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ON THE EQUIVALENCE, VIA RELAXATION–PENALIZATION, BETWEEN VECTOR GENERALIZED SYSTEMS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. A vector generalized system is considered, and a condition is given under which a relaxation of the domain and a penalization of the vector function do not change the set of solutions of the system. Applications are made to Vector Optimization and to Vector Variational Inequalities in a discrete space.

1. Introduction

The great development of the theory of constrained extremum problems and, more recently, that of variational inequalities has led to search for models which embody both theories. A possible answer is offered by the so–called generalized systems.

Such theories have received, in their inners, different (and, sometimes, almost disjoint) developments depending on the kind of space where the problems have been considered. In particular, this has happened in the field of constrained extrema: the so–called Combinatorial Optimization and Continuous Optimization have had a few in common. There is a strong need of establishing connections among such different developments to achieve reciprocal benefits.

As concerns constrained extrema there already exist several results connecting combinatorial and continuous problems; see, for instance, [7, 12, 13]. One of the tools for investigating such connections has been the relaxation of the feasible region; this has provided an equivalence theorem [7], which has been subsequently exploited both in finite dimensional spaces

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[8, 9, 10, 11] and in infinite dimensional ones [3]. An important consequence of such an equivalence has been the possibility to reduce the minimization of a binary problem to that of a strictly concave one for which an elegant method was proposed by H. Tuy as early as in 1964 [10, 14]. Recently [2], in order to extend the above equivalence theorem to fields different from Optimization, as Variational Inequalities, the investigation has been extended to the study of the impossibility of a (scalar) system, which embodies the optimality conditions of a (scalar) constrained extremum problem as well as Variational Inequalities.

The present paper deals with Vector Generalized Systems, which embody at least Vector Optimization and Vector Variational Inequalities; these two fields have been recently [1, 5, 6] connected with the study of the impossibility of a system and, consequently, with the separation of sets.

As concerns the notation, $\subseteq$ will denote inclusion and $\subset$ strict inclusion (inclusion without coincidence); analogously for $\supseteq$ and $\supset$. Moreover, given a convex cone[3] we define[4]

$$x \geq_C y \;\Leftrightarrow\; x - y \in C\,;\; x \ngeq_C y \;\Leftrightarrow\; x - y \notin C;\; x >_C y \Leftrightarrow\; x - y \in \operatorname{int} C\,;$$

the notation $\leq$ and $\nleq$ is defined in analogous way.

2. Generalized systems

Assume we are given the positive integers n and ℓ, the convex cone [5] $C \subset \mathbf{R}^\ell$, the sets $R, Z \subseteq \mathbf{R}^n$, and the vector–valued function $F : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^\ell$. Consider the problem $\mathcal{P}$ which consists in finding $y \in R \cap Z$ such that the system (in the unknown x):

$$F(x; y) \in C, \quad x \in R \cap Z \tag{2.1}$$

be impossible.

Assume we are given a set $X \subset \mathbf{R}^n$, such that $Z \subseteq X$. The replacement of $R \cap Z$ with $R \cap X$ represents a relaxation of the domain of (2.1); of course, this may change the set of solutions of $\mathcal{P}$. This drawback can be

[3] All the cones which will be considered have the apex at the origin. As usual, C is a cone with apex at the origin iff $x \in C$ and $\lambda \in (\mathbf{R}_+ \setminus \{0\})$ imply $\lambda x \in C$.

[4] $\operatorname{int} S$ will denote the interior of the set S.

[5] Next strict inclusion is motivated by the fact that, when $C = \mathbf{R}^\ell$, then the following problem $\mathcal{P}$ has no solutions.

overcome through a suitable change of the system. To this end, let us introduce a vector–valued function $\Phi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^\ell$, and consider the family $\{\mathcal{P}(\mu)\}_{\mu \in \mathbf{R}}$ of problems, where $\mathcal{P}(\mu)$ consists in finding $y \in R \cap X$, such that the system (in the unknown x):

$$F(x; y) + \mu\Phi(x; y) \in C, \quad x \in R \cap X \tag{2.2}$$

be impossible. (2.2) shows, with respect to (2.1), a relaxation of the domain and a penalization of F.

We want to state conditions under which $\mathcal{P}$ and $\mathcal{P}(\mu)$ are equivalent in the sense that they have the same set of solutions (if any, or none of them has solutions). To this end we need a preliminary proposition. For the sake of simplicity, and without any fear of confusion, $\|\cdot\|$ will denote an Euclidean norm both in $\mathbf{R}^\ell$ and in $\mathbf{R}^n$, and a norm of a matrix.

Lemma 1. *Let $\mathcal{C}$, $\mathcal{C}^0 \subset \mathbf{R}^\ell$ be cones such that $\mathcal{C}^0$ be closed, and*[6] $\emptyset \neq (\mathcal{C}^0 \setminus \{0\}) \subseteq \operatorname{int}\mathcal{C}$. *Let $B_\rho := \{x \in \mathbf{R}^\ell : \|x\| \leq \rho\}$ with $\rho \geq 0$, and $U := \{x \in \mathbf{R}^\ell : \|x\| = 1\}$. Then, there exists $\eta_0 \in \mathbf{R}$ such that*

$$V_1 + \eta V_2 \in \operatorname{int}\mathcal{C}\,, \quad \forall \eta > \eta_0\,, \quad \forall V_1 \in B_\rho\,, \quad \forall V_2 \in \mathcal{C}^0 \cap U. \tag{2.3}$$

Proof. Since $\|V_2\| = 1 \ \forall V_2 \in \mathcal{C}^0 \cap U$, and since the scalar product of vectors of unitary norm is ≤ 1, then $\forall \eta > \rho$ we have

(2.4)
$$1 \geq \frac{\langle V_1 + \eta V_2, V_2\rangle}{\|V_1 + \eta V_2\| \cdot \|V_2\|} \geq \frac{\langle V_1, V_2\rangle + \eta\|V_2\|^2}{\|V_1\| \cdot \|V_2\| + \eta\|V_2\|^2} \geq \frac{-\rho + \eta}{\rho + \eta}\,, \ \forall V_2 \in \mathcal{C}^0 \cap U,$$

where the 3rd inequality is a consequence of the inequalities $\|V_1\| \leq \rho$ and $\langle V_1, V_2\rangle \geq -\|V_1\| \geq -\rho$, $\forall V_2 \in \mathcal{C}^0 \cap U$. Since the scalar product of vectors of unitary norm is 1 iff they coincide, passing in (2.4) to the limit as $\eta \to +\infty$, we have that

$$\lim_{\eta \to +\infty} \frac{1}{\|V_1 + \eta V_2\|}(V_1 + \eta V_2) = V_2\,, \quad \forall V_2 \in \mathcal{C}^0 \cap U.$$

Since $\mathcal{C}^0 \cap U$ is a compact set included in $\operatorname{int}\mathcal{C}$, then[7] $(\sim \mathcal{C}) \cap U$ and $\mathcal{C}^0 \cap U$ have positive distance (induced by the norm considered). Hence (2.3) follows. This completes the proof. $\square$

[6] Next inclusion holds as equality iff $\ell = 1$.

[7] $\sim S$ will denote the complement of the set S.

