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APPROXIMATE CONTROLLABILITY WITH POSITIVE CONTROLS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

ABSTRACT. In this paper, controllability of the linear discrete-time systems (A,B,Ω):$x_{k+1}=Ax_k+Bu_k$, $x_k \in X$, $u_k \in \Omega$, is studied, where X is a Banach space and the control set Ω is assumed to be a cone in a Banach space U. Some criteria for approximate controllability are given. The case where the operator A is compact is examined in detail by using the spectral decomposition of the state space X. As a result, a criterion for approximate controllability of (A,B,Ω) is obtained without imposing a restrictive condition that the system with no control constraints (A,B,U) is exactly controllable. The obtained results are then applied to consider the problem of controllability for linear functional differential equations with positive controls. Some necessary and sufficient conditions of approximate controllability to the state space $\mathbf{R}^n \times L_p$ are presented and some illustrating examples are given.

1. INTRODUCTION

Consider the following linear discrete-time system (A, B, Ω)

$$x_{k+1} = Ax_k + Bu_k, \ x_k \in X, \tag{1.1}$$

$$u_k \in \Omega \subset U, \tag{1.2}$$

where X, U are real Banach spaces, $A : X \to X$ and $B : U \to X$ are linear bounded operators, and Ω is a non-empty subset of U.

Infinite-dimensional systems of type (1.1) have been for many years an object of research, see, e.g. [10, 25, 17, 28]. These investigations are motivated not only by the interest in obtaining theoretical results. It has been shown, for example, in [25], [28], that difference equations in abstracts spaces provide an efficient model for studying some qualitative properties, such as stability, observability, controllability, for systems

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described by functional differential equations or abstract evolution equations. The problem of controllability for the system (1.1) with restrained controls (1.2) has been studied, in the abstract setting, by this author in [19], where some criteria for local controllability have been proved. In [20] and [21] the property of null-controllability of the system has been considered. In [22] we give an application of this abstract "discrete-time" approach: some controllability criteria for infinite-dimensional discrete-time systems are used to obtain the corresponding results for periodic systems of ordinary differential equations with restrained controls. Note that our investigations have so far been concerned mainly with the property of exact controllability which is known as quite a restrictive concept. It has been proved, for example, that evolution systems with compact semigroups in infinite-dimensional spaces are never exactly controllable in finite time, [33]. On the other hand, it is well-known, e.g [30], that functional differential equations can be treated as abstract evolution equations generating a C_0-semigroup of operators $S(t)$ which become compact for t large enough. This situation suggests that in order to make the discrete-time approach more applicable to systems of functional differential equations, the concept of approximate controllability of (A, B, Ω) rather than that of exact controllability must be considered. This, in fact, is the object of the present paper.

The main purpose of this paper is to establish some verifiable criteria of approximate controllability for the class of discrete-time systems (A, B, Ω) with a compact operator A and with a control set Ω being a cone. We shall then make use of the obtained criteria to examine the controllability in the state space $M_p = \mathbf{R}^n \times L_p(-h, 0, \mathbf{R}^n)$ of functional differential equations of the form

$$\dot{z}(t) = \int_{-h}^{0} d\eta(\theta) z(\theta + t) + B_0 u(t), \quad z(t) \in \mathbf{R}^n,$$
$$u(t) \in \Omega \subset \mathbf{R}^m \quad \text{a.e.} \quad t \geq 0,$$

where Ω is a cone. We note that the problem of approximate controllability of the above functional differential equations was treated in [23] by another approach, based on the C_0-semigroup theory, and some criteria of approximate controllability have been obtained. It will be shown in this paper that the discrete-time theory not only enables us to obtain the stronger results but also provides us with an insight into the problem.

We list some notations used in this paper. Let $\mathbf{R}$ and $\mathbf{C}$ denote the fields of real and complex numbers, respectively. The symbol $\mathbf{R}^n$ will denote

n-dimensional Euclidean space. If X and Y are Banach spaces then we shall denote by $L(X, Y)$ the Banach space of all bounded linear operators from X into Y, and by $L(X)$ the space $L(X, X)$. The identity operator in $L(X)$ is denoted by I and the spectrum of $A \in L(X)$ is denoted by $\sigma(A)$. Let X be the topological dual of X and $\langle ., . \rangle$ be the duality pairing. If $A \in L(X, Y)$ then $A^* \in L(Y^*, X^*)$ is the adjoint operator of A. It is known that $\sigma(A) = \sigma(A^*)$. The symbol Ker A and Im A will stand for the null space and the range of A, respectively. Now, let M be a subset of a Banach space X. We denote by span M the linear manifold spanned by M; the convex hull, the interior and the closure of M are denoted by co M, int M and cl M, respectively. The negative polar cone of M is defined by

$$M^0 = \{f \in X^* : \langle f, x \rangle \leq 0, \quad \forall x \in M\}.$$

It is clear that $M^0 = (\text{co } M)^0 = (\text{cl } M)^0$. Besides, if M, N are subset of X containing 0 then $(M + N)^0 = (M \cup N)^0 = M^0 \cap N^0$. We say that M is a cone if $\lambda M \subset M$, $\forall \lambda \geq 0$. Finally, the unit ball in X is denoted by S_1 and the set of nonnegative integers is denoted by $\mathbf{N}$.

2. Preliminaries

For a given $k \in \mathbf{N}$, let R_k denote the reachable set at time k of the system (A, B, Ω), i.e. $R_0 = \{0\}$,

$$R_k = \sum_{i=1}^{k} A^{k-i} B\Omega, \quad k \geq 1, \tag{2.1}$$

and let

$$R_\infty = \bigcup \{R_k, \quad k \in \mathbf{N}\}. \tag{2.2}$$

The system (A, B, Ω) is said to be approximately controllable if cl $R_\infty = X$ and exactly controllable if $R_\infty = X$.

In [19] the following criterion of exact controllability is given.

Theorem 2.1. *Suppose that Ω is a convex cone in U with nonempty interior. Then the system (A, B, Ω) is exactly controllable if and only if* (i) *the corresponding system with unconstrained controls (A, B, U) is exactly controllable, or, equivalently,*

$$\exists m \in \mathbf{N} : \quad \text{span } \{BU, ABU, ..., A^{m-1}BU\} = X, \tag{2.3}$$

and (ii)

$$\text{Ker}\,(\lambda I^* - A^*) \cap (B\Omega)^0 = \{0\}, \quad \forall \lambda \geq 0. \tag{2.4}$$

The proof of the above result based on the theorem due to Krein and Rutman concerning properties of a convex cone with nonempty interior invariant under a linear bounded operator. The convexity of Ω and the condition int $\Omega \neq \emptyset$ are necessary to ensure that R_∞ is convex set with nonempty interior in X. It turns out that if one replaces the convexity of Ω by the weaker condition that cl $B\Omega$ is convex, the conditions (2.3) and (2.4) are still sufficient for approximate controllability. More precisely, we have the following.

Theorem 2.2. *Suppose that Ω is a subset of U with nonempty interior and cl $B\Omega$ is a convex cone. If the conditions* (2.3) *and* (2.4) *are satisfied then the system* (A, B, Ω) *is approximately controllable.*

Proof. We give only a sketch since the proof is similar to that of Theorem 2.1 presented in [19]. From the condition int $\Omega \neq \emptyset$ and (2.3) it follows that int $R_\infty \neq \emptyset$. Next, since, clearly,

$$\text{cl } R_\infty = \text{cl } \bigcup \{\text{cl } R_k, k \in \mathbf{N}\} \text{ and cl } R_k = \text{ cl } (\sum_{i=1}^{k} A^{k-i} \text{ cl } (B\Omega)),$$

we conclude that cl R_∞ is a convex cone with nonempty interior. Moreover, since $AR_\infty \subset R_\infty$, it follows that cl R_∞ is invariant under A. Assume to the contrary that cl $R_\infty \neq X$. This implies, by the Krein-Rutman theorem, that there exist a non-zero $f \in$ (cl $R_\infty)^0$ and $\lambda \geq 0$ such that $A^* f = \lambda f$. Since $B\Omega \subset R_\infty$ it follows $f \in (B\Omega)^0$ and, thus, $f \in$ Ker $(\lambda I^* - A^*) \cap (B\Omega)^0$, contradicting (2.4). This completes the proof.

Theorem 2.2, however, is not convenient for applications since, as noted above, the condition (2.3) of exact controllability is not fulfilled for many systems of practical importance. The question arises whether one can replaces the condition (2.3) in Theorem 2.2 by the following weaker condition

$$\text{cl span } \{BU, ABU, ...\} = X. \tag{2.5}$$

The following simple example shows, however, that (2.4) and (2.5), in general, are not sufficient conditions for approximate controllability. Let $X = l_2$, $U = \mathbf{R}^1$, $x = (\xi_1, \xi_2, ...) \in X$, A is the left shift operator: $A(\xi_1, \xi_2, ...) = (\xi_2, \xi_3, ...)$, B is defined by $Bu = bu$, where $b = (1,\ 1/2,$

$1/4, ...)$ and $\Omega = \{u \in \mathbf{R}^1 : u \geq 0\}$. Since A^* has no eigenvectors, the condition (2.4) is satisfied. Besides, it is easily seen that that cl span $\{b, Ab, A^2b, ...\} = X$ and hence (2.5) is also fulfilled. However, since $R_\infty \subset l_2^+$ (the cone of all nonnegative vectors in l_2), the system under consideration is not approximately controllable.

We shall see later that for a class of discrete-time systems associated with functional differential equations, one can actually replace the condition (2.3) by the following inclusion

$$\exists m \in \mathbf{N} : \mathrm{Im}\ A^m \subset \ \mathrm{cl\ span}\ \{BU, ABU, ..., A^{m-1}BU\}, \tag{2.6}$$

which is clearly weaker than (2.5), but in combining with (2.4), forms a necessary and sufficient condition for approximate controllability of the system (A, B, Ω).

First we prove the following simple characterization of approximate controllability.

