ACTA MATHEMATICA VIETNAMICA
Volume 23, Number 1, 1998, pp. 13–24

DIFFERENTIABLE FUNCTIONS AND THE GENERATORS ON A HILBERT-LIE GROUP

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ABSTRACT. A convolution semigroup plays an important role in the theory of probability measure on Lie groups. The basic problem is that one wants to express a semigroup as a Lévy-Khinchine formula. If $(\mu_t)_{t\in\mathbf{R}_+^*}$ is a continuous semigroup of probability measures on a Hilbert-Lie group G, then we define

$$T_{\mu_t} f := \int f_a \mu_t(da) \quad (f \in C_*(G);\ t > 0).$$

It is apparent that $(T_{\mu_t})_{t\in\mathbf{R}_+^*}$ is a continuous operator semigroup on the space $C_*(G)$ with the infinitesimal generator N. The generating functional A of this semigroup is defined by $Af := \lim_{t\downarrow 0} \frac{1}{t}(T_{\mu_t} f(e) - f(e))$. We consider the problem of construction of a subspace $C_{(2)}(G)$ of $C_*(G)$ such that the generating functional A on $C_{(2)}(G)$ exists. This result will be used later to show that Lévy-Khinchine formula holds for Hilbert-Lie groups.

INTRODUCTION

Let $(\mu_t)_{t\in\mathbf{R}_+^*}$ be a continuous convolution semigroup of probability measures on a Hilbert-Lie group G and $C_u(G)$ the Banach space of all bounded left uniformly continuous real-valued functions on G. Then there is associated a strongly continuous semigroup $(T_{\mu_t})_{t\in\mathbf{R}_+^*}$ of contraction operators on $C_u(G)$ with the infinitesimal generator $(N, D(N))$. The generating functional $(A, D(A))$ of the convolution semigroup $(\mu_t)_{t\in\mathbf{R}_+^*}$ is defined by

$$Af := \lim_{t\downarrow 0} \frac{1}{t}\big(T_{\mu_t} f(e) - f(e)\big)$$

Received January 11, 1996
1991 Mathematics Subject Classification. Primary 60B, 58B; Secondary 47B
Key words and phrases. Continuous convolution semigroup, operator semigroup, Hilbert-Lie group, Lévy measure, infinitesimal generator, generating functional.

for all f in its domain $D(A)$. For finite dimensional Lie groups, infinite dimensional Hilbert spaces and Banach spaces of cotype 2, we have

$$C_{(2)}(G) \subset D(A)$$

(cf. [4], [6] and [8] resp.). In this paper we shall prove that the above result is also true for a class of infinite dimensional Hilbert-Lie groups. At several points we shall use ideas and techniques used in [4]. We first obtain the Taylor expansion for functions $f \in C_{(2)}(G)$. In Lemma 2.1 we prove that, for every neighborhood of e in any Hilbert-Lie group G the supremum $\sup\limits_{t \downarrow 0} \frac{1}{t}\mu_t(U^c)$ is finite. Using this result and Banach-Steinhaus Theorem, we prove Theorem 2.8.

1. Preliminaries

$\mathbf{N}$ and $\mathbf{R}$ denote the sets of positive integers and real numbers, respectively. Moreover let $\mathbf{R}_+ := \{r : r \geq 0\}$, $\mathbf{R}_+^* := \{r : r > 0\}$.

Let A be a set and B a subset of A. Then by 1_B we denote the indicator function of B. Let I be a nonvoid set and δ_{ij} the Kronecker delta $(i, j \in I)$.

By G we denote a topological Hausdorff group with identity e. G is called a *Polish group* if G is a topological group with a countable basis of its topology and with a complete left invariant metric d which induces the topology.

