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ON AN EXPANSION OF THE SPECIAL LAGRANGIAN FORM

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. In this paper we present a class of k-forms on C^n induced by k-forms on R^n and investigate their comass and set of maximal directions. In particular, the set of maximal directions of the real part of powers of the complex symplectic form on H^n is described completely.

1. Introduction

The problem of calculating the comass of a covector and determining the set of its maximal directions plays an important role in the theory of calibrated geometries (for a survey on calibrated geometries see [HL1]) and has been dealt with by many authors [D], [HL1], [HL2], [H], [M]. Among forms of constant coefficients on $C^n \cong R^{2n}$ (which can be identified with covectors on $C^n \cong R^{2n}$) the special Lagrangian form $\mathrm{Re}(dz_1 \wedge \ldots \wedge dz_n)$ is the one which have been investigated most (see [HL1], [HL2],...). In this paper we shall study a class of forms on $C^n \cong R^{2n}$ induced by forms on R^n. This class contains the special Lagrangian form. The forms mentioned below are forms of constant coefficients.

Let R^n be the n-dimensional Euclidean space with the standard inner product given by $\langle a, b\rangle = \sum a_i b_i$ where $a = (a_i)$, $b = (b_i)$. Let C^n be the n-dimensional Hermitian space with the Hermitian inner product given by $\langle z, w\rangle_C = \sum z_i \bar{w}_i$ where $z = (z_i), w = (w_i)$. Note that $C^n \equiv R^n + iR^n \cong R^{2n}$ can be considered as the $2n$-dimensional Euclidean space with the real inner product given by $\langle z, w\rangle_R = \mathrm{Re}\langle z, w\rangle_C = \mathrm{Re} \sum z_i \bar{w}_i$ and $R^n \equiv \{(w_i) \in C^n : w_i \in R\}$ with the standard inner product is as its Euclidean subspace. Below we call orthonormal (orthogonal) vectors with respect to the real inner product and with respect to the Hermitian inner product in C^n real orthonormal (orthogonal) vectors and complex orthonormal (orthogonal) vectors, respectively.

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Now let ω be a k-form on R^n, we shall define a complex k-form on C^n as follows.

Let $e_1, \dots, e_n$ be an orthonormal basis of R^n. Then $e_1, \dots, e_n$, $ie_1, \dots, ie_n$ is a real orthonormal basis of $C^n \cong R^{2n}$. Let $dx_1, \dots, dx_n$, $dy_1, \dots, dy_n$ denote the basis dual to the basis $e_1, \dots, e_n$, $ie_1, \dots, ie_n$. For the basis $e_1, \dots, e_n$, ω can be expressed as follows:

$$\omega = \sum a_J dx_J,$$

$J = (i_1, i_2, \dots, i_k)$, $1 \le i_1 < i_2 < \dots < i_k \le n$, $dx_J = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$, $a_J \in R$.

Set

$$\omega^c = \sum a_J dz_J$$

where $dz_\alpha = dx_\alpha + idy_\alpha$ and $dz_J = dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_k}$.

We see that the complex k-form ω^c defined above does not depend on choosing any orthonormal basis $e_1, \dots, e_n$ of R^n (the matrix of transformation between systems of forms $dx_1, \dots, dx_n$ and $dx'_1, \dots, dx'_n$ is just the matrix of transformation between systems of forms $dz_1, \dots, dz_n$ and $dz'_1, \dots, dz'_n$). We call ω^c the complex form induced by ω.

Consider the form $\mathrm{Re}\omega^c$ on C^n which is the real part of ω^c. In the case $k = n, \omega = dx_1 \wedge \dots \wedge dx_n$ is the unit volume form on R^n and $\mathrm{Re}\omega^c = \mathrm{Re}(dz_1 \wedge \dots \wedge dz_n)$ is just the special Lagrangian form on C^n. For investigating the relation between the comass, maximal directions of ω and those of $\mathrm{Re}\omega^c$, we show that $\|\mathrm{Re}\omega^c\|^* = \|\omega\|^*$ for ω being an arbitrary simple separable form (Theorem 2.5). Moreover, we also obtain a complete description for the set of maximal directions of some forms belonging to the above class (Theorem 3.2).