Note that, when $\ell = 1$, the assumptions of Lemma 1 are fulfilled only by $\mathcal{C} = \mathbf{R}_+$ and by $\mathcal{C} = \mathbf{R}_+ \setminus \{0\}$ (and, of course, by their opposite); in both cases $\mathcal{C}^0 = \mathbf{R}_+$ necessarily (or $\mathcal{C}^0 = \mathbf{R}_-$).

In the sequel, $\forall x \in X$, $p(x)$ will denote a vector belonging to the set $\mathrm{proj}_Z(x)$, where $\mathrm{proj}_Z : X \to Z$ is the multi–valued function which projects x on the compact set Z.

Theorem 1. *Let $R \subset \mathbf{R}^n$ be a closed set, $Z \subset X \subset \mathbf{R}^n$, Z and X be compact sets, and let the following hypotheses hold.*

(H_1) *$F : X \times X \to \mathbf{R}^\ell$ is bounded on $X \times X$, and there exist positive reals L, α, and an open set $\Omega \supset Z$, such that*

$$\|F(p(x); x)\| \le L\|x - p(x)\|^\alpha\,, \quad \forall x \in \Omega \cap X, \quad \forall p(x) \in \mathrm{proj}_Z(x).$$

(H_2) *$\Phi : X \times X \to \mathbf{R}^\ell$ is such that*

(i) *Φ is continuous on $X \times X$;*

(ii) *$\forall x, y \in Z$, $\Phi(x; y) = 0$;*

(3i) *there exists a closed cone C^+ with $\emptyset \ne (C^+ \setminus \{0\}) \subseteq \mathrm{int}\, C$, such that*

$$\Phi(x; y) \in (C^+ \setminus \{0\})\,, \quad \forall x \in Z\,, \quad \forall y \in X \setminus Z;$$

(4i) *$\forall z \in Z$ there exist a neighbourhood $S(z)$ of z and a real $\varepsilon(z) > 0$, such that*

$$\|\Phi(p(x); x)\| \ge \varepsilon(z) \cdot \|x - p(x)\|^\alpha\,,\ \forall x \in S(z) \cap (X \setminus Z),\ \forall p(x) \in \mathrm{proj}_Z(x).$$

Then, a real μ_1 exists, such that, $\forall \mu > \mu_1$, a solution of $\mathcal{P}(\mu)$ is a solution of $\mathcal{P}$.

Proof. To prove the thesis it is sufficient to show that $\exists \mu_0 \in \mathbf{R}$ such that, $\forall \mu > \mu_0$, a solution of of $\mathcal{P}(\mu)$ is achieved necessarily at points $z \in R \cap Z$; because of (H_2)(ii) this claim assures that a solution of $\mathcal{P}(\mu)$ is a solution of $\mathcal{P}$ too.

Let us introduce the sets $\hat{X} := R \cap X$, $\hat{Z} := R \cap Z$, $\hat{S}(z) := \Omega \cap S(z)$, where $S(z)$ is precisely that of (H_2)(4i). The family $\{\hat{S}(z), z \in \hat{Z}\}$ is obviously a cover of $\hat{Z}$; since Z is compact and $\hat{Z}$ is a closed subset of Z, there is a finite subfamily, say $\{\hat{S}(z_i),\ i = 1, \dots, k\}$, which is a cover of $\hat{Z}$. Put $S := \bigcup_{i=1}^{k} \hat{S}(z_i)$ and let ρ be the quantity $\max\{L/\varepsilon(z_i), i = 1, \dots, k\}$. Because of ($H_1$) and ($H_2$)(4i), it holds

$$\left\| \frac{1}{\|\Phi(p(x); x)\|} F(p(x); x) \right\| \le \rho, \quad \forall x \in S \cap (\hat{X} \setminus \hat{Z}).$$

Consider the set $U := \{x \in \mathbf{R}^\ell : \|x\| = 1\}$; because of $(\mathrm{H}_2)(3i)$ we have

$$\frac{1}{\|\Phi(p(x);x)\|}\Phi(p(x);x) \in C^+ \cap U\,, \quad \forall x \in \hat{X} \setminus \hat{Z}\,. \tag{2.5}$$

We can apply Lemma 1: V_1, V_2, $\mathcal{C}^0$ and $\mathcal{C}$ are identified with $[1/\|\Phi(p(x);x)\|]F(p(x);x)$, $[1/\|\Phi(p(x);x)\|]\Phi(p(x);x)$, C^+ and C respectively. The assumptions of Lemma 1 being fulfilled, we achieve the existence of a real η_1, such that (2.3) holds, namely, $\forall \eta > \eta_1$ and $\forall x \in S \cap (\hat{X} \setminus \hat{Z})$, we have

$$\frac{1}{\|\Phi(p(x);x)\|}F(p(x);x) + \eta\frac{1}{\|\Phi(p(x);x)\|}\Phi(p(x);x) \in C. \tag{2.6}$$

It follows that, $\forall \mu > \eta_1$, $\mathcal{P}(\mu)$ cannot have solutions in $S \cap (\hat{X} \setminus \hat{Z})$.

Now, let us introduce the compact set $X_0 := \hat{X} \setminus S$, and fix $\hat{z} \in \hat{Z}$. Because of (H_2)(i,3i), Φ is continuous and different from the null vector on the compact set $\{\hat{z}\} \times X_0$, then we find

$$M_\Phi := \min_{x \in X_0} \|\Phi(\hat{z};x)\| > 0.$$

We can apply Lemma 1: we choose $\rho = M_F/M_\Phi$, where V_1, V_2, $\mathcal{C}^0$ and $\mathcal{C}$ are identified with $\left[\frac{1}{\|\Phi(\hat{z};x)\|}\right]F(\hat{z};x)$, $\left[\frac{1}{\|\Phi(\hat{z};x)\|}\right]\Phi(\hat{z};x)$, C^+ and C, respectively, and $M_F = \sup\limits_{(x,y)\in X\times X} \|F(x;y)\|$. Then, the hypotheses of Lemma 1 being satisfied, we achieve the existence of η_2, such that, $\forall \eta > \eta_2$ and $\forall x \in X_0$, we have

$$\frac{1}{\|\Phi(\hat{z};x)\|}F(\hat{z};x) + \eta\frac{1}{\|\Phi(\hat{z};x)\|}\Phi(\hat{z};x) \in C. \tag{2.7}$$

Hence, $\forall \mu > \eta_2$, $\mathcal{P}(\mu)$ cannot have solutions in X_0. If $\mu > \mu_1 := \max\{\eta_1, \eta_2\}$, account taken of (2.6) and (2.7), $\mathcal{P}(\mu)$ cannot have solutions in $\hat{X} \setminus \hat{Z}$. This completes the proof. $\square$

Remark 1. Consider the special case, where

$$F(x;y) = f(y) - f(x), \tag{2.8}$$

with $f : X \to \mathbf{R}^\ell$ bounded. It is trivial to check that, if f fulfills the Hölder Condition on the set Ω, i.e. $\exists\, L, \alpha > 0$ such that

$$\|f(x) - f(y)\| \le L \cdot \|x - y\|^\alpha, \quad \forall x, y \in \Omega \cap X, \tag{2.9}$$

then (2.8) satisfies (H_1). The converse is not true as shown by Example 1.