For every function $f : G \to \mathbf{R}$ and $a \in G$ the functions f^*, $R_a f = f_a$ and $L_a f = {}_a f$ are defined by $f^*(b) = f(b^{-1})$, $f_a(b) = f(ba)$ and ${}_a f(b) = f(ab)$ for all $b \in G$, respectively. Moreover let $\text{supp}\,(f) = \overline{\{a \in G : f(a) \neq 0\}}$ denote the support of f. By $C_u(G)$ we denote the Banasch space of all real-valued bounded left uniformly (or d-uniformly) continuous functions on G furnished with the supremum norm $\|\cdot\|$. A *Hilbert-Lie group* is a separable analytic manifold modeled on a separable Hilbert space, whose group operations are analytic. It is known that the Hilbert-Lie groups are Polish (cf. [2]).

For the exponential mapping $\mathcal{E}xp : T_e \longrightarrow G$ there exists an inverse mapping *log* from a neighborhood U_e of e onto a neighborhood N_0 of zero in T_e, where T_e is the tangential space in $e \in G$ ([5]).

By $B(G)$ we denote the σ-field of Borel subsets of G. Moreover, $\mathcal{V}(e)$ denotes the system of neighborhoods of the identity e of G which are in $B(G)$.

$\mathcal{M}(G)$ denotes the vector space of real-valued (signed) measures on $B(G)$. As is well known, $\mathcal{M}(G)$ is a Banach algebra with respect to convolution * and the norm $\|\cdot\|$ of total variation. $\mathcal{M}_+(G)$ is the set of positive measures in $\mathcal{M}(G)$ and $\mathcal{M}^1(G) = \{\mu \in \mathcal{M}(G) : \mu(G) = 1\}$ is the set of probability measures on G.

Now let $\gamma_X(t) := \mathcal{E}xp(tX)$ for $X \in H$ and $t \in \mathbf{R}^*$.

Definition 1.1. Let $f \in C_u(G)$, $X \in H$ and $a \in G$. f is called *left differentiable* at $a \in G$ with respect to X ("$Xf(a)$ exists" for short), if

$$Xf(a) := \lim_{t \to 0} \frac{1}{t} \Big[L_{\gamma_X(t)} f(a) - f(a) \Big]$$

exists. f is called *continuously left differentiable*, if $Xf(a)$ exists for all $a \in G$ and $X \in H$, and if the mappings $a \longmapsto Xf(a)$, $X \longmapsto Xf(a)$ are continuous.

Derivatives of higher orders are defined inductively. Differentiability from the right is defined by replacing $L_{\gamma_X(h)}$ by $R_{\gamma_X(h)}$.

The following properties of derivatives are well known for continuously left differentiable functions (cf. [1]).

Remark 1.2. Let f, $g \in C_u(G)$, $X \in H$ and $a \in G$,

(i) If $Xf(a)$ exists, then the mapping $X \longmapsto Xf(a)$ is linear.

(ii) If $Xf(a)$ and $Xg(a)$ exists, then $X(f \cdot g)(a)$ exists also and $X(f \cdot g)(a) = Xf(a) \cdot g(a) + f(a) \cdot Xg(a)$.

Now let $f \in C_u(G)$ be a twice continuously left differentiable function. Then the mapping

$$Df(a) : X \longmapsto Xf(a) \quad \big(D^2 f(a) : (X, Y) \longmapsto XYf(a) \big)$$

is continuous and linear (resp. symmetric, continuous and bilinear) functional on H (resp. $H \times H$) for all $a \in G$. Also

$$\langle Df(a), X \rangle = Xf(a) \quad \text{and} \quad \langle D^2 f(a)(X), Y \rangle = XYf(a)$$

for all $a \in G$ and X, $Y \in H$.