2. Complex separable forms

In this section we consider forms ω^c induced by separable forms ω on R^n. First we recall some necessary notions.

Let ω be a k-form (of constant coefficients) on the Euclidean space R^n. The comass $\|\omega\|^*$ of ω is given by

$$\|\omega\|^* = \max\{\omega(\xi) : \xi \in G(k, R^n)\},$$

where the Grassmannian $G(k, R^n)$ consists of all oriented k-planes in R^n and can be identified with the collection of all unit simple k-vectors in R^n. The set $G(\omega)$ of maximal directions of ω is given by

$$G(\omega) = \{\xi \in G(k, R^n) : \omega(\xi) = \|\omega\|^*\}.$$

The comass of the form ω^c on $C^n \cong R^{2n}$ is given by

$$\|\omega^c\|^* = \max\{|\omega^c(\xi)| : \xi \in G_R(k, C^n)\},$$

where $G_R(k, C^n)$ is the set of oriented real k-planes in C^n. Put

$$G(\omega^c) = \{\xi \in G_R(k, C^n) : |\omega^c(\xi)| = \|\omega^c\|^*\}\,.$$

On C^n with the real inner product, a real subspace $V \subset C^n$ is called an isotropic subspace if $iu \perp V$ for any $u \in V$. A real simple k-vector ξ on C^n is called an isotropic k-vector if the real span of ξ, $\mathrm{span}_R\xi$, is an isotropic subspace. Note that any system of vectors $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in C^n$ is complex orthonormal if and only if the system $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ is real orthonormal and $\mathrm{span}_R\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$ is an isotropic subspace of C^n.

Let V be a real subspace of C^n and $\xi \in G_R(k, C^n)$. The canonical form of ξ with respect to V had been given by Harvey-Lawson [HL1-Lemma II.7.5]. Now let V be a complex subspace of C^n and $\xi \in G_R(k, C^n)$, ξ is isotropic. We obtain the following lemma.

Lemma 2.1. *Let $\xi \in G_R(k, C^n)$ be an isotropic k-vector, V - a complex subspace of C^n and $V^\perp$ - the orthogonal supplement of V with respect to the Hermitian inner product in C^n. Then there exist two complex orthonormal systems $e_1, \dots, e_r \in V$ and $f_1, \dots, f_s \in V^\perp$ and numbers $0 \neq a_\alpha, b_\alpha \in C$ satisfying $|a_\alpha|^2 + |b_\alpha|^2 = 1, \quad \alpha = 1, \dots, p$, such that*

$$\begin{aligned}\xi = (a_1e_1 + b_1f_1) \wedge (a_2e_2 + b_2f_2) \wedge \dots \wedge (a_pe_p + b_pf_p) \wedge e_{p+1}\\ \wedge \dots \wedge e_r \wedge f_{p+1} \wedge \dots \wedge f_s\end{aligned}$$

where $p \le r$, $s \le k$ and $r + s - p = k$.

Remark. For the case $\dim_C V = q \le k$ we can take $r = q$ and $a_\alpha, b_\alpha \in C$ satisfying $|a_\alpha|^2 + |b_\alpha|^2 = 1$ for $\alpha \le q$ such that

$$\xi = (a_1e_1 + b_1f_1) \wedge (a_2e_2 + b_2f_2) \wedge \dots \wedge (a_qe_q + b_qf_q) \wedge f_{q+1} \wedge \dots \wedge f_k$$

If $a_\alpha = 0$ (or $b_\alpha = 0$), then e_α (or f_α) is only a formal symbol.

Proof. Let $\xi \in G_R(k, C^n)$ be an isotropic k-vector. Then ξ is of the form $\xi = v_1 \wedge \dots \wedge v_k$, where $v_1, \dots, v_k$ are complex orthonormal vectors. Let $\pi : C^n \longrightarrow V$ denote orthogonal projection with respect to the Hermitian inner product on C^n. Consider the Hermitian form B on the complex span of ξ, $\mathrm{span}_C\xi = \mathrm{span}_C\{v_1, \dots, v_k\}$, defined by $B(u, v) = \langle\pi(u), \pi(v)\rangle_C$. Then the linear operator A defined by $B(u, v) = \langle Au, v\rangle_C$ is self-conjugate.