Lemma 2.3. *Suppose that cl $B\Omega$ is a convex cone. Then the system (A, B, Ω) is approximately controllable if and only if*

$$\bigcap \{(A^k B\Omega)^0, \quad k \in \mathbf{N}\} = \{0\}. \tag{2.7}$$

Proof. Obviously, cl R_∞ is a convex cone. Further, it is readily verified that $(\mathrm{cl}\ R_\infty)^0 = \bigcap\{(A^k B\Omega)^0,\ k \in \mathbf{N}\}$. The assertion now follows immediately by the separation principle for convex cones (see, e.g., Theorem 10, p. 458, [9]).

Since $f \in (A^k B\Omega)^0$ iff $A^{*k} f \in (B\Omega)^0$, we can reformulate the above result in the form of the control-observation duality as follows.

Corollary 2.4. *The system (A, B, Ω) is approximately controllable iff the dual system*

$$f_{k+1} = A^* f_k, \tag{2.8}$$

$$f_k \in (B\Omega)^0 \subset X^* \tag{2.9}$$

is observable (i.e. the only solution of (2.8) *remaining in the observation domain $(B\Omega)^0$ for all $k \in \mathbf{N}$ is $f = 0$).*

Denote

$$H = \ \mathrm{span}\ \{BU, ABU, ...\}. \tag{2.10}$$

Approximate controllability of (A, B, U) is equivalent to that cl $H = X$. For the system with unconstrained controls (A, B, U) we have the following simple criterion of approximate controllability.

Theorem 2.5. *The system (A, B, U) is approximately controllable if and only if*

$$\exists m \in \mathbf{N} \ : \ \text{Im } A^m \subset \ \text{cl } H \tag{2.11}$$

and

$$\text{Ker } A^* \cap \ \text{Ker } B^* = \{0\}. \tag{2.12}$$

Proof. Suppose cl $H = X$, but there exists a nonzero $f \in X^*$ such that $A^* f = B^* f = 0$. Then, since $B^* A^{*i} f = 0$, $\forall i \in \mathbf{N}$ we have, for every $x \in H$, $\langle f, x\rangle = \sum\limits_{i=1}^{k} \langle f, A^{k-i} B u_i\rangle = \sum\limits_{i=1}^{k} \langle B^* A^{*k-i} f, u_i\rangle = 0$ which is impossible. For the sufficiency, suppose (11) and (2.12) hold. Let $f \in X^*$ such that

$$\langle f, x\rangle = 0, \quad \forall x \in H. \tag{2.13}$$

Since H is invariant under $A : AH \subset H$, (2.13) implies $\langle f, A^i x\rangle = 0$, $\forall x \in H$, $\forall i \in \mathbf{N}$, or

$$\langle A^{*i} f, x\rangle = 0, \quad \forall x \in H, \ \forall i \in \mathbf{N}. \tag{2.14}$$

On the other hand, in combination with (11), (2.13) yields $\langle f, A^m x\rangle = \langle A^{*m} f, x\rangle = 0$, $\forall x \in X$, which implies $f \in$ Ker A^{*m}. Denote $g_k = A^{*k} f$ for $k = 0, 1, ..., m$. We have $g_{m-1} \in$ Ker A^*. Since $BU \subset H$, it follows from (2.14) that $\langle g_{m-1}, Bu\rangle = 0$, $\forall u \in U$, or, equivalently, $g_{m-1} \in$ Ker B^*. But then, by (2.12), $g_{m-1} = 0$. By induction, we can prove that $g_k = 0$ for $k = 0, 1, ..., m$. In particular, we get $g_0 = f = 0$. This shows that cl $H = X$ and concludes the proof.

We now recall some properties of the spectral decomposition for compact operators and prove a useful formula for the range of a compact operator whose adjoint satisfies the so-called small solution condition.

It is well-known that if A is a compact operator then A^* is also a compact operator and the spectrum $\sigma(A)$ is a countable set with no accumulation point different from zero. Each nonzero $\lambda \in \sigma(A)$ is an eigenvalue with a finite multiplicity k_λ and λ is also an eigenvalue of A^* with the same multiplicity. It follows that for every $r > 0$, the set $\Lambda = \{\lambda \in \sigma(A) : |\lambda| > r\}$

consists of a finite number of eigenvalues and the spaces X can be decomposed into the direct sum of two A-invariant subspaces

$$X = P_r \oplus Q_r \tag{2.15}$$

where

$$P_r = \oplus\{\text{Ker } (\lambda I - A)^{k_\lambda},\ \lambda \in \Lambda\}, \quad Q_r = \bigcap\{\text{Im } (\lambda I - A)^{k_\lambda},\ \lambda \in \Lambda\}.$$

The subspace P_r is finite-dimensional and, if we denote by A_P and A_Q the restrictions of A to P_r and Q_r, respectively, then $\sigma(A_P) = \Lambda$ and $\sigma(A_Q) = \sigma(A) \setminus \Lambda$. It follows, in particular, that A_P is invertible and

$$\sigma(A_P^{-1}) = \{\lambda^{-1} : \lambda \in \Lambda\}. \tag{2.16}$$

From the definition, it is clear that

$$r_1 \leq r_2 \quad \Rightarrow \quad P_{r_2} \subset P_{r_1} \ \text{ and } \ Q_{r_1} \subset Q_{r_2}. \tag{2.17}$$

Note that the similar decomposition property holds also for A^*. Let $r > 0$ be chosen so that $\{\lambda \in \mathcal{C} : |\lambda| = r\} \cap \sigma(A) = \emptyset$ and let $\Lambda = \{\lambda \in \sigma(A) : |\lambda| > r\}$. From the properties of $\sigma(A)$ we can choose $\delta > 0$ small enough such that $\sigma(A_Q) \subset \{\mu \in \mathcal{C} : |\mu| < r - \delta\}$ and $\Lambda = \sigma(A_P) \subset \{\mu \in \mathcal{C} : |\mu| > r + \delta\}$. Using the Gelfand-Beurling spectral radius formula

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \max\{|\lambda| : \lambda \in \sigma(A)\}, \tag{2.18}$$

one can show easily that there exists a number $N \in \mathbf{N}$ such that

$$\|A_Q^k x_Q\| \leq (r - \delta)^k \|x_Q\|, \quad \forall x_Q \in Q_r,\ \forall k \geq N \tag{2.19}$$

and

$$\|A_P^{-k} x_P\| \leq (r - \delta)^{-k} \|x_P\|, \quad \forall x_P \in P_r,\ \forall k \geq N. \tag{2.20}$$

This implies, in particular, that $\|(A/r)^k x\| \to 0$ as $k \to \infty$ for any $x \in Q_r$. Now, let us define the set

$$X_\infty = \{x \in X : \lim_{k\to\infty} \|(A/r)^k x\| = 0,\ \forall r > 0\}. \tag{2.21}$$

It is clear, that $\bigcup\{\text{Ker } A^i,\ i \in \mathbf{N}\} \subset X_\infty$ and X_∞ is a linear subspace invariant under A. We introduce the following (see, e.g. [27]).

Definition. We say that $A \in L(X)$ satisfies the small solution condition (SSC) if there exists $j \in \mathbf{N}$ such that

$$X_\infty = \text{Ker } A^j. \tag{2.22}$$

We remark that the (SSC) means that each solution of the difference system $x_{k+1} = Ax_k$ which tends to zero more rapidly than any r^k as $k \to \infty$ ("small solution") must vanish identically after a certain finite time. The following lemma is a discrete-time analog to the known results due to Henry [11] and Manitius [14].

Lemma 2.6. *Let $A \in L(X)$ be a compact operator such that A^* satisfies the (SSC). Then there exists $i_0 \in \mathbf{N}$ such that*

$$\text{cl Im } A^{i_0} = \text{cl span}\{P_r,\ r > 0\}, \tag{2.23}$$

where P_r is defined by (2.15) *for each* $r > 0$.

Proof. Since $P_r \subset \text{Im } A^k$, $\forall r \geq 0$, $\forall k \in \mathbf{N}$, it suffices to show that for some $i_0 \in \mathbf{N}$ the inclusion cl Im $A^{i_0} \subset$ cl span $\{P_r,\ r > 0\}$ holds. To this end, we choose a sequence of numbers $\{r_i,\ i \in \mathbf{N}\}$ such that $0 < r_{i+1} < r_i$, $\forall i \in \mathbf{N}$, and $r_i \to 0$ as $i \to \infty$, and $\{\mu \in \mathbf{C} : |\mu| = r_i\} \cap \sigma(A^*) = \emptyset$. By virtue of the property (2.17) and (2.19) we have

$$\bigcap\{Q'_r,\ r > 0\} = \bigcap\{Q'_{r_i},\ i \in \mathbf{N}\} \subset X^*_\infty \tag{2.24}$$

(where X^*_∞ is defined as (2.21) with A replaced by A^*). Now, suppose $\langle f, x\rangle = 0, \forall x \in P_r$, $\forall r > 0$. This implies $\langle f, x\rangle = 0, \forall x \in$ Ker $(\lambda I - A)^{k_\lambda}$, $\forall \lambda \in \sigma(A)$, $\lambda \neq 0$. Since Im $(\lambda I^* - A^*)^{k_\lambda}$ is closed, it follows that $f \in$ Im $(\lambda I^* - A^*)^{k_\lambda}$, $\forall \lambda \in \sigma(A)$, $\lambda \neq 0$, or, equivalently, $f \in \cap\{Q'_r,\ r > 0\}$. In view of (2.24), we have that $f \in X^*_\infty$ and therefore, $f \in$ Ker A^{*i_0} for some $i_0 \in \mathbf{N}$, since A^* satisfies (SSC). The assertion is now immediate.