We denote by $C_2(G)$ the space of all twice continuously left differentiable functions $f \in C_u(G)$ such that the mapping $a \longmapsto D^2 f(a)$ is d-uniformly continuous and $\|Df\| := \sup_{a \in G} \|Df(a)\| < \infty$, $\|D^2 f\| := \sup_{a \in G} \|D^2 f(a)\| < \infty$. It is easy to see that the space $C_2(G)$ is a Banach space with respect to the norm

$$\|f\|_2 := \|f\| + \|Df\| + \|D^2 f\|, \quad f \in C_2(G)$$

and

$$R_a C_2(G) \subset C_2(G)$$

is satisfied for all $a \in G$. However $C_2(G)$ is not dense in $C_u(G)$ (cf. [6]). By $a_i(a) := \langle \log(a), X_i \rangle$ $(i \in \mathbf{N})$ we define maps a_i from the canonical neighborhood U_e in $\mathbf{R}$. Now we call the system $\big(a_i\big)_{i\in\mathbf{N}}$ of maps from U_e in $\mathbf{R}$ a system of canonical coordinates of G with respect to orthonormal base $\big(X_i\big)_{i\in\mathbf{N}}$, if for all $a \in U_e$ the property $a = \mathcal{E}xp\Big(\sum_{i=1}^{\infty} a_i(a)X_i\Big)$ is satisfied.

Lemma 1.3. *Let $f \in C_2(G)$. Then*

(i) $\Big(\sum_{i=1}^{\infty} a_i(a)X_i\Big)f = \sum_{i=1}^{\infty} a_i(a)X_i f$ *for all $a \in U_e$.*

(ii) $\Big(\sum_{i=1}^{\infty} a_i(a)X_i\Big)\Big(\Big(\sum_{j=1}^{\infty} a_j(c)X_j\Big)f\Big) = \sum_{i=1,j=1}^{\infty} a_i(a)a_j(c)X_iX_j f$ *for all $a, c \in U_e$.*

Proof. (i) For any $a \in U_e$ there exists a $X \in H$ with $X = \log(a)$. Then we have $X = \sum_{i=1}^{\infty}\langle X, X_i\rangle X_i = \sum_{i=1}^{\infty} a_i(a)X_i$. Thus

$$\begin{aligned} Xf(e) &= \frac{d}{dt}\Big|_{t=0} f(\gamma_X(t)) = \langle Df(e), X\rangle \\ &= \sum_{i=1}^{\infty} a_i(a)\langle Df(e), X_i\rangle = \sum_{i=1}^{\infty} a_i(a)X_i f(e). \end{aligned}$$

Now let $b \in G$ be an arbitrary point. Then $R_b f \in C_2(G)$, whence the assertion. The statement (ii) can be proved similarly. □

In the following we give the Taylor expansion for functions $f \in C_2(G)$.

Proposition 1.4. *Let $f \in C_2(G)$. Then the Taylor-expansion of second order for f at $e \in G$ is given by*

$$f(a) = f(e) + \sum_{i=1}^{\infty} a_i(a)X_i f(e) + \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i(a)a_j(a)X_iX_j f(\overline{a})$$

for all $a \in U_e$, where $\overline{a}$ is a point of U_e.

Proof. Let $f \in C_2(G)$ and $X \in H$. Then the function $\chi : t \longmapsto f(\gamma_X(t))$ is twice differentiable on $\mathbf{R}$ and therefore admits a Taylor-expansion valid up to second order:

$$\chi(t) = \chi(0) + \chi'(0)\cdot t + \frac{1}{2}\chi''(t)\cdot t^2$$

for $\bar{t} \in [-|t|, |t|]$. Since $\chi'(0) = Xf(e)$ and $\chi''(\bar{t}) = XXf(\gamma_X(\bar{t}))$ it follows from Lemma 1.3 that

$$f(\gamma_X(t)) = f(e) + \sum_{i=1}^{\infty} \langle tX, X_i \rangle X_i f(e) + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle tX, X_i \rangle \ \langle tX, X_j \rangle X_i X_j f(\gamma_X(\bar{t}))$$

for some $\bar{t} \in [-|t|, |t|]$. This yields the assertion. □

Remark 1.5. The Taylor-expansion of $f \in C_2(G)$ can be written in a closed form, i.e.

$$f(a) = f(e) + \langle Df(e), \log(a) \rangle + \frac{1}{2} \langle D^2 f(\bar{a})(\log(a)), \log(a) \rangle$$

for all $a \in U_e$ and for any $\bar{a}$ in the canonical neighborhood U_e.