Hence the eigenvalues $\lambda_1, \dots, \lambda_k$ of A are real numbers and there exists a complex orthonormal system of eigenvectors $\varepsilon_1, \dots, \varepsilon_k$ corresponding to $\lambda_1, \dots, \lambda_k$. Therefore, $\xi = e^{i\theta}\varepsilon_1 \wedge \dots \wedge \varepsilon_k$ $(0 \le \theta \le 2\pi)$. Since $0 \le B(u,u) \le |u|^2$, we have $0 \le \lambda_\alpha \le 1$ for all α. Rearrange the indexes so that $0 < \lambda_\alpha < 1$ for $\alpha = 1, \dots, p$, $\lambda_{p+1} = \dots = \lambda_r = 1$ and $\lambda_{r+1} = \dots = \lambda_k = 0$. Then $\varepsilon_\alpha = a_\alpha e'_\alpha + b_\alpha f'_\alpha$ for $\alpha = 1, \dots, p$, where e'_α, f'_α are unit vectors belonging to V and $V^\perp$, respectively, and $a_\alpha, b_\alpha \in C$, $|a_\alpha|^2 + |b_\alpha|^2 = 1$. Since $|a_\alpha|^2 = |\pi(\varepsilon_\alpha)|^2 = B(\varepsilon_\alpha, \varepsilon_\alpha) = \lambda_\alpha$, $a_\alpha, b_\alpha \neq 0$. Set $e'_{p+1} = \varepsilon_{p+1}, \dots, e'_r = \varepsilon_r$ and $f'_{p+1} = \varepsilon_{r+1}, \dots, f'_s = \varepsilon_k$. Since $B(\varepsilon_\alpha, \varepsilon_\beta) = 0$ for $\alpha \neq \beta$, $\pi(\varepsilon_\alpha)$ and $\pi(\varepsilon_\beta)$ are complex orthogonal for $\alpha \neq \beta$ which proves that $e'_1, \dots, e'_r$ is a complex orthonormal system in V. Moreover,

$$\langle \varepsilon_\alpha - \pi(\varepsilon_\alpha), \varepsilon_\beta - \pi(\varepsilon_\beta) \rangle_C = \langle \varepsilon_\alpha, \varepsilon_\beta \rangle_C - B(\varepsilon_\alpha, \varepsilon_\beta),$$

which vanishes for $\alpha \neq \beta$. Therefore, $f'_1, \dots, f'_s$ is a complex orthonormal system in $V^\perp$. Replacing $e^{i\frac{\theta}{k}}.e'_\alpha$ by e_α for $\alpha = 1, \dots, r$ and $e^{i\frac{\theta}{k}}.f'_\alpha$ by f_α for $\alpha = 1, \dots, s$, then $e_1, e_2, \dots, e_r$ is also a complex orthonormal system in V and $f_1, f_2, \dots, f_s$ is also a complex orthonormal system in $V^\perp$. We have

$$\begin{aligned}\xi = (a_1e_1 + b_1f_1) \wedge (a_2e_2 + b_2f_2) \wedge \dots \wedge (a_pe_p + b_pf_p) \wedge e_{p+1} \\ \wedge \dots \wedge e_r \wedge f_{p+1} \wedge \dots \wedge f_s.\end{aligned}$$

Hence Lemma 2.1 is proved.