3. Criteria of approximate controllability of linear discrete-time systems with positive controls

The following assertion plays a crucial role in our main theorem. The continuous-time counterpart of this result was given in [23]

Lemma 3.1. *Suppose $A \in L(X)$ is a compact operator and* cl $B\Omega$ *is a closed convex cone in X. Let $r > 0$ and π_P denote the projection operator onto the invariant subspace P_r defined by* (2.15). *Let R_∞ be the reachable set of the discrete-time system (A, B, Ω). If $\pi_P(\text{cl } R_\infty) = P_r$, then*

$$P_r \subset \text{ cl } R_\infty. \tag{3.1}$$

Proof. By the definition, $P_r = \pi_P X$ and π_P is completely characterized by the spectral set Λ, $\Lambda = \{\lambda \in \sigma(A) : |\lambda| > r\}$. Since Λ remains clearly unchanged when one replaces r by $r+\varepsilon$ with $\varepsilon > 0$ small enough, we can assume, without loss of generality, that $\{\mu \in \mathcal{C} : |\mu| = r\} \cap \sigma(A) = \emptyset$. Therefore, one can choose $\delta > 0$ to ensure that (2.19) and (2.20) hold for the spectral decomposition $X = P_r \oplus Q_r$. Next, since cl R_∞ is a closed convex cone in X and $\pi_P : X \to P_r$ is an open linear continuous operator, by Proposition 2.4 in [23], there exists $\gamma > 0$ such that

$$\forall x_P \in P_r,\ \exists \hat{x} \in \text{ cl } R_\infty : \|\hat{x}\| \ \le \|x_P\|/\gamma. \tag{3.2}$$

Let π_Q be the projection operator onto Q_r along P_r (i.e. $\pi_Q = I - \pi_P$) and let $x_Q = \pi_Q \hat{x}$, then $x_P + x_Q = \hat{x}$ and we obtain, by (3.2),

$$\|x_Q\| \le \|\pi_Q\|\ \|x_P\|/\gamma. \tag{3.3}$$

Now, given any $x_P^0 \in P_r$ and any $\varepsilon > 0$, we choose $m \in \mathbf{N}$ such that $m > N$ (where N is chosen to ensure (2.19) and (2.20)) and

$$(r-\delta)^m \|\pi_Q\|\ \|x_P^0\|/\gamma(r+\delta)^m < \varepsilon/2. \tag{3.4}$$

Putting $x_P^1 = A_P^{-m} x_P^0$ (recall that the operator $A_P : P_r \to P_r$ is invertible) then, by (3.2) and (3.3), one can find $x_Q^1 \in Q_r$ such that $\hat{x}^1 \triangleq x_P^1 + x_Q^1 \in$ cl R_∞ and

$$\|x_Q^1\| \le \|\pi_Q\|\ \|x_P^1\|/\gamma. \tag{3.5}$$

Since $\hat{x}^1 \in$ cl R_∞, there exists $x^\varepsilon \in R_\infty$ such that

$$\|x^\varepsilon - \hat{x}^1\| < \varepsilon/2\|A^m\|. \tag{3.6}$$

Defining $x = A^m x^\varepsilon$ we have $x \in A^m R_\infty \subset R_\infty$. On the other hand, by virtue of (2.19), (2.20) and (3.3)-(3.5), we can write

$$\begin{aligned}
\|x - x_P^0\| &\le \|x - x_P^0 - A^m x_Q^1\| + \|A^m x_Q^1\| \\
&= \|A^m x^\varepsilon - A^m x_P^1 - A^m x_Q^1\| + \|A^m x_Q^1\| \\
&\le \|A^m\|\ \|x^\varepsilon - x_P^1 - x_Q^1\| + (r-\delta)^m \|x_Q^1\| \\
&\le \varepsilon/2 + (r-\delta)^m \|\pi_Q\|\ \|x_P^1\|/\gamma \\
&\le \varepsilon/2 + (r-\delta)^m \|\pi_Q\|\ \|x_P^0\|/\gamma(r+\delta)^m < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So, given any $x_P^0 \in P_r$ and $\varepsilon > 0$ one can find $x \in R_\infty$ such that $\|x - x_P^0\| < \varepsilon$. This means $P_r \subset \text{ cl } R_\infty$ and the proof is complete.

We are now in a position to prove the main result (compare with Theorem 2.1).

Theorem 3.2. *Let $A \in L(X)$ be a compact operator such that A^* satisfied the (SSC). We assume that int $\Omega \neq \emptyset$ and cl $B\Omega$ is a convex cone. Then the linear discrete-time system (A, B, Ω) is approximately controllable if and only if*

$$\exists m \in \mathbf{N}: \text{ Im } A^m \subset \text{ cl } H \tag{3.7}$$

and

$$\text{Ker } (\lambda I^* - A^*) \cap (B\Omega)^0 = \{0\}, \quad \forall \lambda \geq 0. \tag{3.8}$$

Proof. Necessity. If (A, B, Ω) is approximately controllable then the system with unconstrained controls (A, B, U) is evidently also approximately controllable. This implies (3.7).

Next, suppose that there exist a nonnegative $\lambda \in \sigma(A)$ and a nonzero $f \in X^*$ such that $A^* f = \lambda f$ and $f \in (B\Omega)^0$. Then, obviously, $f \in (A^k B\Omega)^0$, $\forall k \in \mathbf{N}$, which means $f \in R_\infty^0$ and hence the system (A, B, Ω) is not approximately controllable.

Sufficiency. Suppose (3.7) and (3.8) hold. Taking any $r > 0$ we have the spectral decomposition $X = P_r \oplus Q_r$ as described above. Projecting the inclusion (3.7) on P_r along Q_r and taking into account that the operator A_P is onto, we get

$$\pi_P \text{ Im } A^m = \text{Im } A_P^m = P_r \subset \pi_P(\text{cl } H) \subset \text{ cl } (\pi_P H) = \text{cl } H_P \subset P_r,$$

denoting $H_P = \text{ span } \{B_P U, A_P B_P U, ...\}$, $B_P = \pi_P B$. This implies $\text{cl}\, H_P = P_r$. Since P_r is a finite dimensional subspace, there exists $n \geq 1$ such that

$$P_r = \text{ span } \{B_P U, A_P B_P U, ..., A_P^{n-1} B_P U\}, \tag{3.9}$$

which means, by Kalman's criterion, that the system (A_P, B_P, U) is exactly controllable in P_r. Further, from (3.8) it follows

$$\text{Ker } (\lambda I_P^* - A_P^*) \cap (B_P \Omega)^0 = \{0\}, \quad \forall \lambda \geq 0. \tag{3.10}$$

Indeed, if f is a nonzero element of P_r^* such that $A_P^* f = \lambda f$, $\lambda \geq 0$ and $\langle f, B_P u \rangle \leq 0$, $\forall u \in \Omega$, then putting $g = \pi_P^* f$ we see that $g \in X^*$ and g is nonzero (since π_P is onto). Moreover, $A^* g = A^* \pi_P^* f = \pi_P^* A^* f = \lambda \pi_P^* f = \lambda g$ and $\langle g, Bu \rangle = \langle f, \pi_P Bu \rangle = \langle f, B_P u \rangle \leq 0$, $\forall u \in \Omega$. This contradicts (3.8).

Therefore, by Theorem 2.2, the system (A_P, B_P, Ω) is approximately controllable in P_r. Thus, we can write

$$P_r = \text{ cl } \bigcup \{R_k^p,\ k \in \mathbf{N}\} = \text{ cl } \bigcup \{\text{cl } R_k^p,\ k \in \mathbf{N}\},$$

where R_k^p is the reachable set of (A_P, B_P, Ω) in time k. Now, since cl $R_k^p \subset$ cl R_{k+1}^p, $\forall k \in \mathbf{N}$ and cl R_k^p is convex, it follows from the fact that P_r is finite dimensional that there must exist $k_0 \in \mathbf{N}$ such that $P_r =$ cl $R_{k_0}^p$. Since π_P is an open mapping and $R_{k_0}^p = \pi_P R_{k_0}$, we get $P_r = \pi_P(\text{cl } R_{k_0})$ which implies

$$\pi_P(\text{cl } R_\infty) = P_r. \tag{3.11}$$

Remark that we have established (3.11) for an arbitrary $r > 0$. By the properties of $\sigma(A)$, one can choose a sequence $r_i \to 0$ such that $0 < r_{i+1} < r_i$ and $\{\mu \in \mathcal{C} : |\mu| = r_i\} \cap \sigma(A) = \emptyset$, $\forall i \in \mathbf{N}$. By Lemma 3.1, from (3.11) we have $P_{r_i} \subset$ cl R_∞. Since $P_r \subset P_{r_i}$ whenever $r_i < r$, the last inclusion implies cl span $\{P_r, r > 0\} \subset$ cl R_∞. This shows, by virtue of Lemma 2.6, that

$$\exists i_0 \in \mathbf{N} : \text{ cl Im } A^{i_0} \subset \text{ cl } R_\infty. \tag{3.12}$$

Suppose to the contrary that the system (A, B, Ω) is not approximately controllable, i.e. cl $R_\infty \neq X$. Then, since cl R_∞ is a convex cone, Theorem 10, p. 425 in [9], shows that there exists a nonzero $f \in X^*$ such that $f \in R_\infty^0$. Since $B\Omega \subset R_\infty$ we have $f \in (B\Omega)^0$. By (3.12), we get also that $f \in$ Ker A^{*i_0}. Denote $g_k = A^{*k} f$, $k = 0, 1, ..., i_0 - 1$. We have $g_{i_0 - 1} = A^{*i_0 - 1} f \in$ Ker A^*. On the other hand, it is easily seen that the negative polar cone R_∞^0 is invariant under $A^* : A^* R_\infty^0 \subset R_\infty^0$. Therefore, $g_{i_0-1} \in$ Ker $A^* \cap (B\Omega)^0$. In view of (3.8) we conclude that $g_{i_0-1} = 0$. By induction, we can show that $g_k = 0$ for $k = 1, 2, ..., i_0 - 1$, and $g_0 = f \in$ Ker $A^* \cap (B\Omega)^0$. This contradicts (3.8) and completes the proof of the theorem.