Note that y is a solution or vector minimum point (in short, VMP) of $\mathcal{P}$ in case (2.8) iff y is a solution of the vector minimum problem

$$\min_C f(x)\,, \quad x \in R \cap Z, \tag{2.10}$$

where C defines now the partial order and $\min_C$ marks vector minimum: the impossibility of (2.1) defines y as a solution of (2.10).

Example 1. Let us set $n = \ell = 1$, $R = \mathbf{R}$, $Z = [0, 1]$ and $X = [-1, 1]$. Consider (2.8) with $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. (H_1) is satisfied at $\alpha = 1$, $L = 1$ and $\Omega =]-2, 2[$. In fact, if $x \in Z$ so that $p(x) = x$, then $F(p(x); x) = 0$; if $x \in (\Omega \cap X) \setminus Z = X \setminus Z = [-1, 0[$, then $|F(0; x)| = |f(x)| = |x \sin \frac{1}{x}| \leq |x|$. Thus (H_1) holds. Obviously, f does not fulfill (2.9).

Remark 2. Let us consider two special cases. To this aim, let us introduce $G : \mathbf{R}^n \to \mathbf{R}^{\ell \times n}$, and let $\langle G(y), y - x\rangle_\ell$ denote the vector whose i–th component is the scalar product between the i–th row of matrix $G(y)$ and the vector $y - x$. The former special case is:

$$F(x; y) = \langle G(y), y - x\rangle_\ell\ . \tag{2.11a}$$

With the same notation, the latter case is:

$$F(x; y) = \langle G(x), y - x\rangle_\ell\ . \tag{2.11b}$$

If G is bounded on $\mathbf{R}^n$, then both functions (2.11) fulfill (H_1) at $\alpha = 1$ and $L = \|G\| := \sup\limits_{x \in \mathbf{R}^n} \|G(x)\|$, as it is easy to check. Example 2 shows that the converse is not true.

Note that y is a solution of $\mathcal{P}$ in case (2.11a) (or (2.11b)) iff is a solution of the Vector Variational Inequality of Stampacchia type (see [6]): find $y \in R \cap Z$, such that:

$$\langle G(x), y - x\rangle_\ell \not\leq_C 0\,, \quad \forall x \in R \cap Z \tag{2.12a}$$

(or of the Vector Variational Inequality of Minty type (see [6]): find $y \in R \cap Z$, such that

$$\langle G(y), x - y\rangle_\ell \not\geq_C 0\,, \quad \forall x \in R \cap Z). \tag{2.12b}$$

The impossibility of (2.1) defines y as a solution of (2.12a) (or (2.12b)).

Example 2. Let us set $n = \ell = 1$, $C = \mathbf{R}^+ \setminus \{0\}$, $R = \mathbf{R}$, $Z = [0,1]$ and $X = [0,2]$. Consider (2.11a) with $G(y) = 1/\sqrt{|y-1|}$ if $y \neq 1$, and $G(y) = 0$ if $y = 1$; it holds

$$|F(p(x);x)| \leq |x - p(x)|^{1/2}, \quad \forall x \in]1,2[\,;$$

in fact, if $x \in Z$ so that $p(x) = x$, then $F(p(x);x) = 0$; if $x \in]1,2[$, then $p(x) = 1$ so that $|F(p(x);x)| = \sqrt{x-1}$.

Example 3. Let $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $F(x;y) = \sqrt{|x-y|}(x-1)$, $Z = [0,1]$, $X = [0,2]$. Such a function does not fulfill the following condition: there exist a constant L and an open set $\Omega \supset Z$ such that

$$|F(x;y)| \leq |x-y|, \quad \forall x \in \Omega \cap X, \quad \forall y \in Z.$$

Such a function fulfills hypothesis (H_1) of Theorem 1 at $\Omega = \mathbf{R}$, $L = \sqrt{2}$, $\alpha = 3/2$.

Example 4. Let us set $n = \ell = 1$, $R = \mathbf{R}$, $C = \mathbf{R}^+ \setminus \{0\}$, $Z = [0,1]$, $X = [0,2]$, $F : X \times X \to \mathbf{R}$, $F(x;y) = (x-y)^2(1-y)(x-1)$. Such a function F fulfills (H_1), with $\Omega =]-1,2[$, $L = 1$, $\alpha = 2$. In facts, F is bounded; moreover if $x \in]1,2[$, $p(x) = 1$ and $F(1;x) = 0$; if $x \in Z$, $p(x) = x$ and $F(x;x) = 0$. Each $y \in [0,1]$ is a solution of the following problem $\mathcal{P}$: find $y \in Z$ such that

$$F(x;y) \in C\,, \quad x \in [0,1]$$

be impossible.
Let $\Phi : X \times X \to \mathbf{R}$ be a penalty function defined as follows:

$$\Phi(x;y) = \begin{cases} -(1-x)^2, & \text{if } (x,y) \in]1,2] \times [0,1], \\ 0, & \text{if } (x,y) \in [0,1] \times [0,1], \\ (1-y)^2, & \text{if } (x,y) \in [0,1] \times [1,2], \\ -(x-y)^2, & \text{if } (x,y) \in]1,2] \times]1,2],\ y \leq x, \\ (x-y)^2, & \text{if } (x,y) \in]1,2] \times]1,2],\ y > x. \end{cases}$$

Such a function Φ fulfill condition (H_2): it is enough the choose $C^+ = \mathbf{R}_+$, $\varepsilon(z) = 1$, $\forall z \in Z$, $\alpha = 2$, $L = 1$. We show that, $\forall \mu \in \mathbf{R}_+$, $y \in [0,1]$ is not a solution for $P(\mu)$. In fact, $\forall x \in]1,2]$,

$$F(x;y) + \mu\Phi(x;y) = (x-1)[(x-y)^2(1-y) - \mu(x-1)].$$

Observe that $\lim\limits_{x \downarrow 1} (x-y)^2(1-y) - \mu(x-1) = (1-y)^3 > 0$.

We conclude that $y \in [0,1]$ is not a solution of $P(\mu)$.
Observe that F does not fulfill (H_3) of the following Theorem 2 at $\alpha = 2,\ L = 1$. In fact, for $x \geq 1$, $p(x) = 1$, then the inequality

$$|F(p(x);y) - F(x;y)| \leq (x-1)^2, \quad \forall x \in X \cap \Omega, \quad \forall p(x) \in \operatorname{proj}_Z(x),$$

holds if and only if

$$(x-y)^2(1-y) \leq (x-1), \quad \forall x > 1, \quad \forall y \in [0,1].$$

This is impossible for $x = \dfrac{5}{4}$, $y = \dfrac{1}{4}$.
Moreover, observe that $P(\mu)$ has not solution: for each $y \in]1,2]$ and for each $x \in [0,1]$, $\forall \mu > 0$,

$$F(x;y) + \mu\Phi(x;y) \in C.$$

If Z is finite, then the inequality in (H_2)(4i) can be equivalently replaced (in the sense that the thesis of the Theorem 1 is still achieved and the class of the penalty functions Φ which satisfy it is non–empty) with the following condition:
$\forall z \in Z$ there exist a neighborhood $S(z)$ and a real $\varepsilon(z) > 0$, such that

$$\|\Phi(z;x)\| \geq \varepsilon(z)\|x - z\|, \quad \forall x \in S(z) \cap (X \setminus Z).$$

In fact, choosing a suitable neighborhood $S(z)$ of z, we have $p(x) = z$. If Z is not finite, then the above condition might be in contrast with assumptions (H_2)(i,ii).