Lemma 2.2. *Let ω^c be the complex k-form on C^n induced by a k-form ω on R^n and $\xi \in G(\omega^c)$. Then ξ is an isotropic k-vector on C^n.*

Proof. Since ω^c is a skew symmetric complex polylinear form on the complex space $C^n \cong R^{2n}$ we have $\omega^c(\eta) = 0$ for any $\eta \in G_R(k, C^n)$ of the form $\eta = \varepsilon \wedge i\varepsilon \wedge \eta'$ where $\eta' \in G_R(k-2, C^n)$. Let $\xi \in G_R(k, C^n)$. By [HL2-Proposition 2.1] there exists a real orthonormal basis $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, $i\varepsilon_1, i\varepsilon_2, \dots, i\varepsilon_n$ in C^n and angles $0 \le \theta_i \le \dfrac{\pi}{2}$ such that ξ takes the form

$$\pm\xi = \varepsilon_1 \wedge (\cos\theta_1 i\varepsilon_1 + \sin\theta_1\varepsilon_2) \wedge \varepsilon_3 \wedge (\cos\theta_2 i\varepsilon_3 + \sin\theta_2\varepsilon_4) \wedge \varepsilon_5 \wedge \dots .$$

Therefore, using the above remark we have

$$|\omega^c(\xi)| = \sin\theta_1 \sin\theta_2 \dots |\omega^c(\varepsilon_1 \wedge \varepsilon_2 \wedge \dots \wedge \varepsilon_k)|.$$

Now let $\xi \in G(\omega^c)$. From the above equality we get $\sin\theta_1 = sin\theta_2 = \dots = 1$. Therefore, $\xi = \pm\varepsilon_1 \wedge \varepsilon_2 \wedge \dots \wedge \varepsilon_k$ is an isotropic k-vector in C^n. This concludes the proof.

Proposition 2.3. *Let ω^c be the complex k-form on C^n induced by a k-form ω on R^n. Then*

(1) $\|\mathrm{Re}\omega^c\|^* = \|\omega^c\|^*$,

(2) $G(\mathrm{Re}\omega^c) \subset G(\omega^c)$.

Proof. Let $\xi \in G(\omega^c)$. Then there exists $0 \le \theta \le 2\pi$ such that $\omega^c(\xi) = e^{\theta i}\|\omega^c\|^*$. By Lemma 2.2, $\xi = \varepsilon_1 \wedge \ldots \wedge \varepsilon_k$, where $\varepsilon_1, \ldots, \varepsilon_k$ are complex orthonormal vectors. Put $\xi' = (e^{-\theta i}\varepsilon_1) \wedge \ldots \wedge \varepsilon_k$. We have

$$\omega^c(\xi') = e^{-\theta i}e^{\theta i}\|\omega^c\|^* = \|\omega^c\|^*.$$

Hence $\mathrm{Re}\omega^c(\xi') = \|\omega^c\|^*$. On the other hand, since $\|\mathrm{Re}\omega^c\|^* \le \|\omega^c\|^*$, we have $\|\mathrm{Re}\omega^c\|^* = \|\omega^c\|^*$.

Next let $\xi \in G(\mathrm{Re}\omega^c)$. Then

$$|\omega^c(\xi)| \ge |\mathrm{Re}\omega^c(\xi)| = \|\mathrm{Re}\omega^c\|^* = \|\omega^c\|^* \ge |\omega^c(\xi)|$$

therefore $|\omega^c(\xi)| = \|\omega^c\|^*$, that is $\xi \in G(\omega^c)$. Thus, $G(\mathrm{Re}\omega^c) \subset G(\omega^c)$. The proof is complete.

Consider forms $\omega = dx_V \wedge \omega_1 + \omega_2$ on R^n, where dx_V is the unit volume form on a p-dimensional oriented subspace $V \subset R^n$ $(p \ge 2)$ and ω_1, ω_2 are forms on $V^\perp$ (this class of forms has been investigated in [H] and the author called them separable forms with respect to V).