2. Convolution semigroups of probability measures and the generators

For any probability measure μ on G, we define the operator T_μ on $C_u(G)$ by

$$T_\mu f := \int f_a \mu(da) \quad \text{(Bochner-Integral)}.$$

It is easy to see that $T_\mu C_u(G) \subset C_u(G)$ and $T_{\mu * \nu} = T_\mu \circ T_\nu$.

A *convolution semigroup* is a family $(\mu_t)_{t \in \mathbf{R}_+^*}$ in $\mathcal{M}^1(G)$ such that $\mu_0 = \varepsilon_e$ and $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in \mathbf{R}_+^*$. $(\mu_t)_{t \in \mathbf{R}_+^*}$ is called *continuous* if $\lim_{t \mapsto 0} \mu_t = \varepsilon_e$ (weakly). It is well known that the convolution semigroup $(\mu_t)_{t \in \mathbf{R}_+^*}$ is continuous iff the corresponding operator semigroup $(T_{\mu_t})_{t \in \mathbf{R}_+^*}$ is (strongly) continuous. Hille-Yosida theory establishes a bijection between (strongly) continuous operator semigroups $(T_{\mu_t})_{t \in \mathbf{R}_+^*}$ and their infinitesimal generators. N is defined on its domain $D(N)$ which is dense in $C_u(G)$. It is clear that N commutes with the left translations, i.e.

$$L_a D(N) \subset D(N) \quad \text{and} \quad L_a \circ N = N \circ L_a \quad \text{for all } a \in G.$$

A continuous convolution semigroup $(\mu_t)_{t \in \mathbf{R}_+^*}$ in $\mathcal{M}^1(G)$ admits a Lévy measure η, i.e. η is a σ-finite positive measure on $\mathcal{B}(G)$ such that $\eta(\{e\}) =$

0 and

$$\lim_{t\downarrow 0}\frac{1}{t}\int f d\mu_t=\int f d\eta,$$

for all $f\in C_u(G)$ with $e\notin \operatorname{supp}(f)$ (cf. [7]).

Lemma 2.1. *Let $\big(\mu_t\big)_{t\in\mathbf{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$. Then for every $U\in\mathcal{V}(e)$*

$$\sup_{t\in\mathbf{R}_+^*}\frac{1}{t}\mu_t(U^c)<\infty.$$

Proof. Let U and V be two neighborhood in $e\in G$ with $\overline{V}\subset U$. Since G is normal (as a topological space), there exists a function $f\in C_u(G)$ such that

$$0\le f\le 1,\quad f(V)=\{0\}\quad\text{and}\quad f(U^c)=\{1\}.$$

Then we have $\frac{1}{t}\mu_t(U^c)\le\frac{1}{t}\int f d\mu_t$ for all $t\in\mathbf{R}_+^*$. Since $f\in C_u(G)$ with $e\notin\operatorname{supp}(f)$ it implies that

$$\lim_{t\downarrow 0}\frac{1}{t}\int f d\mu_t=\int f d\eta.$$

Hence the assertion. □

Let H be a separable Hilbert space with a complete orthonormal system $\big(X_i\big)_{i\in\mathbf{N}}$ and G a Hilbert-Lie group on H. Moreover, let

$$H_n:=\langle\{X_1,X_2,\dots,X_n\}\rangle$$

be the space of all linear combinations of $X_1,X_2,\dots,X_n$ and $H_n^\perp$ the orthogonal complement of H_n in H (for all $n\in\mathbf{N}$). Then $H/H_n^\perp$ and H_n are isomorphic. Clearly

$$G_n:=\mathcal{E}xp(H_n^\perp)$$

is a closed subgroup of G for all $n\in\mathbf{N}$. The quotient spaces G/G_n are finite-dimensional Hilbert-Lie groups. Now let p_n be the canonical projection from G onto G/G_n and $\{b_i^n : i=1,2,\dots,n\}$ a system of canonical coordinates with respect to $\{X_1,X_2,\dots,X_n\}$. We now define the functions $d_i^n:=b_i^n\circ p_n\in C_2(G)$; then $X_jd_i^n$ exists and

$$X_jd_i^n=X_j(b_i^n\circ p_n)=X_jb_i^n\circ p_n=0$$

holds for all $j > n$ and $i = 1, 2, \dots, n$.