Let us consider the following condition:

$(\mathrm{H}_2)'$ It is possible to find a vector–valued function $\phi : X \to \mathbf{R}^\ell$, such that:
(i) ϕ is continuous on X;
(ii) $\forall x \in Z, \quad \phi(x) = 0$;
(3i) there exists a closed cone C^+ with $\emptyset \neq (C^+ \setminus \{0\}) \subseteq \operatorname{int} C$, such that

$$\phi(x) \in (C^+ \setminus \{0\}), \quad \forall x \in X \setminus Z;$$

(4i) $\forall z \in Z$ there exist a neighborhood $S(z)$ and a real $\varepsilon(z) > 0$, such that

$$\|\phi(x)\| \geq \varepsilon(z) \cdot \|x - p(x)\|^\alpha\ , \ \forall x \in S(z) \cap (X \setminus Z), \quad \forall p(x) \in \operatorname{proj}_Z(x).$$

Note that, if we set $\Phi(x;y) = \phi(y) - \phi(x), \ \ \forall x, y \in X$, then (H_2) of Theorem 1 is fulfilled if $(\mathrm{H}_2)'$ does.

When $\ell = 1$, Theorem 1 collapses to Theorem 1 of [2].

The following theorem consider a special class of problem $\mathcal{P}$: the cone C is equal to $\mathbf{R}^\ell_+$; it gives a condition assuring a solution of $\mathcal{P}$ is a solution also of a suitable problem $\mathcal{P}(\mu)$, for μ large enough.

Theorem 2. *Let $R \subseteq \mathbf{R}^n$ be a closed set, $Z \subset X \subset \mathbf{R}^n$, Z and X be compact sets, and let the following hypotheses hold.*

(H_3) $F : X \times X \to \mathbf{R}^\ell$ *is bounded on $X \times X$ and there exist positive reals L, α and an open set $\Omega \supset Z$, such that*

$$\|F(x;y) - F(p(x);y)\| \leq L \cdot \|x - p(x)\|^\alpha, \ \forall x, y \in \Omega \cap X, \ \forall p(x) \in \mathrm{proj}_Z(x).$$

(H_4) *It is possible to find $\Phi : X \times X \to \mathbf{R}^\ell$, such that:*
(i) Φ *is continuous on $X \times X$;*
(ii) $\forall x, y \in Z$, $\Phi(x;y) = 0$*;* $\forall x \in X$, $\Phi(x;\cdot)$ *is constant on Z;*
(3i) *there exists a closed cone C^-, with $\emptyset \neq (C^- \setminus \{0\}) \subseteq \mathrm{int}\,(-C)$, such that:*

$$\Phi(x;y) \in (C^- \setminus \{0\}), \quad \forall x \in X \setminus Z, \ \forall y \in Z;$$

(4i) $\forall z \in Z$, *there exists a neighbourhood $S(z)$ of z and a positive real $\varepsilon(z)$, such that*

$$\|\Phi(x;p(x))\| \geq \varepsilon(z) \cdot \|x - p(x)\|^\alpha \ , \ \forall x \in S(z) \cap (X \setminus Z) \ , \ \forall p(x) \in \mathrm{proj}_Z(x).$$

Then, there exists $\mu_2 \in \mathbf{R}$, such that, $\forall \mu > \mu_2$, a solution of $\mathcal{P}$ is a solution of $\mathcal{P}(\mu)$.

Proof. Let $S := \bigcup_{i=1}^{k} \hat{S}(z_i)$ be a finite cover of Z. Let us introduce the sets $\hat{X} := R \cap Z$, $\hat{Z} = R \cap Z$, $\hat{S}(z) = \Omega \cap S(z)$, where $S(z)$ is the neighbourhood of (H_4) (4i). Because of (H_4)(3i), $\forall x \in S \cap (\hat{X} \setminus \hat{Z})$ we have

$$\frac{1}{\|\Phi(x;p(x))\|} \Phi(x;p(x)) \in C^- \cap U,$$

where $U := \{x \in \mathbf{R}^\ell : \|x\| = 1\}$. We can apply the Lemma 1. To this purpose let us set $\rho = \max\left\{\frac{L}{\varepsilon(z_i)}, \ i = 1, \ldots, k\right\}$; V_1, V_2, $\mathcal{C}^0$ and $\mathcal{C}$ are identified with

$$[1/\|\Phi(x;p(x))\|][F(x;y) - F(p(x);y)] \, , \quad [1/\|\Phi(x;p(x))\|]\Phi(x;p(x)),$$

C^- and $-C$, respectively. Hence, the assumptions of Lemma 1 being satisfied, we achieve the existence of $\eta_3 \in \mathbf{R}$ such that, $\forall \eta > \eta_3$, we have

$$F(x;y) - F(p(x);y) + \eta\Phi(x;p(x)) \in \operatorname{int}(-C), \quad \forall x \in S \cap (\hat{X} \setminus \hat{Z}),\ \forall y \in \hat{Z}, \quad \forall p(x) \in \operatorname{proj}_Z(x). \tag{2.13}$$

Then, because of (H$_4$) (ii), $\forall \eta > \eta_3$ we have

$$F(x;y) - F(p(x);y) + \eta\Phi(x;y) \in (-C), \quad \forall x \in S \cap (\hat{X} \setminus \hat{Z}), \quad \forall y \in \hat{Z} \quad \forall p(x) \in \operatorname{proj}_Z(x). \tag{2.14}$$

Now, we will prove that $\exists \eta_4 \in \mathbf{R}$, such that, $\forall \eta > \eta_4$ we have

$$F(x;y) + \eta\Phi(x;y) \in (-C),\ \forall x \in X_0 := \hat{X} \setminus \hat{S},\ \forall y \in \hat{Z}. \tag{2.15}$$

In fact, Φ is continuous and different from the null vector on $(X \setminus Z) \times Z$; this fact and (H$_4$) (3i) imply

$$\frac{1}{\|\Phi(x;y)\|}\Phi(x;y) \in C^- \cap U.$$

We can apply the Lemma 1 with $\rho = (1/M)\|F\|$, where

$$M := \min_{(x,y)\in X_0\times Z} \|\Phi(x;y)\|;$$

$V_1, V_2, \mathcal{C}^0$ and $\mathcal{C}$ are identified with $\left[\frac{1}{\|\Phi(x;y)\|}\right]F(x;y)$, $\left[\frac{1}{\|\Phi(x;y)\|}\right]\Phi(x;y)$, C^- and $-C$, respectively. Hence, the assumptions of Lemma 1 being satisfied, we achieve the existence of η_4, such that, $\forall \eta > \eta_4$ (2.14) holds.
Now, let $\bar{y}$ be a solution of $\mathcal{P}$. Then, $\forall x \in X$ and $\forall p(x) \in \operatorname{proj}_Z(x)$, account taken of $C = (\mathbf{R}^\ell_+ \setminus \{0\})$, there exists an index i, such that

$$(F(p(x);\bar{y}))_i \le 0, \tag{2.16}$$

where $(\cdot)_i$ denotes i–th component. We conclude that, $\forall \mu > \mu_2 := \max\{\eta_3, \eta_4\}$, we have, $\forall x \in \hat{X}$,

$$F(x;\bar{y}) + \mu\Phi(x;\bar{y}) \notin C. \tag{2.17}$$

In fact, since $\bar{y}$ is a solution of $\mathcal{P}$ and Φ is null on $Z \times Z$, (2.17) holds for $x \in \hat{Z}$; (2.15) implies (2.17) for $x \in X_0$; (2.14) and (2.16) imply (2.17) for $x \in S \cap (\hat{X} \setminus \hat{Z})$. This completes the proof. □