Now we consider the complex form ω^c on C^n induced by a separable form ω on R^n, that is

$$\omega^c = dz_{V^c} \wedge \omega_1^c + \omega_2^c,$$

where dz_{V^c} is the complex form on $V^c = V \oplus iV$ induced by the form dx_V and ω_1^c, ω_2^c are complex forms on $(V^c)^\perp$ (the orthogonal supplement of V^c with respect to the Hermitian inner product in C^n) induced by forms ω_1, ω_2, respectively. We call such forms ω^c complex separable forms. Applying the above lemmas we obtain

Proposition 2.4. *Let $\omega^c = dz_{V^c} \wedge \omega_1^c + \omega_2^c$ be the complex separable form on C^n induced by a separable form $\omega = dx_V \wedge \omega_1 + \omega_2$ on R^n. Then*

$$\|\omega^c\|^* = \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\}.$$

Proof. Using Lemma 2.2 for the form ω^c we have

$$\begin{aligned}\|\omega^c\|^* &= \max\{|\omega^c(\xi)| : \xi \in G_R(k, C^n)\}\\ &= \max\{|\omega^c(\xi)| : \xi \in G_R(k, C^n) \text{ and } \xi \text{ is isotropic}\}\end{aligned}$$

By Lemma 2.1 any isotropic k-vector $\xi \in G_R(k, C^n)$ is of the form

$$\xi = (a_1\varepsilon_1 + b_1 f_1) \wedge (a_2\varepsilon_2 + b_2 f_2) \wedge \ldots \wedge (a_p\varepsilon_p + b_p f_p) \wedge f_{p+1} \wedge \ldots \wedge f_k,$$

where $\varepsilon_1, \ldots, \varepsilon_p$ is a complex orthonormal basis of the complex space $V^c, f_1, \ldots, f_k$ is a complex orthonormal system of the complex space $(V^c)^\perp$ and $a_i, b_i \in C, |a_i|^2 + |b_i|^2 = 1,\ i = 1, \ldots, p$. Therefore

$$\begin{aligned}\omega^c(\xi) &= a_1 \ldots a_p dz_{V^c}(\varepsilon_1 \wedge \ldots \wedge \varepsilon_p)\omega_1^c(f_{p+1} \wedge \ldots \wedge f_k) \\ &\quad + b_1 \ldots b_p \omega_2^c(f_1 \wedge \ldots \wedge f_k).\end{aligned}$$

Since $|a_i| \leq 1,\ |b_i| \leq 1$ for all i and $|dz_{V^c}(\varepsilon_1 \wedge \ldots \wedge \varepsilon_p)| = 1$ [HL1-Proposition III.1.14], we have

$$\begin{aligned}|\omega^c(\xi)| &\leq (|a_1||a_2| + |b_1||b_2|) \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\} \\ &\leq (|a_1|^2 + |b_1|^2)(|a_2|^2 + |b_2|^2) \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\} \\ &= \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\}.\end{aligned}$$

Notice that there exists ξ such that $|\omega^c(\xi)| = \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\}$. Consider two following cases:

Case $\|\omega_1^c\|^* \geq \|\omega_2^c\|^*$. Take $\xi = \theta \wedge \eta$, where $dz_{V^c}(\theta) = 1$ and $\eta \in G(\omega_1^c)$. Then

$$|\omega^c(\xi)| = |\omega_1^c(\eta)| = \|\omega_1^c\|^* = \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\}.$$

Case $\|\omega_1^c\|^* < \|\omega_2^c\|^*$. Take $\xi \in G(\omega_2^c)$. Then

$$|\omega^c(\xi)| = |\omega_2^c(\xi)| = \|\omega_2^c\|^* = \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\}.$$

Therefore

$$\|\omega^c\|^* = \max\{\|\omega_1^c\|^*, \|\omega_2^c\|^*\}.$$

Hence Proposition 2.4 is proved.

Next we shall consider a special class of complex separable forms. Let $R^n = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ be an orthogonal decomposition of R^n. For any multi-index $I = (i_1, \ldots, i_q)$ we denote by dx_I the p-form $dx_{V_{i_1}} \wedge \ldots \wedge dx_{V_{i_q}}$, where $p = |I| = \sum\limits_{j \in I} \dim V_j$ and dx_V is the unit volume form on V.