Remark 3.3. The requirement int$\Omega \neq \emptyset$ in the above theorem is rather restrictive and is not fulfilled for some cases of practical interest. For example, the set Ω of controls in the space $U = L_p(0, h, \mathbf{R}^m)$ define as

$$(3.13) \qquad \Omega = \{u(.) \in U : u_i(t) \geq 0 \text{ a.e. on } [0,h],\ i = 1,...,m\}$$

has no interior points. In such a situation, the following observation may be useful. Suppose that there exist a Banach space U^1 and a subset $\Omega^1 \subset U^1$ such that $U^1 \subset U$, $\Omega^1 \subset \Omega$ and cl $U^1 = U$, cl $\Omega^1 = \Omega$ (the closure is taken in the topology of U). Suppose further that the embedding operator $E : U^1 \to U$ is continuous. In this case, if Ω^1 has a nonempty interior in U^1, then the condition (3.7) and (3.8) are also sufficient for approximate controllability of the system (A, B, Ω). To see this, we notice that the condition int$\Omega \neq \emptyset$ has been used in the proof of Theorem 3.2 only one time: namely, it was required when we applied Theorem 2.2 to assert that the spectral subsystem (A_P, B_P, Ω) is approximately controllable. Here, we can obtain this as follows. Notice first that the linear operator $B^1 \triangleq BE : U^1 \to X$ is continuous. Since cl $U^1 = U$ and P_r is finite dimensional, from (3.9) it follows that

$$P_r = \text{ span}\{B^1_P U^1, A_P B^1_P U^1, ..., A_P^{n-1} B^1_P U^1\}.$$

Further, since cl $\Omega^1 = \Omega$ it is easy to show that (3.10) is also satisfied when replacing $B_P\Omega$ by $B^1_P\Omega$, that is

$$\text{Ker } (\lambda I^*_P - A^*_P) \cap (B^1_P\Omega)^0 = \{0\}.$$

Therefore, by Theorem 2.2, the system (A_P, B^1_P, Ω^1) is approximately controllable. Since $B^1\Omega^1 \subset B_P\Omega$ it follows that the reachable set of the system (A_P, B^1_P, Ω^1) is contained in the reachable set of the system (A_P, B_P, Ω). Consequently, (A_P, B_P, Ω) is also approximately controllable.

For example, in the case of control set (3.11) we can take $U^1 \triangleq L_\infty(0, h, \mathbf{R})$ and

$$\Omega^1 = \{u(.) \in U^1 : u_i(t) \geq 0 \ \text{ a.e.}, i = 1, ..., m\}$$

and the above remark applies.

Finally, we notice that the condition (3.7) is certainly weaker than (2.5). Hence, we obtain the following.

Corollary 3.4. *Under the assumptions of Theorem 3.2, the system* (A, B, Ω) *is approximately controllable if the corresponding system with unconstrained controls* (A, B, U) *is approximately controllable and* Ker $(\lambda I^* - A^*) \cap (B\Omega)^0 = \{0\}$, $\forall \lambda \geq 0$.

As will be shown in the next section, for the discrete-time system associated with a functional differential equation, the condition (3.7) is equivalent to the approximate null-controllability of the system with no restrictions on controls (A, B, U).

We conclude this section by giving an example, illustrating the use of the above results. Consider a system (A, B, Ω) in $X = l_2$ with $\Omega \subset U = \mathbf{R}^2$. Let $A \in L(X)$ be defined by $Ax = (\xi_1, (-1/2)\xi_2,\ (-1/3)\xi_3, ...)$ for $x = (\xi_1, \xi_2, \xi_3, ...) \in l_2$ and $B \in L(U, X)$ be defined by $Bu = b_1 u_1 + b_2 u_2$, for $u = (u_1, u_2) \in U$, where $b_1 = (1, 1/2, 1/3, ...)$ and $b_2 = (1, 0, 0, ...)$. Notice first that this system is not exactly controllable, even when $\Omega = U$, since the operator B is compact. However, it is easily verified that cl span $\{b_1, Ab_1, A^2 b_1, ...\} = X$, and hence cl span $\{BU, ABU, ...\} = X$. Therefore, the condition (3.7) is satisfied. Further, we have $\sigma(A^*) = \{1, -1/2, -1/3, ...\}$, Ker $A^{*i} = \{0\}$, $\forall i \in \mathbf{N}$ and $f = \pm(1, 0, 0, ...)$ are the eigenvectors of A^* corresponding to the eigenvalue $\lambda = 1$. Consequently, the condition (3.8) is satisfied if we take

$$\Omega = \{(u_1, u_2) \in U : u_1 \geq 1/2|u_2| \geq 0\}, \tag{3.14}$$

but (3.8) fails for the set of positive controls

$$\Omega = \{(u_1, u_2) \in U : u_1 \geq 0,\ u_2 \geq 0\}. \tag{3.15}$$

Next, we show that the operator A^* satisfies the (SSC). Taking any $x = (\xi_1, \xi_2, \xi_3, ...) \in X_\infty^*$ we have, for $r > 0$,

$$(A/r)^k x = ((1/r)^k \xi_1, (-1/2r)^k \xi_2, (-1/3r)^k \xi_3, ...).$$

Suppose that $x \neq 0$ or, equivalently, $\xi_p \neq 0$ for some $p \in \mathbf{N},\ p \geq 1$. Then, taking $r = (p+1)^{-1}$, we have

$$\|(A/r)^k x\| \geq |\xi_p|((p+1)/p)^k \to \infty \text{ as } k \to \infty,$$

conflicting with the definition of X_∞^*. Therefore $X_\infty^* = \{0\} = \text{Ker } A^*$. Hence, from the Corollary 3.4, we have that the system (A, B, Ω) under consideration is approximately controllable if Ω is defined by (3.14) and is not approximately controllable if Ω is defined by (3.15).

Let A be now the left shift operator $A : l_2 \to l_2,\ A(\xi_1, \xi_2, ...) = (\xi_2, \xi_3, ...)$. Assume that the operator B is as above and the control set Ω is defined by (3.14). Then (3.7) and (3.8) are satisfied since cl span $\{b_1, Ab_1, A^2 b_1, ...\} = X$ and A^* has no eigenvectors. However, the system

(A, B, Ω) is not approximately controllable. Indeed, since $A^i b_2 = 0,\ i = 1, 2, ...,$ we have that any vector x in R_∞ can be written as

$$x = b_2 u_2 + \sum_{i=1}^{k} A^{k-i} b_1 u_{1,i},$$

with some $k \in \mathbf{N}, u_2 \in \mathbf{R}$ and $u_{1,i} \geq 0,\ i = 1, ..., k$. Therefore, taking $f = (0, -1, 0, 0, ...) \in X^*$ we verify that $\langle f, x\rangle \leq 0,\ \forall x \in R_\infty$. The reason for which the approximate controllability fails is that in this case the adjoint operator A^* does not satisfy the (SSC) and, hence, the Theorem 3.2 does not apply. Indeed, we have Ker $A^{*i} = \{0\},\ \forall i \in \mathbf{N}$. On the other hand, since $\sigma(A^*) = \{0\}$, it can be easily shown (by using the Beurling-Gelfand spectral radius formula) that the set $X^*_\infty \triangleq \{f \in X^* : \lim\limits_{k\to\infty} \|(A^*/r)^k f\| = 0,\ \forall r > 0\}$ coincides with the whole space.

4. Approximate controllability of linear retarded systems

In this section we shall use the results obtained for linear discrete-time systems in the previous section to examine the controllability of the linear autonomous retarded functional differential equations (FDE) of the general form

$$\dot{z}(t) = L(z_t) + B_0 u(t),\ z(t) \in \mathbf{R}^n, \tag{4.1}$$

$$u(t) \in \Omega \subset \mathbf{R}^m, \tag{4.2}$$

where B_0 is a $n \times m$ real matrix. We shall suppose that L is a bounded linear functional from $C = C(-h, 0, \mathbf{R}^n)$ into $\mathbf{R}^n$ given by

$$L(\varphi) = \int_{-h}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(\cdot)$ is a $n \times n$ real matrix function of bounded variation such that $\eta(\theta) = 0$ for $\theta \geq 0$, $\eta(\theta) = \eta(-h)$ for $\theta \leq -h$ and η is left-sided continuous on $(-h, 0)$. Ω is assumed to be a nonempty subset of $\mathbf{R}^m$ such that int co $\Omega \neq \emptyset$ and $0 \in \Omega$. For every $T > 0$, the set of admissible controls on $[0, T]$ is defined as

$$\widetilde{\Omega}_T = \{u(.) \in L_p(0, T, \mathbf{R}^m) : u(t) \in \Omega \text{ a.e. } t \in [0, T]\}.$$

The function space controllability for functional differential equations (4.1) (with no restrictions on controls) has been studied during the last decade by many authors (see e.g., [1, 15, 16, 17, 24, 30]). One of the most successfull approaches to this problem is based on the use of the C_0-semigroup theory in the state space $M_p = \mathbf{R}^n \times L_p(-h, 0, \mathbf{R}^n)$. For details and further references on this topic, the reader is referred to [14] and [30] where some verifiable criteria of approximate controllability for general systems of type (4.1) are also given. The controllability of FDE with restrained controls has been so far studied in only very few papers. In [6], some conditions for exact controllability with positive controls to the state space $W_2^1(-h, 0, \mathbf{R}^n)$ are given. In [32] the author studied the problem of approximate controllability to the state space C for the systems (4.1)-(4.2) with $\Omega = \{u \in \mathbf{R}^m,\ u_i \geq 0,\ i = 1, ..., m\}$. Some criteria of controllability are proved by using the analytic methods developed in [12]. In [5] a problem of exact null-controllability of linear delay systems has been examined, provided that the control set $\widetilde{\Omega}_T$ contain 0 in its interior.

In this section, we investigate the approximate controllability with positive controls for (4.1)-(4.2) in the space M_p $(1 < p < \infty)$. The choice of the space is motivated by the existence of the well-developed theory of FDE in this space (see [3, 14, 30]). The consideration of the retarded FDE (4.1) as an abstract evolution equation generating a C_0-semigroup in M_p allows us to make a natural discretization for the solutions and to use the controllability tests for discrete-time systems which have been established in the previous section. This approach enables us to obtain the criteria of controllability with restrained controls in a simple way. As a corollary, a verifiable controllability criterion is obtained for the FDE, whose generalized eigenfunctions are complete. This criterion extends the results [4] to the FDE.