Definition 2.2. Let G be a Hilbert-Lie group on H, and $\big(X_i\big)_{i\in\mathbf{N}}$ an orthonormal basis in H. For any $n \in \mathbf{N}$ we define

$$C_{(2),n}(G) := \Big\{ f \in C_2(G) : X_i f = 0 \text{ for all } i > n \text{ and } X_i X_j f = 0 \text{ for all } i > n \text{ or } j > n \Big\}.$$

Remark 2.3. Let $f \in C_u(G)$ be a left uniformly differentiable function with respect to X which satisfies the condition $X_i f = 0$ for all $i > n$ ($n \in \mathbf{N}$). Let π_n be the orthogonal projection from H onto H_n. Then we have

$$Xf = \pi_n(X) f \quad \text{for all} \quad X \in H.$$

So f is continuously left differentiable and clearly $\big(C_{(2),n}(G)\big)_{n\in\mathbf{N}}$ is a monotonic increasing sequence of Banach subalgebras of the Banach algebra $C_2(G)$. Further properties of $C_{(2),n}(G)$ ($n \in \mathbf{N}$):

(i) $C_{(2),n}(G)$ are $\|\cdot\|_2$-closed in $C_2(G)$

and

(ii) For any probability measure $\mu \in \mathcal{M}^1(G)$, we have

$$T_\mu C_{(2),n}(G) \subset C_{(2),n}(G) \quad \text{for all } n \in \mathbf{N}.$$

Thus, $\overline{C_{(2),n}(G) \cap D(N)}^{\|\cdot\|_2} = C_{(2),n}(G)$. Now consider the subspace

$$C_{(2)}(G) := \bigcup_{n\in\mathbf{N}} C_{(2),n}(G).$$

$C_{(2)}(G)$ is obviously a linear subspace of $C_2(G)$ with $T_\mu C_{(2)}(G) \subset C_{(2)}(G)$ for probability measures $\mu \in \mathcal{M}^1(G)$. Especially $\overline{C_{(2)}(G)}^{\|\cdot\|_2}$ is a Banach space with $T_\mu \overline{C_{(2)}(G)}^{\|\cdot\|_2} \subset \overline{C_{(2)}(G)}^{\|\cdot\|_2}$.

Definition 2.4. For $n \in \mathbf{N}$ let $\{b_i^n : i = 1, 2, \dots, n\}$ be a system of extended canonical coordinates with respect to $\{X_1, X_2, \dots, X_n\}$. Then we say that the Hilbert-Lie group G *has the property* (K), if

$$b_i^n \in C_{(2),n}(G) \quad \text{for all } i = 1, 2, \dots, n, \quad n \geq n_0,$$

and for any $n_0 \in \mathbf{N}$.

Every commutative Hilbert-Lie group and every finite dimensional Lie group have clearly the property (K). In the finite dimensional case we have $n_0 = \dim(G)$. Since $C_{(2),n}(G) \subset C_{(2),n+1}(G)$, a system $\{b_i^n, b_{n+1}^{n+1} : i = 1, 2, \ldots, n\} \subset C_{(2),n+1}(G)$ of canonical coordinates exists with respect to $\{X_1, X_2, \ldots, X_{n+1}\}$. We also have the following

Proposition 2.5. *Let G be a Hilbert-Lie group with the property (K). Then a system $(d_n)_{n\in\mathbf{N}}$ of functions in $C_{(2)}(G)$ exists with*

$$d_i = b_i^{n_0} \quad \text{for all } i = 1, 2, \ldots, n_0$$

and

$$d_n = b_n^n \quad \text{for all } n > n_0.$$

This system $(d_n)_{n\in\mathbf{N}}$ is called a system of local canonical coordinates with respect to $(X_i)_{i\in\mathbf{N}}$.