Note that hypothesis (H_3) of Theorem 2 can be weakened by replacing the condition "$x, y \in \Omega \cap X$" with "$x \in \Omega \cap X$, $y \in Z$."

3. The case of a discrete domain

We will now analyse a special but very important case, namely that of a discrete problem. Indeed, we will consider the case $Z = \mathbf{B}^n$, where $\mathbf{B} := \{0, 1\}$; the case where Z is a bounded subset of $\mathbf{Z}^n$ can be reduced to the present one by well known transformations [12]. More precisely, we consider now the following case:

(3.1)

$$C = \mathbf{R}^\ell_+ \backslash \{0\},\ Z = \mathbf{B}^n,\ C^+ = \{v \in \mathbf{R}^\ell_+ : v_1 = v_2 = \cdots = v_\ell\},\ C^- = -C^+.$$

The "natural" relaxation of Z is now $X = X_Q := \{x \in \mathbf{R}^n : 0 \leq x \leq e\}$, where [8]

$e^T := (1, \ldots, 1)$. As "penalty term" we choose: $\Phi : X \times X \to \mathbf{R}^\ell$,

$$\Phi(x; y) = \begin{pmatrix} \varphi(y) - \varphi(x) \\ \vdots \\ \varphi(y) - \varphi(x) \end{pmatrix}, \tag{3.2}$$

where $\varphi : \mathbf{R}^n \to \mathbf{R}$, $\varphi(x) := x^T(e - x)$.

Under assumptions (3.1), the function Φ defined in (3.2) fulfills, at $\alpha = 1$, the conditions (H_2) of Theorem 1 and (H_4) of Theorem 2. In fact, (H_2)(i), (ii) and (3i) and (H_4) (i), (ii) and (3i) are obvious. As concerns (H_2)(4i) and (H_4)(4i), let us note that, if $S(z)$ is small enough so that $p(x) = z$, then φ satisfies the following condition: $\forall z \in Z$ there is a neighbourhood $S(z)$ of z and a real $\hat{\varepsilon}(z)$, such that

$$\varphi(x) \geq \hat{\varepsilon}(z) \cdot \|x - z\|, \quad \forall x \in S(z) \cap (X \setminus Z),$$

as it has been proved in [G_1, Th. 3.1; G_3, Th. 4]; therefore, $\forall z \in Z$ and $\forall x \in S(z) \cap (X \setminus Z)$ we find

$$\|\Phi(p(x); x)\| = \|\Phi(x; p(x))\| = \left\| \begin{pmatrix} \varphi(x) \\ \vdots \\ \varphi(x) \end{pmatrix} \right\| = \sqrt{\ell}|\varphi(x)| \geq \sqrt{\ell}\hat{\varepsilon}(z)\|x - z\|$$

[8] T as apex will mark transposition.

which proves (H_2)(4i) and (H_4)(4i), by setting $\varepsilon(z) = \sqrt{\ell}\hat{\varepsilon}(z)$, and completes the proof. Hence, we have proved the following:

Theorem 3. *Under the case* (3.1)-(3.2), *let the function F verify at $\alpha = 1$, the hypothesis* (H_3) *of Theorem 2. Furthermore, F be such that $F(x;x) = 0$, $\forall x \in \Omega$. Then, there exists $\mu_3 \in \mathbf{R}$ such that, $\forall \mu > \mu_3$, $\mathcal{P}$ and $\mathcal{P}(\mu)$ have the same solutions.*

Remark 3. When $F : X \times X \to \mathbf{R}^\ell$ is of the kind

$$F(x;y) = f(y) - f(x),$$

where $f : X \to \mathbf{R}^\ell$, or

$$F(x;y) = \langle G(y), y - x\rangle,$$

where $G : \mathbf{R}^n \to \mathbf{R}^{\ell\times n}$, then it fulfills the condition $F(x;x) = 0,\ \forall x \in \Omega$.

In the Theorem 3 we have considered the hypothesis (H_3) at $\alpha = 1$ since this is enough with the special Φ we have chosen. Concerning such a choice, note that the above theorem is still valid if we select any strictly concave functions $\varphi_1, \dots, \varphi_\ell$, such that, $\forall_i$, $\varphi_i : \mathbf{R}^n \to \mathbf{R}$, $\varphi_i(x) = 0\ \ \forall x \in Z$, $\varphi_i(x) > 0\ \ \forall x \in X \setminus Z$; moreover, $\forall y \in Z$, there exist a neighbourhood $S_i(y)$ and a real constant $\varepsilon_i(y)$ such that

$$|\varphi_i(x)| \geq \varepsilon_i(y)\|x - y\|^\alpha, \quad \forall x \in S_i(y) \cap (X \setminus Z).$$

Note that, $\forall i$, the above condition is a slight generalization of the condition on the function φ in [7], and it is equivalent to (H_2) of Theorem 1 at $\ell = 1$. Then, we can put $\phi = (\varphi_1, \dots, \varphi_\ell)$, and, $\forall x, y \in X$,

$$\Phi(x,y) = \phi(y) - \phi(x). \tag{3.3}$$

Condition (H_2) (4i) for Φ follows choosing, $\forall z \in Z$, $S(z) = \cap_{i=1}^{\ell} S_i(z)$ and $\varepsilon(z) = \sqrt{\ell}\min\{\varepsilon_i(y),\ i = 1, \dots, \ell\}$.

The following theorem gives a condition which assures that, $\forall y \in X$, the function $F(\cdot;y) + \mu\Phi(\cdot;y)$ is component–wise strictly convex.

This is a straightforward extension of Theorem 3.2 of [7] and Theorem 2 of [8].