We need some more notations, besides those introduced in Section 1. The state space for the system (4.1) is denoted by X, $X \overset{\Delta}{=} M_p = \mathbf{R}^n \times L_p(-h, 0, \mathbf{R}^n)$, $1 < p < \infty$. The element $\varphi \in X$ will be denote by (φ^0, φ^1) where $\varphi^0 \in \mathbf{R}^n$, $\varphi^1 \in L_p(-h, 0, \mathbf{R}^n)$. X is a Banach space with the norm $\|\varphi\| = \|\varphi^0\|_{\mathbf{R}^n} + \|\varphi^1\|_{L_p}$. The symbol W_p^1 will denote the Sobolev space of absolutely continuous functions from $(-h, 0)$ to $\mathbf{R}^n$ with the derivative in $L_p(-h, 0, \mathbf{R}^n)$. We shall denote the system (4.1)-(4.2) briefly by (L, B_0, Ω). The transposed system

$$\dot{z}(t) = L^+(z_t) + B_0 u(t), \quad z(t) \in \mathbf{R}^n,$$
$$u(t) \in \Omega \subset \mathbf{R}^m,$$

where $L^+(\varphi) \overset{\Delta}{=} \int_{-h}^{0} d\eta^\top(\theta)\varphi(\theta)$, will be denoted briefly by (L^+, B_0, Ω).

The inner product in $\mathbf{R}^n$ is denoted by $(.,.)_{\mathbf{R}^n}$.

We now recall some elements from theory of linear functional differential equations which will be used in this paper. It is well-known (e.g., [3]) that the homogeneous equation

$$\dot{z}(t) = L(z_t) \tag{4.3}$$

induces a strongly continuous semigroup $\{S(t),\ t \geq 0\}$ on X. Let $A : \mathcal{D}(A) \to X$ be the infinitesimal generator of $S(t)$, then

$$\mathcal{D}(A) = \{\varphi \in X : \dot{\varphi}^1 \in W_p^1,\ \varphi^1(0) = \varphi^0\},$$
$$A\varphi = (L(\varphi^1), \varphi^1) \text{ for } \varphi \in \mathcal{D}(A).$$

Let $\Delta(\lambda)$ be the characteristic matrix of (4.3), i.e.

$$\Delta(\lambda) = \lambda I - \int_{-h}^{0} d\eta(\theta) e^{\lambda\theta}.$$

Then the spectrum of the generator A is a point spectrum and given by the selfconjugate set of zeros of det $\Delta(\lambda)$:

$$\sigma(A) = \{\lambda \in \mathcal{C} :\ \det\ \Delta(\lambda) = 0\}.$$

For each eigenvalue $\lambda \in \sigma(A)$, the eigenspace is given by

$$\text{Ker } (\lambda I - A) = \{(\varphi^0, \varphi^1) \in X : \Delta(\lambda)\varphi^0 = 0,\ \varphi^1 = \varphi^0 e^{\lambda\theta},\ \theta \in [-h, 0]\}.$$

and the generalized eigenspace is $M_\lambda \triangleq \text{Ker } (\lambda I - A)^{k_\lambda}$, where k_λ is the multiplicity of λ. Let $\{S^+(t),\ t \geq 0\}$ denote the C_0-semigroup induced by the transposed equation

$$\dot{z}(t) = L^+(z_t),\ z(t) \in \mathbf{R}^n.$$

Then the corresponding generator A^+ is given by

$$\mathcal{D}(A^+) = \{\psi \in X^* : \dot{\psi}^1 \in W_q^1,\ \psi(0) = \psi^0\},$$
$$A^+\psi = (L^+(\psi^1), \psi^1).$$

Clearly, $\sigma(A^+) = \sigma(A)$.

We shall make use the notion of the *structural operators* $F, G \in L(X)$ associated with the homogeneous equation (4.3). They are defined as follows (see, e.g. [14]) Given $\varphi = (\varphi^0, \varphi^1) \in X$, then

$$(G\varphi)^1(\theta) = X(h+\theta)\varphi^0 + \int_{-h}^{0} X(h+\theta+s)\varphi^1(s)ds,\ \theta \in [-h, 0],$$
$$(G\varphi)^0 = (G\varphi)^1(0),$$

where $X(t)$ denotes the fundamental matrix of the equation (4.3), and

$$(F\varphi)^0 = \varphi^0,$$
$$(F\varphi)^1(\theta) \triangleq (H\varphi^1)(\theta),$$

where the operator $H : L_p(-h, 0, \mathbf{R}^n) \to L_p(-h, 0, \mathbf{R}^n)$ is defined by

$$(H\varphi^1)(\theta) = \int_{-h}^{0} d\eta(s)\varphi^1(s-\theta), \quad \theta \in [-h, 0].$$

It is important to notice that the adjoint operators F^* and G^* are of the same type as F and G, respectively, except for the transposition of matrices. Further, by Proposition 3.1 in [14],

$$\text{(4.4)} \qquad \operatorname{Im} G^* = \mathcal{D}(A^+) \text{ and } \operatorname{Ker} G^* = \{0\}.$$

The following relations between the structural operators will be used in the sequel.

$$\text{(4.5)} \qquad S(h) = GF, \quad S^+(h) = G^*F^*, \quad S^*(h) = F^*G^*,$$
$$\text{(4.6)} \qquad S^*(t)F^* = F^*S^+(t), \quad \forall t \geq 0,$$
$$\text{(4.7)} \qquad S^*(t) = G^{*-1}S^+(t)G^*, \quad \forall t \geq 0.$$

Let us now consider the linear control system (L, B_0, Ω) described by (4.1)-(4.2). Let $z(t)$ be a solution of (4.1) corresponding to some initial condition $z(0) = \varphi^0$, $z(\theta) = \varphi^1$, $\theta \in [-h, 0)$, $\varphi = (\varphi^0, \varphi^1) \in X$, and to some control $u(.) \in L_p(0, T, \mathbf{R}^m)$. Then $x(t) = (z(t), z_t)$ is called the mild solution of the abstract differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = \varphi, \quad t \in [0, T],$$

and $x(\cdot)$ can be represented in the form

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds$$

where $x_0 = \varphi$ and $B : \mathbf{R}^m \to X$ is a bounded linear operator defined as

$$Bu = (B_0 u, 0).$$

For a given $T > 0$, the reachable set in time T of the system (L, B_0, Ω), corresponding to $\varphi = 0 \in X$, is

$$R_T = \Big\{ \int_0^T S(T-t)Bu(t)dt,\ u(.) \in \widetilde{\Omega}_T \Big\}.$$

We define

$$R_\infty = \bigcup \{R_T,\ T > 0\}.$$

We also consider the set of approximately null-controllable states in time T,

$$C_T = \{\varphi \in X : -S(T)\varphi \in \text{ cl } R_T\},$$

and we set

$$C_\infty = \bigcup \{C_T,\ T > 0\}.$$

Definition. The system (L, B_0, Ω) is said to be approximately controllable if $\operatorname{cl} R_\infty = X$ and approximately null-controllable if $C_\infty = X$.

In what follows we shall frequently make use of properties of the reachable set. Some of them are collected in the following

Lemma 4.1. *The reachable set of (L, B_0, Ω) has the following properties:* (i) $\operatorname{cl} R_T$ *is convex and* $\operatorname{cl} R_T$ *remain unchanged when one replaces* Ω *by* $\operatorname{cl} \operatorname{co} \Omega$; (ii) $\operatorname{cl} R_{T_1} \subset \operatorname{cl} R_{T_2}$ *when* $T_1 \leq T_2$ *and, hence,* $\operatorname{cl} R_\infty$ *is also convex;* (iii) R_∞ *is invariant under the semigroup* $S(t)$, *i.e.*

$$S(t)R_\infty \subset R_\infty, \quad \forall t \geq 0; \tag{4.8}$$

(iv) *If* $\Omega = \mathbf{R}^m$ *then there exists* $T_1 \leq (n+1)h$ *such that*

$$\text{cl } R_\infty = \text{ cl } R_t, \quad \forall t \geq T_1. \tag{4.9}$$

Proof. Properties (i)-(iii) have been proved in [7] and [13]. The property (iv) has been proved in [30] for a class of systems more general than (4.1).

In order to establish a criterion of approximate controllability for (L, B_0, Ω), we shall to construct a linear discrete-time system, which is equivalent to the retarded system (L, B_0, Ω). Let denote $U = L_p(0, h, \mathbf{R}^m)$. Elements of U will be denoted by $u(.)$ or simple by u. We define the operator

$$\mathcal{A} : X \to X, \quad \mathcal{A}x = S(h)x,$$

$$\mathcal{B} : U \to X, \quad \mathcal{B}u = \int_0^h S(h-\theta)Bu(\theta)d\theta,$$

and the control set

$$\widetilde{\Omega} \triangleq \widetilde{\Omega}_h = \{u \in U : u(t) \in \Omega \text{ a.e. } t \in [0,h]\},$$
$$\widetilde{\text{co}}\Omega = \{u \in U : u(t) \in \text{ clco}\,\Omega \text{ a.e. } t \in [0,h]\}.$$

We note that $\mathcal{A}$ is a compact operator, by Proposition 4.1 in [3]. Moreover, from Lemma 4.1 it follows that cl $\mathcal{B}\widetilde{\Omega}$ is convex and

$$\text{cl } \mathcal{B}\widetilde{\Omega} = \text{cl } \mathcal{B}\ \widetilde{\text{co}}\,\Omega. \tag{4.10}$$

Now, consider the linear discrete-time system $(\mathcal{A}, \mathcal{B}, \widetilde{\Omega})$

$$x_{k+1} = \mathcal{A}x_k + \mathcal{B}u_k,\ x_k \in X$$
$$u_k \in \widetilde{\Omega}.$$

We say that $(\mathcal{A}, \mathcal{B}, \widetilde{\Omega})$ is associated with (L, B_0, Ω). The reachable ser of $(\mathcal{A}, \mathcal{B}, \widetilde{\Omega})$ in time k is: $R_k^d = \{0\}$ for $k = 0$ and, for $k \geq 1$,

$$R_k^d = \sum_{i=1}^{k} \mathcal{A}^{k-i}\mathcal{B}\widetilde{\Omega}.$$

Let us denote $R_\infty^d = \bigcup\{R_k^d,\ k \in \mathbf{N}\}$. Using the transitive property of $S(t)$ it is easy to show the following

Lemma 4.2. *For every $k \in \mathbf{N}$*

$$R_{kh} = R_k^d,$$

(where R_{kh} denotes the reachable set of the system (L, B_0, Ω) in time $T = kh$).

