapping, consider any $A : X \rightrightarrows X^*$ satisfying the normalization condition $0 \in A(0)$.

Proposition 3.3. *Let $A : X \rightrightarrows X^*$ be monotone and stable. Then, A is positive semidefinite.*

Proof. Take any $(x, y) \in Gr\ A$. Due to the stability of A, there exists a real number $\alpha \neq 1$ such that $(\alpha x, \alpha y) \in Gr\ A$. The monotonicity of A yields in this case

$$\langle y, \alpha x - x \rangle + \langle \alpha y, x - \alpha x \rangle\ \leq 0\ ,$$

or equivalently,

$$(\alpha - 1)^2\ \langle y, x \rangle\ \geq 0\ .$$

This proves that $\langle y, x \rangle\ \geq 0$.

Remark 3.2. The singled-valued mapping $x \in \mathbf{R} \longmapsto A(x) = x + 1$ is monotone, but not positive semidefinite. In this case the stability condition (3.3) fails.

In some practical situations, the monotonicity of A may be too stringent a requirement. As a relaxation of this assumption, one may consider the concept of quasimonotonicity as introduced by Luc [7, 8].

Definition 3.3. $A : X \rightrightarrows X^*$ is said to be *quasimonotone* if

$$\min\left\{\langle y_1, x_2 - x_1\rangle\ ,\ \langle y_2, x_1 - x_2\rangle\right\} \le 0 \quad \text{for all} \quad \begin{array}{l}(x_1, y_1) \in Gr\ A\ ,\\ (x_2, y_2) \in Gr\ A\ .\end{array}$$

Further information on this concept can be found in a recent paper by Penot and Quang [10]. As a proto-type of quasimonotone mapping, consider the Clarke-Rockafellar subdifferential of a proper lower-semicontinuous quasiconvex function defined over a Banach space (cf. Luc [7, Theorem 3.2]).

Since quasimonotonicity is a weaker assumption than monotonicity, the concept of stability needs to be reinforced if one wishes to obtain a result similar to that of Proposition 3.3.

Definition 3.4. $A : X \rightrightarrows X^*$ is said to be *negatively stable* if the set $Gr\ A$ is negatively stable, i.e.

$$(x, y) \in Gr\ A \Longrightarrow \exists \alpha < 0 \quad \text{such that} \quad \alpha(x, y) \in Gr\ A\ . \tag{3.4}$$

As a simple example of negatively stable mapping, consider any $A : X \rightrightarrows X^*$ satisfying the oddness condition

$$A(-x) = -A(x) \quad \text{for all} \quad x \in X\ .$$

Proposition 3.4. *Let $A : X \rightrightarrows X^*$ be quasimonotone and negatively stable. Then, A is positive semidefinite.*

Proof. Take any $(x, y) \in Gr\ A$. Due to the negative stability of A, there exists $\alpha < 0$ such that $(\alpha x, \alpha y) \in Gr\ A$. The quasimonotonicity of A yields in this case

$$\min\ \left\{\langle y, \alpha x - x\rangle\ ,\ \langle \alpha y, x - \alpha x\rangle\right\}\ \le 0\ ,$$

that is to say,

$$\min\ \left\{(\alpha - 1)\ \langle y, x\rangle\ ,\ \alpha(1 - \alpha)\ \langle y, x\rangle\right\}\ \le 0\ .$$

If $\langle y, x\rangle$ is strictly negative, then one should have

$$\min; \left\{1 - \alpha\ ,\ \alpha(\alpha - 1)\right\}\ \le 0\ .$$

Since the last inequality contradicts the facts that $\alpha < 0$, it follows that $\langle y, x\rangle \geq 0$. □

Remark 3.3. Propositions 3.3 and 3.4 can not be compared. On the one hand, the mapping

$$x \in \mathbf{R} \rightrightarrows A(x) = \begin{cases} 0 & \text{if} \quad x < 0\,, \\ [0,1] & \text{if} \quad x = 0\,, \\ 1 + x & \text{if} \quad x > 0\,, \end{cases}$$

is monotone and stable, but not negatively stable. On other hand, the singled-valued mapping

$$x \in \mathbf{R} \longmapsto A(x) = \begin{cases} -2 - x & \text{if} \quad x \in]-2,-1[\,, \\ x & \text{if} \quad x \in [-1,1]\,, \\ 2 - x & \text{if} \quad x \in]1,2[\,, \\ 0 & \text{otherwise}, \end{cases}$$

is quasimonotone and negatively stable, but not monotone.