Theorem 4. *In the case* (3.1)–(3.2), *let $F : X \times X \to \mathbf{R}^\ell$ be a function fulfilling the hypotheses of Theorem 3 at $\alpha = 1$. If $F \in [C^2(X \times X)]^\ell$, then there exists a real μ_4 such that, $\forall \mu > \mu_4$, $\mathcal{P}$ and $\mathcal{P}(\mu)$ have the same* *solutions, and, $\forall y \in X$, the function $F(\,\cdot\,; y) + \mu\Phi(\,\cdot\,; y)$ is component–wise strictly convex.*

Proof. $\forall y \in X$, let $H_i(x; y)$ and $\hat{H}_i(x; y)$ be the Hessian matrices at x, of the i–th component of $F(\cdot; y)$ and $F(\cdot; y) + \hat{\mu}\Phi(\cdot; y)$, respectively. Hence, $\forall i = 1, \dots, \ell$, we have

$$\hat{H}_i(x; y) = H_i(x; y) + 2\hat{\mu} I_n,$$

where I_n denotes the $n \times n$ identity matrix. $\forall i \in \{1, \dots, \ell\}$, $\forall r \in \{1, \ll, n\}$ let $\lambda_{ir} : X \times X \to \mathbf{R}$ be the function where $\lambda_{ir}(x; y)$, $r = 1, \dots, n$ are the eigenvalue of the Hessian $H_i(x; y)$. Because of the continuity of F, λ_{ir} is continuous. Hence

$$\eta_4 := \max_{i,r} \max_{X \times X} \ |\lambda_{ir}(x; y)| < +\infty.$$

Moreover, let us observe that, $\forall x, y$, $\nu_i(x, y)$ is an eigenvalue of $\hat{H}_i(x; y)$ if and only if $\nu_i(x; y) - 2\hat{\mu}$ is an eigenvalue of $H_i(x; y)$. Then, for any $\hat{\mu} > \mu_4 := \frac{1}{2}\max\{\mu_1, \mu_2, \eta_4\}$, $\forall (x, y) \in X \times X$,

$$\nu_{ir}(x; y) = \lambda_{ir}(x; y) + 2\hat{\mu}$$

is eigenvalue of $\hat{H}_i(x; y)$ and it results $\nu_{ir}(x; y) > 0$. This completes the proof. $\square$

4. Discrete vector optimization and variational inequality

Let us now consider the special case of (2.10), where $Z = \mathbf{B}^n$ and $C = \mathbf{R}^\ell_+ \setminus \{0\}$, namely the problem:

$$\min\nolimits_C f(x)\,, \quad x \in R \cap \mathbf{B}^n. \tag{4.1}$$

y is a vector minimum point of (4.1) iff the system (in the unknown x):

$$f(y) - f(x) \in C\,, \quad x \in R \cap \mathbf{B}^n \tag{4.2}$$

is impossible. When $\ell = 1$, then (4.1) becomes a scalar binary optimization and (4.2) becomes the obvious inequality: $f(y) \le f(x)$, $\forall x \in R \cap \mathbf{B}^n$.

With the positions $Z = \mathbf{B}^n$ and $F(x; y) = f(y) - f(x)$, (4.2) becomes (2.1). Now consider the vector minimum problem:

$$\min_C [f(x) + \mu\phi(x)]\,, \quad x \in R \cap X_Q, \tag{4.3}$$

where $\phi : \mathbf{R}^n \to \mathbf{R}^\ell$, $\phi = (\varphi, \ldots, \varphi)$, $\varphi(x) = x^T(e - x)$ and $\mu \in \mathbf{R}$.

Corollary 1. *Let the following hypothesis be satisfied.*

(H_5) $f : \mathbf{R}^n \to \mathbf{R}\ \ell$ *is bounded on* X_Q*, and there exist a positive real* L *and an open set* $\Omega \supset \mathbf{B}^n$ *which make true the inequality*

$$|f_i(x) - f_i(y)| \le L \cdot \|x - y\|,\ \forall x, y \in \Omega \cap X_Q,\ i = 1, \ldots, \ell,$$

where f_i *denotes the* i*–th component of* f*.*

Then, there exists a real $\mu_5 \in \mathbf{R}$*, such that,* $\forall \mu > \mu_5$*, (4.1) and (4.3) have the same solutions. If, in addition,* $f \in C^2(X)$*, then there exists* $\mu_6 \in \mathbf{R}$*, such that,* $\forall \mu > \mu_7 := \max\{\mu_5, \mu_6\}$*,* $f + \mu\Phi$ *is component–wise strictly concave.*

Proof. Put $\Phi : X \times X \to \mathbf{R}^\ell$, $\Phi(x; y) = \phi(y) - \phi(x)$. According to Remark 1, under assumption (H_5) the function $F(x; y) = f(y) - f(x)$ satisfies (H_1) of Theorem 1 and (H_3) of Theorem 2. Moreover, assumption (H_2) of Theorem 1 and (H_4) of Theorem 2 are fulfilled by the present Φ given by (3.2), because of what has been shown in Sect. 3. As concerns the 2nd part of the thesis, it is enough to note that (4.1) and (4.3) are equivalent to $\mathcal{P}$ and $\mathcal{P}(\mu)$, respectively. Hence, Theorem 4 can be applied. This completes the proof. □

In the special – but important – case where R is a polyhedron, note that Corollary 1 shows a class of vector minimization problems with strictly concave objective function, i.e. (4.3), having [9] a (vector) minimum point necessarily at a vertex of the feasible region. In general, this is not true as the following Theorem 5 and Examples 5, 6 show.

Theorem 5. *Let* $g : \mathbf{R}^n \to \mathbf{R}^\ell$ *be component–wise concave, and* $P \subset \mathbf{R}^n$ *be a non–empty polytope. Then, at least a vector minimum point of the problem:*

$$\min_C g(x)\,, \quad x \in P \tag{4.9}$$

happens at a vertex of P*.*

Proof. Consider the problems

$$a_i := \min_{x \in S_{i-1}} g_i(x)\,, \quad i = 1, \ldots, \ell; \quad S_0 := P,$$

[9] Because of their equivalence with (4.1) whose solutions are obviously vertices.

and the sets

$$S_i := \operatorname*{argmin}_{x \in S_{i-1}} g_i(x)\,, \quad i = 1, \dots, \ell.$$

We obviously have

$$S_i \subseteq S_{i-1}\,, \quad i = 1, \dots, \ell,$$

and, because of the concavity of g, $S_1, \dots, S_\ell$ are unions of faces of P. We will show that each element of S_ℓ is a VMP of (4.9), so that the thesis will follow. Consider any $x^0 \in S_\ell$. Ab absurdo, suppose that x^0 be not solution of (4.9). Then, $\exists y_{x^0} \in P$, such that:

$$g(y_{x^0}) \leq_C g(x^0),$$

so that $\exists i_{x^0} \in \{1, \dots, \ell\}$ such that

$$g_{i_{x^0}}(y_{x^0}) < g_{i_{x^0}}(x^0) = a_{i_{x^0}}; \tag{4.10a}$$

$$g_i(y_{x^0}) \leq g_i(x^0)\,, \quad \forall i = 1, \dots, \ell, \quad i \neq i_{x^0}. \tag{4.10b}$$

y_{x^0} must belong to S_ℓ. In fact, $\forall i = 1, \dots, \ell - 1$,

$$y_{x^0} \in S_i \setminus S_{i+1} \;\Rightarrow\; g_{i+1}(y_{x^0}) > g_{i+1}(x^0) = a_{i+1},$$

which contradicts (4.10b). Since

$$g(x) = (a_1, \dots, a_\ell)\,, \quad \forall x \in S_\ell,$$

we have $g(y_{x^0}) = (a_1, \dots, a_\ell)$ which contradicts (4.10a). Finally, observe that the sets $S_1, \dots, S_\ell$ are unions of faces of P. This completes the proof. $\square$

Remark 4. The above proof shows that the set of solutions to (4.9) contains a union of faces. If for any $k \in \{1, \dots, \ell\}$ S_k is a singleton, then obviously its (unique) element is a VMP of (4.9), and the subsequent S_i are equal S_k. Such a proof suggests a method for finding a solution of (4.9); indeed, this method does not require the concavity of g. However, it does not necessarily find all VMP; for instance, if g is component–wise strictly concave, then the method does not find the VMP (if any) which fall in $\operatorname{int} P$ whatever the ordering of the components of g may be. Moreover, note that the thesis of the above theorem can be achieved with the same proof under the assumption that only one component of g (which in the proof must be considered as g_1) be strictly concave.