For the case when $\Omega = \mathbf{R}^m$ (or, equivalently, $\widetilde{\Omega} = U$), from (4.9) and Lemma 4.2 it follows that cl $R_\infty = \mathrm{cl}\, R^d_\infty = \mathrm{cl}\, R^d_{n+1}$, which is equivalent to

$$\text{(4.11)} \qquad \text{cl span } \{\mathcal{A}U, \mathcal{AB}U, ...\} = \text{ cl span } \{\mathcal{B}U, \mathcal{AB}U, ..., \mathcal{A}^n\mathcal{B}U\}.$$

In [11] it has been shown that any solution of a linear autonomous retarded FDE (4.3) (in the state space C) which tends to zero more rapidly than any exponential must vanish in finite time, not exceeding nh. Since every solution will be in the state space C after the time $t = h$, the mentioned result is also valid in the space M_p, where it reads: if $\lim\limits_{t\to\infty} e^{\omega t}\|S(t)\varphi\| = 0,\ \forall \omega > 0$ then $\varphi \in \mathrm{Ker}\, S(t_0)$, for some $t_0 \le (n+1)h$. This implies that the operator $\mathcal{A} = S(h)$ satisfies the small solution condition (SSC) defined in Section 3 of this paper. The analogous property is certainly valid for $\mathcal{A}^+ \triangleq S^+(h)$ which read: there exists $j \in \mathbf{N}$, $j \ge (n+1)h$ such that

$$X^+_\infty \triangleq \{f \in X : \lim_{k\to\infty} \|(\mathcal{A}^+/r)^k f\| = 0,\ \forall r > 0\} = \text{ Ker } (\mathcal{A}^+)^j.$$

Lemma 4.3. *For the discrete-time system $(\mathcal{A}, \mathcal{B}, \widetilde{\Omega})$ associated with the system (L, B_0, Ω), the adjoint operator $\mathcal{A}^*$ satisfies the small solution condition.*

Proof. Since, obviously, Ker $(\mathcal{A}^*)^i \subset X^*_\infty = \{f \in X^* : \lim\limits_{k\to\infty} \|(\mathcal{A}^*/r)^k f\| = 0,\ \forall r > 0\}$, $\forall i \in \mathbf{N}$, it suffices to show that the inverse inclusion holds for some $j \in \mathbf{N}$. As shown in [14] (Corollary 3.4) there exist a bounded invertible operator $V \in L(X^*)$ such that

$$(\mathcal{A}^*)^k = V^{-1}(\mathcal{A}^+)^k V, \quad \forall k \in \mathbf{N}.$$

Taking any $f \in X^*_\infty$ and denoting $g = Vf$, we have, for every $r > 0$ and $k \in \mathbf{N}$,

$$\|(\mathcal{A}^+/r)^k g\| = \|(\mathcal{A}^+/r)^k V f\| = \|V(\mathcal{A}^*/r)^k f\| \le \|V\|\ \|(\mathcal{A}^*/r)^k f\|.$$

Since $\mathcal{A}^+$ satisfies the (SSC), the above inequality shows that $g \in X^+_\infty$ and, hence, $g = Vf \in \mathrm{Ker}\,(\mathcal{A}^+)^j$, since $\mathcal{A}^+$ satisfies (SSC). Thus, we obtain $(\mathcal{A}^*)^j f = V^{-1}(\mathcal{A}^+)^j V f = 0$, and so $X^*_\infty \subset \mathrm{Ker}\,(\mathcal{A}^*)^j$, as required.

This result combining with (4.10) and (4.11), yields the following.

Lemma 4.4. *Let Ω be a cone in $\mathbf{R}^m$ such that int* $\operatorname{co}\Omega \neq \emptyset$*. The discrete-time system $(\mathcal{A}, \mathcal{B}, \widetilde{\Omega})$ associated with the FDE (L, B_0, Ω) is approximately controllable if and only if*

$$\exists k \geq n+1 : \text{ Im } \mathcal{A}^* \subset \text{ cl span } \{\mathcal{B}U, \mathcal{AB}U, ..., \mathcal{A}^{k-1}\mathcal{B}U\}, \tag{4.12}$$

and

$$\text{Ker } (\lambda I^* - \mathcal{A}^*) \cap (\mathcal{B}\widetilde{\Omega})^0 = \{0\}, \quad \forall \lambda \geq 0.$$

Proof. Put $U^1 = L_\infty(0, h, \mathbf{R}^m)$ and $\widetilde{\Omega}^1 = \{u(.) \in U^1 : u(t) \in \text{ cl co}\,\Omega\}$ a.e. $t \in [0, h]$. Then, clearly, $\widetilde{\Omega}^1$ has nonempty interior in U^1 and $\operatorname{cl} U^1 = U$, $\operatorname{cl}\widetilde{\Omega}^1 = \widetilde{\operatorname{co}}\,\Omega$ (the closure is taken in the topology of U). The assertion is now immediate from Theorem 3.2 and Remark 3.3 of the previous section.

We note that the condition (4.12) means that the system $(\mathcal{A}, \mathcal{B}, \widetilde{\Omega})$ is approximately null-controllable in time k. Turning to the continuous time system (L, B_0, Ω) we first note that from Lemma 4.2 it follows obviously

Lemma 4.5. *The retarded FDE (L, B_0, Ω) is approximately controllable (respectively, approximately null-controllable in time $T = kh$) iff the associated discrete-time system $(\mathcal{A}, \mathcal{B}, \Omega)$ is approximately controllable (respectively, approximately null-controllable in time k).*

Therefore, Theorem 3.3 gives the criterion for approximate controllability of (L, B_0, Ω) which we can reformulate as follows.

Theorem 4.6. *Let Ω be a cone in $\mathbf{R}^m$ such that int* $\operatorname{co}\Omega \neq \emptyset$*. The retarded FDE (L, B_0, Ω) is approximately controllable if and only if*

(i) *The corresponding system with no restrictions on controls $(L, B_0, \mathbf{R}^m)$ is approximately null-controllable in some time $t \geq (n+1)h$,*

(ii)

$$\operatorname{Ker} S^*(h) \cap (\mathcal{B}\widetilde{\Omega})^0 = \{0\}, \tag{4.13}$$

(iii)

$$\operatorname{Ker} (\lambda I^* - S^*(h) \cap (\mathcal{B}\widetilde{\Omega})^0 = \{0\}, \quad \lambda > 0. \tag{4.14}$$

The remainder of this section is devoted to translating the above abstract controllability conditions into those expressed in terms of the original system matrices.

First, it is well-known ([26], [30]) that the condition (i) is equivalent to the condition of spectral controllability

$$\text{rank } (\Delta(\lambda), B_0) = n, \ \forall\lambda \in \mathbf{C}.$$

To deal with the conditions (ii) and (iii) we introduce the following operator $D : U \to X$,

$$\begin{aligned}&(Du)^0 = 0,\\ &(Du)^1(\theta) = B_0 u(-\theta), \ -h \leq \theta \leq 0,\end{aligned}$$

for $u \in U$. The adjoint operator $D^* : X^* \to U^*$ is given by

$$(D^*\phi)(\theta) = B_0^\top \phi^1(-\theta), \ 0 \leq \theta \leq h.$$

It can be easily verified that

$$\mathcal{B} = GD,$$

where G is the structural operator defined as above (see, e.g. [31]). Therefore, from the definition of the negative polar cone, we have

$$\varphi \in (\mathcal{B}\widetilde{\Omega})^0 \Leftrightarrow G^*\varphi \in (D\widetilde{\Omega})^0. \tag{4.15}$$

The following lemma gives the characterization for $(D\widetilde{\Omega})^0$.

Lemma 4.7.*Suppose Ω is a closed cone in $\mathbf{R}^m$. Given a vector $g = (g^0, g^1) \in X^*$, we have*

$$g \in (D\widetilde{\Omega})^0 \Leftrightarrow (g^1(\theta), B_0 u)_{\mathbf{R}^n} \leq 0 \text{ a.e. } \theta \in [-h, 0], \quad \forall u \in \Omega.$$

Proof. From the definition of D, $g \in (D\widetilde{\Omega})^0$ if and only if

$$\int_{-h}^{0} (g^1(\theta), B_0 u(-\theta))_{\mathbf{R}^n} d\theta \leq 0, \ \forall u(.) \in \widetilde{\Omega}.$$

Therefore it suffices to show that the implication " $\Rightarrow$ " holds. Suppose to the contrary that there exists $\delta > 0$ such that the set

$$E_\delta = \{\theta \in [-h, 0] : \ \exists v_\delta \in \Omega, \ (g^1(\theta), B_0 v_\delta)_{\mathbf{R}^n} \geq \delta\}$$

is of positive measure: $\mu E_\delta > 0$. Define, for $\theta \in E_\delta$,

$$K(\theta) = \{v \in S_1 \cap \Omega : (g^1(\theta), B_0 v)_{\mathbf{R}^n} \geq \delta\},$$

where S_1 is the unit ball in $\mathbf{R}^m$. Since Ω is a cone, $K(\theta)$ is nonempty for every $\theta \in E_\delta$. Thus, K has nonempty compact values and is measurable. By Theorem 1.7.7 in [34] (due to von Neumann-Aumann-Castaing) there exists a measurable selection $v_\delta(\theta) \in K(\theta)$, $\theta \in E_\delta$. Define

$$v(\theta) = \begin{cases} v_\delta(\theta), & \theta \in E_\delta, \\ 0, & \theta \in [-h, 0] \setminus E_\delta, \end{cases}$$

and $\hat{u}(\theta) = v(-\theta)$, $\theta \in [0, h]$. We observe that $\hat{u} \in \widetilde{\Omega}$ and

$$\int_{-h}^{0} (g^1(\theta), B_0 \hat{u}(-\theta))_{\mathbf{R}^n} \, d\theta = \int_{E_\delta} (g^1(\theta), B_0 v_\delta(\theta))_{\mathbf{R}^n} \, d\theta \geq \delta \mu E_\delta > 0,$$

which means that g does not belong to $(D\widetilde{\Omega})^0$, concluding the proof.

We now prove the main result of this section.