There are several operations which preserve the positive semidefinite character of a set-valued mapping. As way of example, consider the next result whose proof is immediate.

Proposition 3.5. *Let $A_1, A_2 : X \rightrightarrows X^*$ be two positive semidefinite set-valued mappings. Then, their direct sum*

$$x \in X \rightrightarrows (A_1 + A_2)(x) = A_1(x) + A_2(x),$$

and their inverse sum

$$x \in X \rightrightarrows (A_1 \,\square\, A_2)(x) = \bigcup_{x_1 + x_2 = x} \{A_1(x_1) \cap A_2(x_2)\}$$

are also positive semidefinite.

Recall that the *Schur complement* of $A : X \rightrightarrows X^*$, relative to a continuous linear operator $L : X \to Z$, is the set-valued mapping $A_L : Z \rightrightarrows Z^*$ defined by

$$A_L(z) := \bigcup_{Lx = z} \{w \in Z^* : L^* w \in A(x)\}\,,$$

where $L^* : Z^* \to X^*$ stands for the adjoint operator of L.

Proposition 3.6. *Let $L : X \to Z$ be a continuous linear operator. If the set-valued mapping $A : X \rightrightarrows X^*$ is positive semidefinite, then so does its Schur complement $A_L : Z \rightrightarrows Z^*$.*

Proof. Take any $(z, w) \in Gr\ A_L$. In this case

$$z = Lx \quad \text{and} \quad (x, L^* w) \in Gr\ A$$

for some $x \in X$. Since A is positive semidefinite, one gets

$$0 \leq \langle L^* w, x \rangle = \langle w, Lx \rangle = \langle w, z \rangle . \qquad \square$$

4. Recession eigenvalues and recession eigenvectors

Recall that the recession (or asymptotic) cone Q_∞ of a nonempty convex set Q is defined by

$$Q_\infty := \cap \{Q - q : q \in Q\} . \tag{4.1}$$

It is known that Q_∞ is a convex cone containing the origin.

Definition 4.1. Let $A : X \rightrightarrows X^*$ be a convex mapping in the sense that $Gr\ A$ is a convex set. Then the *recession mapping* $A_\infty : X \rightrightarrows X^*$ of A is defined by

$$A_\infty(u) := \{v \in X^* : (u, v) \in [Gr\ A]_\infty\} .$$

In other words, $Gr\ A_\infty = [Gr\ A]_\infty$.

Thus, $A_\infty : X \rightrightarrows X^*$ is a mapping whose graph is a convex cone containing the origin. The latter property means that A_∞ satisfies the normalization condition $0 \in A_\infty(0)$. The concept of recession mapping is not new. It has been used in a different context by Borwein [4] and Luc [6], among others.

In a parallel way to (4.1), one can also consider the expression

$$Q^\diamond := \cup \{Q - q : q \in Q\} . \tag{4.2}$$

If Q is a nonempty convex set, then $Q^\diamond$ is a convex set containing the origin. However, $Q^\diamond$ is not necessarily a cone.

Definition 4.2. Let $A : X \rightrightarrows X^*$ be a convex mapping. The *companion* mapping $A^\diamond : X \rightrightarrows X^*$ of A is defined by

$$A^\diamond(u) := \{v \in X^* : (u, v) \in [Gr\ A]^\diamond\} .$$

Both mappings A_∞ and $A^\diamond$ are of interest in connection with the spectral analysis of A. Consider also the *traslated* mapping $A_{x,y} : X \rightrightarrows X^*$ defined by

$$A_{x,y}(u) := A(x+u) - y \quad \text{for all} \quad u \in X \ .$$

The term "traslated" refers to the property

$$Gr\ A_{x,y} = Gr\ A - (x,y) \ .$$

Theorem 4.1. *Let $A : X \rightrightarrows X^*$ be a convex mapping. For a given $(\lambda, u) \in \mathbf{R} \times X$, consider the following four conditions :*

(a) $0 \in (A_\infty - \lambda I)(u)$;

(b) $0 \in (A_{x,y} - \lambda I)(u)$ *for each* $(x,y) \in Gr\ A$;

(c) $0 \in (A_{x,y} - \lambda I)(u)$ *for some* $(x,y) \in Gr\ A$;

(d) $0 \in (A^\diamond - \lambda I)(u)$.