Unlike the case $\ell = 1$, when $\ell > 1$ a VMP of (4.9) is not necessarily a vertex of P (in spite of the strict concavity of g), as Example 5 shows; this conclusion does not change, if we make the further assumption that the (global) maximum points of the several g_i fall in the interior of P, as Example 6 shows.

Example 5. Let us set $n = 1$, $P = [0, 1]$, $C = \mathbf{R}_+ \setminus \{0\}$, $g_1(x) = 1 - x^2$, $g_2(x) = x(2 - x)$. It is easy to check that every element of P is a VMP of (4.9).

Example 6. Let us set $n = 1$, $P = [-3, 3]$, $C = \mathbf{R}_+ \setminus \{0\}$, $g_1(x) = (x + 3)(7 - x)$, $g_2(x) = (3 - x)(x + 7)$. It is easy to check that the VMP of (4.9) are now $x = \pm 3$ and all the elements of $]-1, 1[$.

The case where a VMP of (4.9) is necessarily a vertex of P is a very special one. For instance, it happens if the function g is component–wise strictly concave and $vert\, P \subseteq lev_{=\beta}\; g_i$, $i = 1, \ldots, \ell$, where *lev* denotes level set.

Corollary 1 suggests a method for solving (4.1), which is based on the theory introduced in [14] (see also [10]), and will be shortly outlined. To this end we will consider the special, but wide, case where $R = P$. Because of Corollary 1 the combinatorial vector problem (4.1) can be replaced with the continuous vector problem (4.3). If μ is large enough (i.e., $\mu > \mu_7$), then, because of Theorem 5 (see Remark 4), a VMP of (4.3) is a vertex of $P \cap X_Q$. Therefore, a method can start by finding a vertex, say x^0, of $P \cap X_Q$. It is not restrictive to assume that x^0 be a local VMP of (4.3); otherwise this can be achieved by jumping from one vertex to an adjacent one until it has been obtained. Now, consider the family of strictly concave (scalar) problems:

$$(4.11) \qquad \min g_i(x; \mu)\,, \quad x \in P \cap X_Q\ ;\ i = 1, \ldots, \ell,$$

where $g_i(x; \mu) := f_i(x) + \mu\varphi(x)$. If i is such that x^0 is a local[10] (scalar) minimum point of (4.11), then Tuy Theory [10, 14] gives us a "cutting halfspace", say H_i, such that

$$(4.12) \qquad x^0 \notin H_i\ ;\ \left\{ \begin{array}{l} x \in P \cap X_Q \\ g_i(x; \mu) < g_i(x^0; \mu) \end{array} \right\} \Rightarrow\ x \in (P \cap H_i) \cap X_Q.$$

This condition means that, if there exists an x at which g_i takes a value

[10] In the sense of not necessarily global.

less that $g_i(x^0;\mu)$, then x must belong to H_i. For all other indexes i Tuy's "cutting halfspace" collapses to a supporting halfspace of $P \cap X_Q$; it will be denoted again by H_i. The former (latter) set of indexes will be denoted by I^+ (respectively I^-). If $I^+ \neq \emptyset$, then from (4.12) we easily deduce that:

$$\left.\begin{cases} x \in P \cap X_Q \\ g(x;\mu) <_C g(x^0;\mu) \end{cases}\right\} \Rightarrow \; x \in P \cap (\bigcap_{i \in I^+} H_i) \cap X_Q. \tag{4.13}$$

This condition means that, if there exists an x at which g takes a value less (in vector sense; with respect to C) than $g(x^0;\mu)$, then x must belong to $\bigcap_{i \in I^+} H_i$; hence such an intersection plays a role for vector problems as Tuy's cut does for scalar ones. The case $I^+ = \emptyset$ is a degenerate one for all Tuy's cut, and requires a special analysis: the present vertex can be replaced with any of the adjacent vertices, since they are alternative local VMP. From (4.13) we have that the condition:

$$I^+ \neq \emptyset\,, \quad P_1 \cap X_Q = \emptyset, \tag{4.14}$$

where

$$P_1 := P \cap \Big(\bigcap_{i \in I^+} H_i \Big),$$

is a sufficient condition for x^0 to be a VMP of (4.1) at $R = P$. If (4.14) is not satisfied, then we can replace, in (4.11), P with P_1 and repeat the above reasoning. Noting that the set

$$P \cap \Big(\bigcap_{i \in I^+} \sim H_i \Big) \cap X_Q$$

does not contain any alternative VMP of (4.3); while they might happen in the sets

$$P \cap (\sim H_r) \cap \Big(\bigcap_{i \in I^+ \setminus \{r\}} H_i \Big) \cap X_Q\,, \quad r \in I^+.$$

According to Remark 4, an alternative method for finding a VMP of (4.1) may consists in solving ℓ scalar problems, having a strictly concave objective function and a union of vertices of P as feasible region.

Let us now consider the special case of (2.12a) where $Z = \mathbf{B}^n$ and $C = \mathbf{R}^\ell_+ \setminus \{0\}$, namely the following Vector Variational Inequality: find $y \in R \cap \mathbf{B}^n$, such that

$$\langle G(y), x-y \rangle_\ell \not\leq_C 0\,, \quad \forall x \in R \cap \mathbf{B}^n. \tag{4.15}$$

With the positions $Z = \mathbf{B}^n$ and $F(x;y) = \langle G(y), x-y \rangle_\ell$, (4.15) becomes (2.1). Now, consider the problem which consists in finding $y \in R \cap X_Q$, such that

$$\langle G(y), x-y \rangle_\ell - \mu \Phi(x;y) \not\leq_C 0\,, \quad \forall x \in R \cap X_Q, \tag{4.16}$$

where Φ is the function in (3.2) and $\mu \in \mathbf{R}$.

Corollary 2. *Let $G : \mathbf{R}^n \to \mathbf{R}^{\ell \times n}$ be bounded on X_Q. Then, there exists a real $\mu_8 \in \mathbf{R}$, such that, $\forall \mu > \mu_8$,* (4.15) *and* (4.16) *have the same solutions.*

Proof. Let $F : X_Q \times X_Q \to \mathbf{R}^\ell$ be defined by $F(x;y) = \langle G(y), y-x \rangle$. According to Remark 2, such a function fulfils (H_1) of Theorem 1. (H_3) of Theorem 2 holds, since

$$\begin{aligned} \|F(x_1;y) - F(x_2;y)\| &= \|\langle G(y), y-x_1 \rangle - \langle G(y), y-x_2 \rangle\| \\ &\leq \|G\| \cdot \|x_1 - x_2\| \quad \forall x_1, x_2, y \in X_Q\,. \end{aligned}$$

The present Φ fulfils (H_2) of Theorem 1 and (H_4) of Theorem 2 as shown in Sect. 3. Hence, Theorems 1 and 2 give the existence of a real $\mu_8 \in \mathbf{R}$ such that (4.15) and (4.16) have the same solutions. □

In the special – but important – case where R is a polyhedron, note that Corollary 2 shows a class of Vector Variational Inequalities with bounded operator, i.e. (4.16), having – because of the equivalence with (4.15) – a solution necessarily at a vertex of the domain. In general this is not true as simple examples show.