Theorem 4.8. *Let Ω be a cone in $\mathbf{R}^m$ such that int $\operatorname{co} \Omega \neq \emptyset$. The retarded functional differential equation (L, B_0, Ω) is approximately controllable if and only if*

(i)$_1$ $$\text{rank } (\Delta(\lambda), B_0) = n, \quad \forall \lambda \in \mathbf{C},$$

(ii)$_1$ *There exists no nonzero vector $\phi \in L_q(-h, 0, \mathbf{R}^n)$ such that*

$$(H^* \phi)(\theta) = \int_{-h}^{0} d\eta^\top(s) g^1(s - \theta) = 0, \quad \theta \in [-h, 0]$$

and

$$(\phi(\theta), B_0 u)_{\mathbf{R}^n} \leq 0 \quad a.e. \ \theta \in [-h, 0], \quad \forall u \in \Omega, \tag{4.16}$$

(iii)$_1$ *There exists no nonzero vector* $\varphi^0 \in \mathbf{R}^n$ *such that* $\Delta^\top(\lambda)\varphi^0 = 0$ *for some* $\lambda \in \mathbf{R}$ *and*

$$(\varphi^0, Bu)_{\mathbf{R}^n} \le 0, \quad \forall u \in \Omega.$$

Proof. It suffices to show that the conditions (ii)$_1$ and (iii)$_1$ of Theorem 4.8 are equivalent, respectively, to the conditions (ii) and (iii) of Theorem 4.6. In view of Lemma 4.1, we can assume without loss of generality that Ω is closed and convex. Then, by Lemma 4.7, (ii)$_1$ is equivalent to the condition

$$\text{Ker } F^* \cap (D\widetilde{\Omega})^0 = \{0\}, \tag{4.17}$$

where F is the structural operator defined as above. Further, if $\varphi^0 \in \mathbf{R}^n$ satisfies (iii)$_1$ then the function $\varphi = (\varphi^0, \varphi^1)$, $\varphi^1(\theta) = \varphi^0 e^{\lambda\theta}$, $\theta \in [-h, 0]$ is an eigenfunction of A^+, i.e. $\varphi \in \text{Ker}\,(\lambda I^* - A^+)$, and, clearly, $(\varphi^1(\theta), B_0 u)_{\mathbf{R}^n} \le 0$, $\forall u \in \Omega$, $\forall \theta \in [-h, 0]$. Therefore, by Lemma 4.7, $\varphi \in (D\widetilde{\Omega})^0$. The converse is also true. In other words, (iii)$_1$ is equivalent to the condition

$$\text{Ker } (\lambda I^* - A^+) \cap (D\widetilde{\Omega})^0 = \{0\}, \quad \forall \lambda \in \mathbf{R}. \tag{4.18}$$

Consequently, the proof of the theorem is reduced to showing that (4.17) is equivalent to (4.13) and (4.18) is equivalent to (4.14).

(4.13) $\Leftrightarrow$ (4.17): Suppose that there exists a nonzero $f \in X^*$ $f \in \text{Ker}\, S^*(h) \cap (\mathcal{B}\widetilde{\Omega})^0$. Then taking $g = G^* f$ we have $g \neq 0$ since G^* is injective, and $F^* g = F^* G^* f = S^*(h) f = 0$, by (4.5). Further, in view of (4.15), $f \in (\mathcal{B}\widetilde{\Omega})^0$ implies $G^* f \in (D\widetilde{\Omega})^0$. Hence, $g \in \text{Ker}\, F^* \cap (D\widetilde{\Omega})^0$. Conversely, let there exist a nonzero $g \in X^*$ such that $F^* g = 0$ and $g \in (D\widetilde{\Omega})^0$. Then, we have $g = (0, g^1)$, $g^1 \in L_q(-h, 0, \mathbf{R}^n)$, $g^1 \neq 0$, and

$$(H^* g^1)(\theta) = \int_{-h}^{0} d\eta^\top(s) g^1(s - \theta) = 0, \quad \theta \in [-h, 0].$$

Define

$$\varphi^1(\theta) = \int_{\theta}^{0} g^1(s) ds, \quad \theta \in [-h, 0],$$

and set $\varphi = (0, \varphi^1)$. Then, clearly, $\varphi \in \mathcal{D}(A^+)$ and $\varphi \neq 0$. Moreover, it can be verified that $\varphi \in \operatorname{Ker} F^*$ (see the proof of Theorem 1 in [16]). Further, since $g \in (D\widetilde{\Omega})^0$, in view of Lemma 4.7, we have

$$(\varphi^1(\theta), B_0 u)_{\mathbf{R}^n} = \int_\theta^0 (g^1(s), B_0 u)_{\mathbf{R}^n} ds \leq 0, \quad \forall u \in \Omega,$$

which implies, again by Lemma 4.7 that $\varphi \in (D\widetilde{\Omega})^0$. Next, since $\varphi \in \mathcal{D}(A^+)$, there exists, by (4.4), a nonzero $\psi \in X^*$ such that $\varphi = G^*\psi$. Therefore, we obtain, by (4.5), that $S^*(h)\psi = F^*G^*\psi = F^*\varphi = 0$ and $\varphi = G^*\psi \in (D\widetilde{\Omega})^0$. This gives $\psi \in \operatorname{Ker} S^*(h) \cap (\mathcal{B}\widetilde{\Omega})^0$, as was to be shown.

(4.14) $\Leftrightarrow$ (4.18) : Suppose that there exists a nonzero $f \in X^*$ such that $S^*(h)f = \lambda f$, $\lambda > 0$ and $f \in (\mathcal{B}\widetilde{\Omega})^0$. Then, clearly, $f \in (S(h)^k\mathcal{B}\widetilde{\Omega})^0$, $\forall k \in \mathbf{N}$. On the other hand, by Lemma 4.2, we have $R_\infty = R_\infty^d = \bigcup\{R_k^d,\ k \in \mathbf{N}\}$, where $R_k^d = \sum_{i=1}^{k} S(h)^{k-i}\mathcal{B}\widetilde{\Omega}$. Therefore $f \in R_\infty^0$. Put $\gamma = (1/h)\ln\lambda$, we have

$$S^*(h)f = e^{\gamma h} f.$$

Let us define an element $g \in X^*$ by

$$\langle g, x\rangle = \int_0^h \langle f, e^{-\gamma\theta} S(\theta)x\rangle d\theta, \quad \forall x \in X.$$

Then, clearly, $f \neq 0$. Besides, by (4.8), it follows that $g \in R_\infty^0$. In particular, $g \in (\mathcal{B}\widetilde{\Omega})^0$. Further, for any $x \in X$ and $t \geq 0$, we have

$$\begin{aligned}
\langle S^*(t)g, x\rangle &= \int_0^h \langle f, e^{-\gamma\theta} S(\theta + t)x\rangle d\theta \\
&= e^{\gamma t}\Big(\int_t^h \langle f, e^{-\gamma\theta} S(\theta)x\rangle d\theta + \int_h^{h+t} \langle f, e^{-\gamma\theta} S(\theta)x\rangle d\theta\Big) \\
&= e^{\gamma t}\Big(\int_t^h \langle f, e^{-\gamma\theta} S(\theta)x\rangle d\theta + \int_0^t \langle S^*(h)f, e^{-\gamma(\theta+h)} S(\theta)x\rangle d\theta\Big).
\end{aligned}$$

Hence we have

$$\langle S^*(t)g, x\rangle \ = e^{\gamma t}\int_0^h \langle f, e^{-\gamma\theta}S(\theta)x\rangle d\theta = e^{\gamma t}\langle g, x\rangle,$$

which implies $S^*(t)g = e^{\gamma t}g, \ \forall t \geq 0$.

Therefore, by (4.7), $S^*(t)g = G^{*-1}S^+(t)G^*g = e^{\gamma t}g$ and consequently, $S^+(t)G^*g = e^{\gamma t}G^*g$, $\forall t \geq 0$. Denote $\varphi = G^*g$. Then, by (4.4), $\varphi \neq 0$ and $\varphi \in \mathcal{D}(A^+)$. Hence $A^+\varphi = \gamma\varphi$. Thus, we have found a nonzero $\varphi \in X^*$ such that $\varphi \in \operatorname{Ker}(\gamma I^* - A^+) \cap (D\widetilde{\Omega})^0$.

Conversely, let there exist a nonzero $\varphi \in X^*$ such that $A^+\varphi = \gamma\varphi$, $\gamma \in \mathbf{R}$ and $\varphi \in (D\widetilde{\Omega})^0$. Since $\varphi \in \mathcal{D}(A^+$, by (4.4), we can find a nonzero $\psi \in X^*$ such that $G^*\psi = \varphi$. We have $S^+(h)\varphi = S^+(h)G^*\psi = e^{\gamma h}G^*\psi$ and hence, by (4.7), $S^*(h)\psi = \lambda\psi$, with $\lambda = e^{\gamma h} > 0$. Furthermore, $\varphi = G^*\psi \in (D\widetilde{\Omega})^0$ implies, by (4.15), $\psi \in (\mathcal{B}\widetilde{\Omega})^0$. Thus $\psi \in \operatorname{Ker}(\lambda I^* - S^*(h)) \cap (\mathcal{B}\widetilde{\Omega})^0$. This completes the proof of the theorem.

We note that the above result is stronger than Theorem 4.3 in [23] because it gives us a necessary and sufficient condition of approximate controllability without imposing the condition that the homogeneous FDE (4.3) is complete.

We consider some corollaries of the above theorem. First, notice that if $\Omega = \mathbf{R}^m$ then, obviously, the condition (iii)$_1$ is included in the condition (i)$_1$. Moreover, in this case, $(D\widetilde{\Omega})^0 = \operatorname{Ker} D^*$ and, hence, (4.17) (or equivalently, (ii)$_1$) is equivalent to that $\operatorname{Ker} F^* \cap \operatorname{Ker} D^* = \{0\}$. Therefore, Theorem 4.8 yields the following known result due to Manitius [16] and Salamon [31].

Corollary 4.9. *The retarded system with unconstrained controls* $(L, B_0, \mathbf{R}^m)$ *is approximately controllable to* M_p *if and only if*

(i)$_2$ $$\operatorname{rank}(\Delta(\lambda), B_0) = n, \ \forall \lambda \in \mathbf{C},$$

(ii)$_2$ $$\operatorname{Ker} F^* \cap \operatorname{Ker} D^* = \{0\}.$$

It worths noticing that the above result can not be derived from Theorem 4.3 in [23]. Next, we say that the FDE (4.3) is *complete* if the generalized eigenfunctions of the corresponding generator A forms a complete set in the state space M_p, i.e. cl span $\{M_\lambda, \lambda \in \sigma(A)\} = M_p$. It is known (e.g. [14]) that the completeness is equivalent to the condition $\operatorname{Ker} H^* = \{0\}$. Therefore, for the complete system, the condition (ii)$_1$ is automatically satisfied, and we obtain

Corollary 4.10. *Let the FDE* $\dot{z} = L(z_t)$ *be complete. Then the system* (L, B_0, Ω) *is approximately controllable iff the conditions* (i)$_1$ *and* (iii)$_1$ *of*

Theorem 4.8 are satisfied.