In [H] Hoang Xuan Huan has considered forms $\omega = \sum\limits_I a_I dx_I$, where $\dim V_j \geq 2$ for all $j \leq k$, called simply separable forms (with respect to $(V_1, V_2, \ldots, V_k)$) and he has proved that

$$\|\omega\|^* = \max_I\{|a_I|\}.$$

On $C^n = V_1^c \oplus V_2^c \oplus \ldots \oplus V_k^c$ each simply separable form $\omega = \sum_I a_I dx_I$ mentioned above induces a form ω^c of the form

$$\omega^c = \sum_I a_I dz_I, \quad I = (i_1, \ldots, i_q), dz_I = dz_{V_{i_1}^c} \wedge \ldots \wedge dz_{V_{i_q}^c}.$$

Theorem 2.5. *Let $\omega^c = \sum_I a_I dz_I$ be the complex form on C^n induced by a simply separable form $\omega = \sum_I a_I dx_I$ on $R^n = V_1 \oplus V_2 \oplus \ldots \oplus V_k$. Then*

$$\|\mathrm{Re}\omega^c\|^* = \|\omega\|^* = \max_I\{|a_I|\}.$$

Proof. Using Proposition 2.4 and the equality $\|dz_I\|^* = 1$ for complex forms dz_I (see [HL1-Proposition III.1.14]), we have $\|\omega^c\|^* = \max_I\{|a_I|\}$ by induction on k. Combining the equalities $\|\mathrm{Re}\omega^c\|^* = \|\omega^c\|^*$ (Proposition 2.3) and $\|\omega\|^* = \max_I\{|a_I|\}$ ([H-Theorem 3.10]) we obtain the conclusion.

We have the following result concerning the relation between maximal directions of forms ω and $\mathrm{Re}\omega^c$.

Theorem 2.6. *Let* $\mathrm{Re}\omega^c$ *be the real part of the complex form ω^c on C^n induced by a form ω on R^n such that $\|\mathrm{Re}\omega^c\|^* = \|\omega\|^*$. Then*

$$\bigcup_{\xi \in G(\omega)} G(\mathrm{Re}dz_\xi) \subset G(\mathrm{Re}\omega^c),$$

where dz_ξ is the complex form induced by the unit volume form dx_ξ on the oriented subspace $\mathrm{span}\xi \subset R^n$. *In particular, the above inclusion of sets happens when ω is a simply separable form on R^n.*

Proof. Let $\xi = \varepsilon_1 \wedge \ldots \wedge \varepsilon_p \in G(\omega)$, where $\varepsilon_1, \ldots, \varepsilon_p$ is an orthonormal system in R^n and let dx'_α be the basis dual to an orthonormal basis $(\varepsilon_\alpha), \alpha = 1, \ldots, n$ of R^n containing $\varepsilon_1, \ldots, \varepsilon_p$. Then ω can be expressed as follows:

$$\omega = \|\omega\|^* dx'_1 \wedge \ldots \wedge dx'_p + \sum_J a_J dx'_J,$$

where $dx'_J = dx'_{i_1} \wedge \ldots \wedge dx'_{i_p}$, $J = (i_1, \ldots, i_p) \neq (1, \ldots, p)$.

On C^n with the real orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n, i\varepsilon_1, \ldots, i\varepsilon_n$ and the dual basis $dx'_1, \ldots, dx'_n, dy'_1, \ldots, dy'_n$ we have

$$\omega^c = \|\omega\|^* dz'_1 \wedge \ldots \wedge dz'_p + \sum_J a_J dz'_J,$$

where $dz'_\alpha = dx'_\alpha + idy'_\alpha$, $\alpha = 1, \dots, n$.

Let $dz_\xi = dz'_1 \wedge \dots \wedge dz'_p$ be the complex form on C^n induced by the form $dx_\xi = dx'_1 \wedge \dots \wedge dx'_p$ on R^n and let $\eta \in G(\mathrm{Re} dz_\xi)$. Then $\mathrm{span}_R \eta \subset \mathrm{span}_R\{\varepsilon_1, \dots, \varepsilon_p, i\varepsilon_1, \dots, i\varepsilon_p\}$, therefore $a_J dz'_J(\eta) = 0$ for all $J = (i_1, \dots, i_p) \neq (1, \dots, p)$. Hence

$$\mathrm{Re}\omega^c(\eta) = \|\omega\|^* = \|\mathrm{Re}\omega^c\|^*,$$

that is $\eta \in G(\mathrm{Re}\omega^c)$. So $G(\mathrm{Re} dz_\xi) \subset G(\mathrm{Re}\omega^c)$ for any $\xi \in G(\omega)$. The theorem is proved.