Now let G be a Hilbert-Lie group with the property (K). We define for any $n \in \mathbf{N}$ the functions

$$\Phi_n(a) := \sum_{i=1}^{n} d_i(a)^2, \quad a \in G,$$

where $(d_i)_{i=1,2,\ldots,n}$ is a system of local canonical coordinates with respect to $\{X_1, X_2, \ldots, X_n\}$. Then $\Phi_n \in C_{(2),n}(G)$ and $\Phi_n(a) > 0$ for all $a \in G \setminus \{\Phi_n = 0\}$. Therefore

$$X_i\Phi_n(e) = 0, \quad X_iX_j\Phi_n(e) = 2\delta_{ij}, \quad i, j = 1, 2, \ldots, n$$

(cf. [3], Lemma 4.1.9 and 4.1.10).

The following lemma is a consequence of Banach-Steinhaus Theorem and Hille-Yosida theory (cf. [3], Lemma 4.1.11).

Lemma 2.6. *For every $f \in C_{(2),n}(G)$ and every $\varepsilon > 0$ there is a $g := g_e \in C_{(2),n}(G) \cap D(N)$ such that $\|f - g\|_2 < \varepsilon$, $f(e) = g(e)$, $X_if(e) = X_i(g(e)$ and $X_iX_jf(e) = X_iX_jg(e)$ for $i, j = 1, 2, \ldots, n$.*

Proposition 2.7. *Let G be a Hilbert-Lie group with the property (K), $(\mu_t)_{t\in\mathbf{R}_+^*}$ a convolution semigroup in $\mathcal{M}^1(G)$ and Φ_n, $(n \in \mathbf{N})$ be as above. Then the suprema*

$$\sup_{t\in\mathbf{R}_+^*} \frac{1}{t} \int \Phi_n d\mu_t$$

are finite for every $n \in \mathbf{N}$.

Proof. An application of Lemma 2.6 to the function $\Phi_n \in C_{(2),n}(G)$ yields the existence of a function $\Psi_n \in C_{(2),n}(G) \cap D(N)$ with the property

$$\|\Phi_n - \Psi_n\|_2 < \varepsilon, \quad \Psi_n(e) = \Phi_n(e) = 0, \quad X_i\Psi_n(e) = X_i\Phi_n(e) = 0$$
$$\text{and } X_iX_j\Psi_n(e) = X_iX_j\Phi_n(e) = 2\delta_{ij}, \quad i,j = 1,2,\dots,n.$$

The Taylor expansion of $\Psi_n \in C_{(2),n}(G) \cap D(N)$ in a neighborhood W_1 of e with $W_1 \subset U_e$ has the form

$$\Psi_n(a) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_i(a)d_j(a)X_iX_j\Psi_n(\overline{a}),$$

for all $a \in W_1$ with $\overline{a} \in W_1$. Since $\|\Phi_n - \Psi_n\|_2 < \varepsilon$ and $X_iX_j\Psi_n(e) = 2\delta_{ij}$, $i,j = 1,2,\dots,n$ there exists a neighborhood W_2 of e with the properties

$$-\varepsilon \le X_iX_j\Psi_n(a) \le \varepsilon \quad \text{for all } i,j = 1,2,\dots,n,\ i \neq j,$$
$$2-\varepsilon \le X_iX_i\Psi_n(a) \le 2+\varepsilon \quad \text{for all } i = 1,2,\dots,n,$$

whenever $a \in W_2$. Putting $\delta_n := \delta_n(e) := \frac{1}{2}(2 - \varepsilon - \varepsilon(n-1))$ and $W := W_1 \cap W_2$, we obtain

$$\Psi_n(a) \ge \delta_n \cdot \sum_{i=1}^{n} d_i(a)^2 \quad \text{for all } a \in W.$$