Then, one has the relationship (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longleftrightarrow$ (d). Moreover, if the duality mapping I is singled-valued at u, then the implication (b) $\Longrightarrow$ (a) is also true.

Proof. (a) $\Longrightarrow$ (b). Suppose $0 \in (A_\infty - \lambda I)(u)$, i.e. there exists $v \in I(u)$ such that $\lambda v \in A_\infty(u)$. From the definition of A_∞, it follows that

$$(u, \lambda v) \in \cap\{Gr\ A - (x,y) : (x,y) \in Gr\ A\} \ .$$

Thus, condition (a) is equivalent to the sentence

$$(4.3) \qquad \exists v \in I(u) \text{ such that } \forall (x,y) \in Gr\ A \text{ one has } (u, \lambda v) \in Gr\ A_{x,y} \ ,$$

which is, of course, stronger than

$$(4.4) \qquad \forall (x,y) \in Gr\ A \quad \exists v \in I(u) \quad \text{such that} \quad (u, \lambda v) \in Gr\ A_{x,y} \ .$$

The latter sentence corresponds to the condition (b). The implication (b) $\Longrightarrow$ (c) is trivial. To prove the equivalence (c) $\Longleftrightarrow$ (d), observe that (c) amounts to saying that

$$\exists (x,y) \in Gr\ A \ , \ \exists v \in I(u) \quad \text{such that} \quad (u, \lambda v) \in Gr\ A_{x,y} \ ,$$

or equivalently

$$\exists v \in I(u) \quad \text{such that} \quad (u, \lambda v) \in Gr\ A^\diamond = \cup\{Gr\ A_{x,y} : (x,y) \in Gr\ A\} \ .$$

In other words, $0 \in (A^\diamond - \lambda I)(u)$. Finally, if I is singled-valued at u, then (4.4) is equivalent to (4.3). □

The next two results are obtained straightforwardly from Theorem 4.1.

Corollary 4.1. *Let $A : X \rightrightarrows X^*$ be a convex mapping. Then*

$$\sigma(A_\infty) \subset \bigcap_{(x,y)\in Gr\ A} \sigma(A_{x,y}) \subset \bigcup_{(x,y)\in Gr\ A} \sigma(A_{x,y}) \subset \sigma(A^\diamond)\ ,$$

and, for all $\lambda \in \mathbf{R}$,

$$E_\lambda(A_\infty) \subset \bigcap_{(x,y)\in Gr\ A} E_\lambda(A_{x,y}) \subset \bigcup_{(x,y)\in Gr\ A} E_\lambda(A_{x,y}) \subset E_\lambda(A^\diamond)\ .$$

Moreover, if the duality mapping I is singled-valued, then

$$E_\lambda(A_\infty) = \bigcap_{(x,y)\in Gr\ A} E_\lambda(A_{x,y}) \quad \text{for all} \quad \lambda \in \mathbf{R}\ .$$

Corollary 4.2. *Let $A : X \rightrightarrows X^*$ be convex and normalized, i.e. $0 \in A(0)$. Then,*

$$\sigma(A_\infty) \subset \sigma(A) \subset \sigma(A^\diamond)\ , \tag{4.5}$$

and

$$E_\lambda(A_\infty) \subset E_\lambda(A) \subset E_\lambda(A^\diamond) \quad \text{for all} \quad \lambda \in \mathbf{R}\ . \tag{4.6}$$