3. Maximal directions of the real part of powers of a complex symplectic form

In this section we describe the set of maximal directions of the form $\mathrm{Re}\omega^c$ on C^n induced by a particular simply separable form ω on R^n.

We consider H^n as a (left) quaternionic vector space and consider a quaternionic inner product $\langle .,. \rangle_H$ on H^n defined by $\langle p, q \rangle_H = \sum\limits_{i=1}^{n} p_i \bar{q}_i$.

Since $H \equiv C \oplus Cj$, the identification $H^n \equiv C^n \oplus C^n j \cong C^{2n}$ provides a complex linear isomorphism $H^n \cong C^{2n}$, where the left multiplication by $i \in H$ defines the complex structure on C^{2n}. Then $(q_1, \dots, q_n) \in H^n$ is identified with $(z_1, \dots, z_n, w_1, \dots, w_n) \in C^{2n}$, where $q_\alpha = z_\alpha + w_\alpha j$, $\alpha = 1, \dots, n$.

We consider the standard Hermitian inner product on $H^n \cong C^{2n}$ given by

$$\langle (q_1, \dots, q_n), (q'_1, \dots, q'_n) \rangle_C = \sum_1^n (z_\alpha \bar{z}'_\alpha + w_\alpha \bar{w}'_\alpha)$$

and consider the complex symplectic form σ on $H^n \cong C^{2n}$ given by

$$\sigma\big((q_1, \dots, q_n), (q'_1, \dots, q'_n)\big) = \sum_1^n (z_\alpha w'_\alpha - w_\alpha z'_\alpha)$$

where $q_\alpha = z_\alpha + w_\alpha j$, $q'_\alpha = z'_\alpha + w'_\alpha j$, $\alpha = 1, \dots, n$.

Then the quaternionic inner product on H^n can be reexpressed as

$$\langle .,. \rangle_H = \langle .,. \rangle_C - \sigma(.,.)j.$$

Note that the canonical basis $e_1, \dots, e_n$ of H^n is orthonormal with respect to the quaternionic inner product $\langle .,. \rangle_H$ (quaternionic orthonormal) and the basis $e_1, \dots, e_n, je_1, \dots, je_n$ of $H^n \cong C^{2n}$ is orthonormal with respect to the Hermitian inner product $\langle .,. \rangle_C$. Let $Sp(n)$ denote the group of H-linear transformations on H^n preserving the quaternionic inner product above. Then σ (and therefore all powers σ^s) is $Sp(n)$-invariant.

Let $dz_1, \ldots, dz_n, dw_1 \ldots, dw_n$ be the complex basis dual to the complex basis $e_1, \ldots, e_n, je_1, \ldots, je_n$ of $H^n \cong C^{2n}$. Then we have

$$\sigma = dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + \ldots + dz_n \wedge dw_n.$$

We see that σ^s is just the complex $2s$-form on $H^n \cong C^{2n}$ induced by the simply separable $2s$-form $(dx_1 \wedge dx_{n+1} + dx_2 \wedge dx_{n+2} + \cdots + dx_n \wedge dx_{2n})^s$ on V, where $V = \mathrm{span}_R\{e_1, \ldots, e_n, je_1, \ldots, je_n\}$ is the $2n$-dimensional Euclidean subspace of $H^n \cong R^{4n}$ with the real inner product

$$\langle .,. \rangle_R = \mathrm{Re}\langle .,. \rangle_C (= \mathrm{Re}\langle .,. \rangle_H),$$

and $dx_1, \ldots, dx_n, dx_{n+1} \ldots, dx_{2n}$ is the basis dual to the real orthonormal basis $e_1, \ldots, e_n, je_1, \ldots, je_n$ of V. Applying Theorem 2.5 we have $\|\mathrm{Re}\frac{1}{s!}\sigma^s\|^* = 1$.