Since $\Psi_n \in C_{(2),n}(G) \cap D(N)$, we obtain $\sup_{t\in\mathbf{R}_+^*} \frac{1}{t}\Big|\int_W \Psi_n d\mu_t\Big| < \infty$ by Lemma 2.1. Thus $\sup_{t\in\mathbf{R}_+^*} \frac{1}{t}\int_W \Phi_n d\mu_t < \infty$, and since Φ_n is bounded, the assertion follows from Lemma 2.1. □

Now let G be a Hilbert-Lie group with the property (K) and $\big(d_i\big)_{i\in\mathbf{N}}$ a system of local canonical coordinates with respect to $\big(X_i\big)_{i\in\mathbf{N}}$. By Lemma 2.6 there exist functions $z_i \in C_{(2),n}(G) \cap D(N)$, $(n \in \mathbf{N})$ with the property

$$z_i(e) = d_i(e) = 0, \quad X_jz_i(e) = X_jd_i(e) = \delta_{ij}, \quad i,j = 1,2,\dots,n.$$

Theorem 2.8. *Let G be a Hilbert-Lie group with the property (K) and $\big(\mu_t\big)_{t\in\mathbf{R}_+^*}$ a convolution semigroup in $\mathcal{P}(G)$. Then the generating function A of $\big(\mu_t\big)_{t\in\mathbf{R}_+^*}$ on $C_{(2)}(G)$ exists, i.e.*

$$C_{(2)}(G)\subset D(A).$$

Proof. Let $f\in C_{(2),n}(G)$ $(n\in\mathbf{N})$ and setting

$$g(a):=f(a)-f(e)-\sum_{i=1}^{n}z_i(a)\cdot X_if(e)\quad\text{for all } a\in G,$$

where the function z_i, $i=1,2,\ldots,n$ are as in above. Then $g\in C_{(2),n}(G)$ with $g(e)=0$, $X_jg(e)=X_jf(e)-\sum\limits_{i=1}^{n}X_jz_i(e)\cdot X_if(e)=X_jf(e)-\sum\limits_{i=1}^{n}\delta_{ij}\cdot X_if(e)=0$. The Taylor expansion of g in a neighborhood $W\subset U_e$ gives

$$g(a)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}d_i(a)d_j(a)X_iX_jg(\overline{a}),\quad a\in W.$$

Thus there is a constant $k_1\in\mathbf{R}_+^*$ such that

$$|g(a)|\leq k_1\cdot\|g\|_2\cdot\Phi_n(a)\quad\text{for all } a\in W.$$

It follows from Proposition 2.7 that

$$\sup_{t\in\mathbf{R}_+^*}\Big|\frac{1}{t}\int\limits_{W}gd\mu_t\Big|\leq k_1\cdot\|g\|_2\cdot\sup_{t\in\mathbf{R}_+^*}\int\Phi_nd\mu_t<\infty. \tag{1}$$

Clearly, $\Big|\frac{1}{t}\int\limits_{W^c}gd\mu_t\Big|\leq\|g\|_2\cdot\frac{1}{2}\mu_t(W^c)$, and $\sup\limits_{t\in\mathbf{R}_+^*}\Big|\frac{1}{t}\int\limits_{W^c}gd\mu_t\Big|<\infty$. Hence, there exists a constant $k_2\in\mathbf{R}_+^*$ independent of t such that

$$\Big|\frac{1}{t}\int\limits_{W^c}gd\mu_t\Big|\leq k_2\cdot\|g\|_2\quad\text{for all } t\in\mathbf{R}_+^*. \tag{2}$$