Remark 4.1. The inclusion $\sigma(A_\infty) \subset \sigma(A)$ may fail if A is not normalized. To see this, consider the convex mapping $A : \mathbf{R} \rightrightarrows \mathbf{R}$ defined by

$$Gr\ A = \{(x,y) \in \mathbf{R}\times\mathbf{R} : x > 0\ ,\ xy \geq 1\}\ .$$

In this case $\sigma(A_\infty) = \mathbf{R}_+$ and $\sigma(A) = \mathbf{R}_+ \setminus \{0\}$. The same example shows that also the inclusion $E_\lambda(A_\infty) \subset E_\lambda(A)$ may fail. Here

$$E_\lambda(A_\infty) = \begin{cases} \mathbf{R}_+ & \text{if} \quad \lambda \geq 0\ , \\ \{0\} & \text{if} \quad \lambda < 0\ , \end{cases}$$

and

$$E_\lambda(A) = \begin{cases} [1/\sqrt{\lambda}\,,\ \infty[& \text{if} \quad \lambda > 0\,, \\ \emptyset & \text{if} \quad \lambda \le 0\,. \end{cases}$$

Motivated by the inclusions established in Corollary 4.2, we proceed now to a classification of the eigenvalues and eigenvectors of A.

Definition 4.3. Let $A : X \rightrightarrows X^*$ be convex and normalized. Then, $\sigma(A_\infty)$ is referred to as the set of *recession eigenvalues* of A. Any element in $\sigma(A)\backslash\sigma(A_\infty)$ is called a *nonrecession eigenvalue* of A. Each nonzero vector in $E_\lambda(A_\infty)$ is called a *recession eigenvector* of A. If $u \neq 0$ belongs to $E_\lambda(A) \setminus E_\lambda(A_\infty)$, then u is said to be a *nonrecession eigenvector* of A.

Example 4.1. Consider the convex normalized mapping $A : \mathbf{R} \rightrightarrows \mathbf{R}$ defined by

$$Gr\ A = \{(x, y) \in \mathbf{R} \times \mathbf{R} : e^{-x} - y \le 1\}\,.$$

In this case the spectrum $\sigma(A) = \mathbf{R} \setminus \{-1\}$ is decomposed into the set $\sigma(A_\infty) = [0, \infty[$ of recession eigenvalues, and the set $\sigma(A) \setminus \sigma(A_\infty) =]-\infty, -1[\cup]-1, 0[$ of nonrecession eigenvalues.

Under the hypotheses of Corollary 4.2, one clearly has

$$\left.\begin{array}{r} E_\lambda(A) \neq \{0\} \\ \\ E_\lambda(A) \quad \text{is bounded} \end{array}\right\} \Longrightarrow \lambda \quad \text{is a nonrecession eigenvalue of} \quad A\,,$$

and

$$\lambda \quad \text{is recession eigenvalue of} \quad A \Longrightarrow E_\lambda(A) \quad \text{is unbounded}\,.$$

The reverse implications can be proven only in a more restrictive setting. To start with, observe that the first inclusion in (4.6) can be sharpened as indicated below.

Proposition 4.1. *Let $A : X \rightrightarrows X^*$ be convex and normalized. Then,*

$$E_\lambda(A_\infty) \subset \bigcap_{\alpha > 0} \frac{1}{\alpha}\, E_\lambda(A) \quad \textit{for all} \quad \lambda \in \mathbf{R}\,. \tag{4.7}$$

Proof. It suffices to combine $E_\lambda(A_\infty) \subset E_\lambda(A)$ and the fact that $E_\lambda(A_\infty)$ is a cone. □

Under some extra assumptions on the space X and the mapping A, it can be shown that the intersection appearing in (4.7) corresponds to the recession cone of $E_\lambda(A)$. The next theorem provides another justification for the use of the term "recession" while refering to some of the eigenvectors of A.

Theorem 4.2. *Let $(X, \ ||\cdot||)$ be a Hilbert space. Suppose the graph of $A : X \rightrightarrows X$ is a closed convex set containing the origin. Then*

$$E_\lambda(A_\infty) = [E_\lambda(A)]_\infty \quad \textit{for all} \quad \lambda \in \mathbf{R}\ .$$

In particular, each $E_\lambda(A_\infty)$ is a closed convex cone containing the origin.