5. Further developments

Let us consider the special case of (2.10), where $R = \mathbf{R}^n$, $Z = \bigcup_{k=1}^{r} Z_k$, with Z_k convex and compact, $f : \mathbf{R}^n \to \mathbf{R}^\ell$ component–wise convex, $C = \mathbf{R}^\ell_+ \setminus \{0\}$, namely the problem

$$\min\nolimits_C f(x)\,, \quad x \in \bigcup_{k=1}^{r} Z_k. \tag{5.1}$$

Obviously, y is a solution of (5.1) iff the system (in the unknown x)

$$f(y) - f(x) \in C\,, \quad x \in \bigcup_{k=1}^{r} Z_k \tag{5.2}$$

is impossible.

With the positions $R = \mathbf{R}^n$, $Z = \bigcup_{k=1}^{r} Z_k$ and $F(x;y) = f(y) - f(x)$, (5.2) becomes a special case of (2.1). Assume we are given the compact sets $X_k \supseteq Z_k$ and the functions $\phi_k : X \to \mathbf{R}$, $k = 1, \ll, r$, (where $X = \bigcup_{k=1}^{h} X_k$) such that $\exists \alpha \in \mathbf{R}$ for which each ϕ_k fulfils $(\mathrm{H}_2)'$ of Sect. 2 at $\ell = 1$, $C^+ = \mathbf{R}_+$. It is easily seen that $\varphi : X \to \mathbf{R}$, with

$$\varphi(x) := \prod_{k=1}^{r} \phi_k(x)^{\alpha/r} \tag{5.3}$$

fulfils $(\mathrm{H}_2)'$ of Sect. 2. In fact, it is trivial to verify (i), (ii) and (3i). In the following formulas k as index will denote that we are referred to Z_k and X_k instead of Z and X. φ fulfils also (4i):

$$\begin{aligned}
\varphi(x) &= \prod_{k=1}^{r} \phi_k(x)^{1/r} = \prod_{k=1}^{r} \phi_k(x)^{1/r} \geq \prod_{k=1}^{r} [\varepsilon_k(z) \cdot \|x - p_k(x)\|]^{\alpha/r} \\
&= \prod_{k=1}^{r} \varepsilon_k(z)^{\alpha/r} \cdot \prod_{k=1}^{r} \|x - p_k(x)\|^{\alpha/r} \geq \prod_{k=1}^{r} \varepsilon_k(z)^{\alpha/r} \cdot \|x - p(x)\|^{\alpha} \\
&= \tilde{\varepsilon}(z) \cdot \|x - p(x)\|^{\alpha} \ , \quad \tilde{\varepsilon}(z) := \prod_{k=1}^{r} \varepsilon_k(z)^{\alpha/r},
\end{aligned}$$

where the 1st inequality comes from $(\mathrm{H}_2)'$ at $\phi = \phi_k$, $k = 1, \ldots, r$, and the 2nd inequality is due to the fact that $\|x - p(x)\| = \min\{\|x - p_k(x)\|,\ k = 1, \ldots, r\}$.

The function $\Phi : X \times X \to \mathbf{R}^\ell$, defined by

$$\Phi(x;y) = \begin{pmatrix} \varphi(y) - \varphi(x) \\ \vdots \\ \varphi(y) - \varphi(x) \end{pmatrix} \tag{5.4}$$

can be chosen as "penalty term". The "decomposition" (5.3) may help in setting up φ, and in conceiving solving methods. For instance, if Z_k is a polytope defined by

$$Z_k := \{x \in \mathbf{R}^n : A^k x \geq b^k\}, \quad k = 1, \ldots, r,$$

where $A^k \in \mathbf{R}^{m_k \times n}$, $b^k \in \mathbf{R}^{m_k}$, then, with obvious notation, we can set

$$\phi_k(x) := \max \Big\{ 0, \exp \Big(- \alpha_i^k \sum_{j=1}^{n} a_{ij}^k x_j - b_i^k \Big) - 1, \quad i = 1, \ldots, m_k \Big\},$$

where $A^k = (a_{ij}^k,\ i=1,\dots,m_k, j=1,\dots,n)$, $b^k = (b_1^k,\dots,b_{m_k}^k)^T$, α_i^k being positive parameters. A particular – but interesting – case is that where Z is not convex, while the sets Z_i, X_i and $\bigcup_{k=1}^{r} X_i$ are all convex. For instance, at $n=2, r=2$ it happens to the sets:

$$Z := ([0,2]\times[0,1])\cup([0,1]\times[0,2]),\ Z_1 := ([0,2]\times[0,1]),\ Z_2 := ([0,1]\times[0,2]),$$

$$X_1 := Z_1,\ X_2 := \{(x_1,x_2)\in \mathbf{R}_+^2 : x_2 \le 2,\ x_1 - x_2 \le 1,\ x_1 + x_2 \le 3\}.$$

It would be interesting to define a decomposition of the Tuy Vector Method outlined in Sect. 4 which corresponds to the decomposition (5.3). To this end it might be useful to investigate about the properties of φ given by (5.3) and those of the ϕ_k; in particular as concerns the (strict) concavity and the fulfilment of (H_2) and (H_4).

An interesting application of the above decomposition should be to the case where (5.1) is replaced with one of the Variational Inequalities (2.12).

As concerns further developments of the topics analyzed in the preceding sections, we stress the importance of extending Theorem 4 to other "penalty functions" than (3.2).

Some real problems lead to vector systems, in particular Vector Variational Inequalities, when we want to analyze equilibrium aspects. The extensions of the present results would be of much interest. Connections between the present kind of penalization and the classic one for Variational Inequalities has to be investigated. When the operator of a Variational Inequality escapes from know classes which allow us to solve it, then a result like Theorem 4 would lead to a strictly antitone operator which might less worse that a generic nonmonotone operator.

Another particular case of (2.1) is that of Complementarity Systems. Strictly connected with Variational Inequality, they are very important in several fields of applications and, in special, in Structural Mechanics, beside in equilibrium problems. Their investigation in vector form is at the beginning.

An interesting extension of Theorems 1 and 2 would be to a metric space. In [3] there is a result for an optimal control problem in infinite dimensional space.

The coefficients μ_j of the penalty terms must be chosen, in the numerical applications, large enough. This might be a drawback. Hence, it would be useful to find their infima or meaningful upper bounds of these infima; possible connections between μ_1 and Lagrangian Theory of multipliers might help.

At least from computational point of view it would be useful to investigate connections with fixed–point problems and with the theory of gap functions.

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