Adding the inequalities (1) and (2)

$$\Big|\frac{1}{t}\big[T_{\mu_t}f(e)-f(e)\big]-\frac{1}{t}\sum_{i=1}^{n}X_if(e)\cdot T_{\mu_t}z_i(e)\Big|\leq k_3\cdot\|f\|_2,\quad\forall t\in\mathbf{R}_+^*,$$

where k_3 is a constant (independent of t). Since $z_i \in D(N)$ and $z_i(e) = 0$, we have $\sup_{t\in\mathbf{R}_+^*} \left|\frac{1}{t} T_{\mu_t} z_i(e)\right| < \infty$ for all $i = 1, 2, \ldots, n$. Hence we obtain a constant $k(n) \in \mathbf{R}_+^*$ depending only on n such that

$$\left|\frac{1}{t}\big(T_{\mu_t} f(e) - f(e)\big)\right| \leq k(n) \cdot \|f\|_2$$

for all $t \in \mathbf{R}_+^*$ and $f \in C_{(2),n}(G)$. By Theorem of Banach-Steinhaus the limit

$$\lim_{t\downarrow 0} \frac{1}{t}\big[T_{\tilde{\mu}_t} f(e) - f(e)\big]$$

exists for every $f \in C_{(2)}(G)$. □

Remark 2.9. Let G be a *commutative* Hilbert-Lie group and $\big(\mu_t\big)_{t\in\mathbf{R}_+^*}$ a convolution semigroup in $\mathcal{M}^1(G)$. As in the proof of Proposition 2.8, we can find a constant $k(n) \in \mathbf{R}_+^*$ (independent of $a \in G$ and $t \in \mathbf{R}_+^*$) such that

$$\begin{aligned}\left|\frac{1}{t}\big(T_{\mu_t} f(a) - f(a)\big)\right| &= \left|\frac{1}{t}\big[T_{\mu_t}(L_a f)(e) - (L_a f)(e)\big]\right| \\ &\leq k(n) \cdot \big\|L_a f\big\|_2 = k(n) \cdot \|f\|_2,\end{aligned}$$

for all $f \in C_{(2),n}(G)$ and $a \in G$. Banach-Steinhaus Theorem now yields the existence of the limit

$$Nf(a) = \lim_{t\downarrow 0} \frac{1}{t}\big[T_{\mu_t} f(a) - f(a)\big]$$

uniformly in $a \in G$. This implies the existence of the infinitesimal generator N on $C_{(2)}(G)$.

Remark 2.10. Let $G = H$ be a separable Hilbert space and $C_u^{(2)}(H)$ the space of all twice Fréchet differentiable functions $f \in C_u(H)$ such that $\|f'\| := \sup_{x\in H} \|f'(x)\| < \infty$, $\|f''\| := \sup_{x\in H} \|f''(x)\| < \infty$ and f'' is uniformly continuous in x. Then we have $C_u^{(2)}(H) \subset D(N)$ (cf. [6]) and $C_2(H) = C_u^{(2)}(H)$.

Acknowledgement

The author would like to thank Professor E. Siebert for several helpful discussions and comments.

References

1. H. Boseck and G. Czichowski, *Grundfunktionen und verallgemeinerte Funktionen auf topologischen Gruppen I*, Math. Nach. **58** (1973), 215-240.

2. E. Coşkun, *Faltungshalbgruppen von Wahrscheinlichkeitsmaßen auf einer Hilbert-Lie-Grouppe*, Dissertation der Mathematischen Fakultät der Universität Tübingen, 1991.

3. H. Heyer, *Probability Measures on Locally Compact Groups*, Springer, Berlin-Heidelberg-New York, 1977.

4. G. A. Hunt, *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264-293.

5. M. Maissen, *Lie-Gruppen mit Banachräumen als Parameterräume*, Acta Math. **108** (1962), 229-270.

6. J. D. Samur, *On semigroup of convolution operators in Hilbert space*, Pacific J. Math. **115** (1984), 463-479.

7. E. Siebert, *Jumps of stochastic processes with values in a topological group*, Prob. Math. Statist. **5** (1985), 197-209.

8. T. Zak, *A representation of infinitesimal operators of semigroups of measures on Banach spaces of cotype 2*, Bull. Pol. Acad. Scien. **31** (1983), 71-74.

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