Proof. Take any $\lambda \in \mathbf{R}$. From Proposition 4.1, one knows already that $E_\lambda(A_\infty)$ is contained in the intersection

$$S_\lambda := \bigcap_{\alpha>0} \frac{1}{\alpha}\, E_\lambda(A)\ .$$

The assumptions of the theorem imply that $E_\lambda(A)$ is a closed convex set containing the origin. In such a case, the set S_λ coincides with the recession cone of $E_\lambda(A)$. To prove the reverse inclusion $[E_\lambda(A)]_\infty \subset E_\lambda(A_\infty)$, take any $u \in [E_\lambda(A)]_\infty$. In this case

$$u \in \frac{1}{\alpha}\, E_\lambda(A) \quad \text{for all} \quad \alpha > 0\ ,$$

or equivalently

$$(u, \lambda u) \in \bigcap_{\alpha>0} \frac{1}{\alpha}\, Gr\ A\ .$$

Since $Gr\ A$ is a closed convex set containing the origin, the above intersection coincides with $[Gr\ A]_\infty$. This proves that $(u, \lambda u) \in Gr\ A_\infty$, i.e. $u \in E_\lambda(A_\infty)$. □

The conclusion of Theorem 4.2 can be stated in the following terms:

$$\begin{aligned} &u \quad \text{is a recession eigenvector of } A \text{ associated to } \lambda \iff \\ &u \quad \text{is a nonzero vector in the recession cone of } E_\lambda(A)\ . \end{aligned}$$

This observation leads to a simple characterization of the recession eigenvalues of A. In fact, one has:

Corollary 4.3. *Let $(X, \ ||\cdot||)$ be a finite dimensional Hilbert space. Suppose the graph of $A : X \rightrightarrows X$ is a closed convex set containing the origin. Then*

$$\lambda \text{ is a recession eigenvalue of } A \iff E_\lambda(A) \text{ is unbounded} \,.$$

Similarly,

$$\lambda \text{ is a nonrecession eigenvalue of } A \iff E_\lambda(A) \neq \{0\} \text{ and } E_\lambda(A) \text{ is bounded} \,.$$

Proof. $E_\lambda(A)$ is a closed convex set in a finite dimensional space. Thus (cf. [11, Theorem 8.4])

$$E_\lambda(A) \quad \text{is bounded} \iff [E_\lambda(A)]_\infty = \{0\} \,.$$

It suffices then to combine the above result and Theorem 4.2. □

5. Conclusions

Two important classes of set-valued mappings with only nonnegative eigenvalues have been singled out in Section 3. The first class consists of those monotone mappings $A : X \rightrightarrows X^*$ which have a stable graph, and the second class is formed by the quasimonotone mappings which have a negatively stable graph. The results of Section 3 are all related to the concept of positive semidefinite set-valued mapping.

Recession eigenvalues and recession eigenvectors were introduced and studied in Section 4. Among other results, it was shown that $\sigma(A_\infty) \subset \sigma(A)$, whenever $A : X \rightrightarrows X^*$ is convex and normalized. This yields in particular the bounds

$$\overline{\lambda}(A_\infty) \leq \overline{\lambda}(A) \quad \text{and} \quad \underline{\lambda}(A) \leq \underline{\lambda}(A_\infty)$$

for the extremal values

$$\overline{\lambda}(A) := \sup \{\lambda \in \mathbf{R} : \lambda \in \sigma(A)\} \,,$$
$$\underline{\lambda}(A) := \inf \{\lambda \in \mathbf{R} : \lambda \in \sigma(A)\} \,,$$

of the spectrum of A. The quantities $\overline{\lambda}(A_\infty)$ and $\underline{\lambda}(A_\infty)$ can be estimated, in principle, by using Leizarowitz's variational formulation of the extremal eigenvalues of a convex process [5].

Recession eigenvalues and recession eigenvectors can be introduced also if the graph of $A : X \rightrightarrows X^*$ is not necessarily convex. This case, however, is much more involved and requires further investigation (cf. [12]).

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