In particular, if $L(z_t) = A_0 z(t)$, where A_0 is $n \times n$ real matrix, then $\Delta(\lambda) = \lambda I - A_0$. In this case, the condition $(\mathrm{i})_1$ is equivalent to the Kalman rank condition rank $\mathcal{B}_0, A_0 B_0, ..., A_0^{n-1} B_0\} = n$ and $(\mathrm{iii})_1$ reads: there exists no real eigenvector of $A_0^\top$ supporting $B_0 \Omega$ at the origin. Therefore, Corollary 4.10 can be considered as an extension of the result of [4] to linear autonomous retarded systems.

The completeness of the retarded FDE has been studied in [8] and [14] where some verifiable criteria for this property are given.

We assume now that η is of the form

$$\eta(\theta) = -A_0 \chi_{(-\infty,0)}(\theta) - \sum_{i=1}^{N} A_i \chi_{(-\infty,-h_i)}(\theta) - \int_{\theta}^{0} A_{01}(\alpha) d\alpha, \tag{4.19}$$

where χ_I denotes the characteristic function of the set I, A_i are $n \times n$ real matrices, $A_{01}(.) \in L_p(-h, 0, \mathbf{R}^{n\times n})$ and $0 = h_0 < h_1 < ... < h_N = h$. In this case, the system (L, B_0, Ω) has the form

$$\dot{z}(t) = A_0 z(t) + \sum_{i=1}^{N} A_i z(t - h_i) + \int_{-h}^{0} A_{01}(\alpha) z(t+\alpha) d\alpha + B_0 u, \tag{4.20}$$

$$u(t) \in \Omega \subset \mathbf{R}^m. \tag{4.21}$$

Then, the null space $\operatorname{Ker} H^*$ is given by the set of all functions $\phi \in L_q(-h, 0, \mathbf{R}^n)$ satisfying the equation

$$\sum_{i=1}^{N} A_i^\top \phi(-\theta - h_i) \chi_{[h_i,0]}(\theta) + \int_{-h}^{\theta} A_{01}^\top(\alpha) \phi(\alpha - \theta) d\alpha = 0 \tag{4.22}$$

a.e. $\theta \in [-h, 0]$, see [8].

Theorem 4.11. *Consider the system* (4.20)-(4.21). *Let Ω be a cone with int $\operatorname{co}\Omega \neq \emptyset$. Suppose, in addition, that $A_{01}(\alpha) \equiv 0$ in $[-h, -h+\varepsilon)$ for some $\varepsilon > 0$. Then the system is approximately controllable if and only if*

$(\mathrm{i})_3$ *rank* $(\Delta(\lambda), B_0) = n, \ \forall \lambda \in \mathbf{C}$,

$(\mathrm{ii})_3$ *There exists no nonzero vector $\phi \in \mathbf{R}^n$ such that*

$$A_N^\top \phi = 0 \quad and \quad (\phi^0, B_0 u)_{\mathbf{R}^n} \le 0, \quad \forall u \in \Omega, \tag{4.23}$$

(iii)$_3$ *There exists no nonzero vector* $\varphi^0 \in \mathbf{R}^n$ *such that* $\Delta^\top(\lambda)\varphi^0$ *for some* $\lambda \in \mathbf{R}$ *and* $(\varphi^0, B_0 u)_{\mathbf{R}^n} \leq 0, \ \forall u \in \Omega$.

Proof. It suffices to show that (ii)$_1$ implies (ii)$_3$. Indeed, if there exists a nonzero vector $\phi^0 \in \mathbf{R}^n$ satisfying (4.23) then, setting $\theta_0 = \min\{\varepsilon, h_1\}$ and

$$\phi(\theta) = \begin{cases} \phi^0, \ \theta \in [-h, -h+\theta_0] \\ 0, \ \theta \in (-h+\theta_0, 0] \end{cases}$$

we readily verify that ϕ is an element in Ker H^* (given by (4.23)) satisfying (4.16).

Therefore, in particular, we obtain

Corollary 4.12. *For differential-difference system (i.e. when* η *is given by* (4.19) *with* $A_{01} \equiv 0$, $\theta \in [-h, 0]$*), the conditions* (i)$_3$-(iii)$_3$ *are necessary and sufficient for approximate controllability.*

Note that for differential-difference systems, the characteristic matrix has the form

$$\Delta(\lambda) = \lambda I - \sum_{i=0}^{N} A_i e^{-\lambda h_i}.$$

Finally, consider the differential-difference system with a single positive input $(L, b, \mathbf{R}^1_+)$

$$\dot{z}(t) = A_0 z(t) + \sum_{i=1}^{N} A_i z(t-h_i) + bu, \ u \geq 0.$$

We observe that for any nonzero vector $y \in \mathbf{R}^n$, either y order $-y$ belongs to the negative polar cone of the ray $\{bu, u \geq 0\}$. Therefore, in this case, the condition (ii)$_3$ is equivalent to

$$\det A_N \neq 0 \tag{4.24}$$

and the condition (iii)$_3$ is equivalent

$$\det \Delta(\lambda) \neq 0, \quad \forall \lambda \in \mathbf{R}.$$

As has been shown in [14], (4.24) is a necessary and sufficient condition of completeness of differential difference equations. Thus, we obtain

Corollary 4.13. *Differential-difference system* $(L_0, b, \mathbf{R}^1_+)$ *is approximately controllable iff*

(i)$_4$ $$\text{rank } (\Delta(\lambda), b) = n, \ \forall \lambda \in \mathbf{C},$$

(ii)$_4$ $$\det A_N \neq 0,$$

(iii)$_4$ *The characteristic polynom det* $\Delta(\lambda)$ *has no real zeros.*

We illustrate the results by two simple examples.

Example 1. Consider two-dimensional system with two delays $h_1 = 1$, $h_2 = 2$. Let

$$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \; B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$\Delta(\lambda) = \begin{pmatrix} \lambda + e^{-\lambda} & 1 + e^{-2\lambda} \\ -1 & 1 - e^{-\lambda} \end{pmatrix},$$

$\det\,\Delta(\lambda) = \lambda^2 + 1$. Therefore, this system satisfies the condition (i)$_3$ and (iii)$_3$. Since any vector ϕ^0 in $\mathbf{R}^2$ satisfying $A_2^\top \phi^0 = 0$ has the form $\phi^0 = \gamma(0 \;\; 1)^\top$, $\gamma \in \mathbf{R}$, the condition (ii)$_3$ fails if

$$\Omega = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \geq 0, \; u_2 \geq 0 \right\}, \tag{4.25}$$

but is satisfied if we take, for example,

$$\Omega = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \geq |u_2|, \; u_2 \in \mathbf{R} \right\},$$

Thus, by Corollary 4.12, we obtain that the system is not approximately controllable (to M_p) with the positive controls $u_1 \geq 0$, $u_2 \geq 0$, but is approximately controllable with the controls $u_1 \geq 0$ and $u_2 \in \mathbf{R}$ satisfying $u_1 \geq |u_2|$.

Let now A_2 depend of a parameter α:

$$A_2(\alpha) = \begin{pmatrix} 0 & -1 \\ 0 & \alpha \end{pmatrix}, \quad \alpha \in \mathbf{R}.$$

The vectors ϕ^0, satisfying the equation $A_2^\top(\alpha)\phi^0 = 0$, have the form $\phi^0 = \gamma(\alpha \;\; 1)^\top$, $\gamma \in \mathbf{R}$. Therefore, for the set of positive controls (4.25), the conditions (ii)$_3$ is satisfied for every $\alpha < 0$ and fails whenever $\alpha \geq 0$. Computation $\det\Delta_\alpha(\lambda)$ gives

$$\det\,\Delta_\alpha(\lambda) = \lambda^2 - \alpha\lambda + 1 - \alpha e^{-\lambda}.$$

One can easily verify that, for $\alpha < 0$, $\det \Delta_\alpha(\lambda) > 0$, $\forall \lambda \in \mathbf{R}$, and hence $(\text{iii})_3$ is also satisfied. Consequently, the system (with the matrix $A_2(\alpha)$) is approximately controllable with positive controls (4.25) for all $\alpha < 0$, but is not approximately controllable whenever $\alpha \geq 0$.

Example 2. Consider a two-dimensional system with one delay $h = 1$ and a single input $u \in \mathbf{R}^1$. Let

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_0 = b = \begin{pmatrix} 1 \\ 3+\alpha \end{pmatrix}.$$

We have $\det \Delta(\lambda) = \lambda^2 - 3\lambda - e^{-\lambda} - e^{-2\lambda} + 2$ and $\det \Delta(\lambda)$ has 0 as the only real zero. As pointed out in [15], this system is approximately controllable with controls $u \in \mathbf{R}$ for every $\alpha \neq 0$. This system is however not approximately controllable with positive controls $u \geq 0$ since the condition $(\text{iii})_4$ of Corollary 4.13 is not satisfied: $\det \Delta(0) = 0$. Let take now

$$B_0 = \begin{pmatrix} 1 & 0 \\ 3+\alpha & 1 \end{pmatrix}, \quad \text{and} \quad \Omega = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \geq 0, \ u_2 \geq 0 \right\}.$$

We have $\Delta(0) = \begin{pmatrix} 0 & -1 \\ 0 & -3 \end{pmatrix}$ and, hence, the vectors φ^0 satisfying the equation $\Delta^\top(0)\varphi^0 = 0$ have the form $\varphi^0 = \gamma(-3 \ \ 1)^\top$, $\gamma \in \mathbf{R}$. Consequently, $(\text{iii})_4$ is satisfied. Since $(\text{i})_4$ and $(\text{ii})_4$ also hold, we conclude that in this case the system is approximately controllable.

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