Now we shall investigate the set of maximal directions of the form $\mathrm{Re}\frac{1}{s!}\sigma^s$ (or $\mathrm{Re}\sigma^s$) by using the quaternionic structure on H^n. Notice that each real subspace $V \subset H^n$ is a quaternionic subspace of H^n if and only if V is simultaneously invariant with respect to the complex structures defined by the unit imaginar quaternions $i, j, k \in H$.

Theorem 3.1. *Let σ be the complex symplectic form on $H^n \cong C^{2n}$ mentioned above and $\xi \in G(\frac{1}{s!}\sigma^s)$. Then* $\mathrm{span}_R\xi \subset V$, *where V is a s-dimensional quaternionic subspace of H^n.*

Proof. We prove by induction on s.

Case $s = 1$: Let $\xi \in G(\sigma)$ and $\xi = \varepsilon_1 \wedge \eta$, where ε_1, η are orthonormal vectors (with respect to the real inner product) in $H^n \cong R^{4n}$. Complete ε_1 into a quaternionic orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$ in H^n, that is $\langle \varepsilon_\alpha, \varepsilon_\beta \rangle_H = \delta^\alpha_\beta$, for α, $\beta = 1, \ldots, n$. Let $dz'_1, \ldots, dz'_n, dw'_1, \ldots, dw'_n$ be the complex basis dual to the complex orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n, j\varepsilon_1, \ldots, j\varepsilon_n$ of $H^n \cong C^{2n}$. Since σ is $Sp(n)$-invariant, we also have

$$\sigma = dz'_1 \wedge dw'_1 + \ldots + dz'_n \wedge dw'_n.$$

Because of $dz'_\alpha(\varepsilon_1) = 0\,, dw'_\alpha(\varepsilon_1) = 0$ for $\alpha = 2, \ldots, n$, we get

$$|\sigma(\xi)| = |(dz'_1 \wedge dw'_1 + \ldots + dz'_n \wedge dw'_n)(\xi)| = |dz'_1 \wedge dw'_1(\xi)| = 1$$

this implies that there exists $\theta\,, 0 \leq \theta \leq 2\pi$ such that $dz'_1 \wedge dw'_1(\xi) = e^{i\theta}$. Since

$$dz'_1 \wedge dw'_1(\xi) = dz'_1(\varepsilon_1).dw'_1(\eta) - dz'_1(\eta).dw'_1(\varepsilon_1) = dw'_1(\eta),$$

we have $dw'_1(\eta) = e^{i\theta}$. Hence $\eta = e^{i\theta}.j\varepsilon_1 \in \mathrm{span}_H\{\varepsilon_1\}$ (here $e^{i\theta} = \cos\theta + i\sin\theta, i \in H$) and therefore $\mathrm{span}_R\xi = \mathrm{span}_R\{\varepsilon_1, \eta\} \subset \mathrm{span}_H\{\varepsilon_1\}$ (where $\mathrm{span}_H\{\varepsilon_1\}$ is the quaternionic subspace spanned by ε_1). So we have proved the case $s = 1$.

Assume that the statement has been proved for $s = t - 1$. We will show that the statement is true for $s = t$ by induction on n with respect to $n \geq t$ as follows.

If $n = t$, then

$$\frac{1}{t!}\sigma^t = dz_1 \wedge dw_1 \wedge \ldots \wedge dz_t \wedge dw_t.$$

Hence, the statement is immediate.

Assume that the statement has been proved for $n = m - 1 \geq t$. We have

$$\begin{aligned}\frac{1}{t!}\sigma^t &= \frac{1}{t!}(dz_1 \wedge dw_1 + \ldots + dz_m \wedge dw_m)^t \\ &= dz_1 \wedge dw_1 \wedge \Big[\frac{1}{(t-1)!}(dz_2 \wedge dw_2 + \ldots + dz_m \wedge dw_m)^{t-1}\Big] \\ &\quad + \frac{1}{t!}(dz_2 \wedge dw_2 + \ldots + dz_m \wedge dw_m)^t.\end{aligned}$$

Put

$$\begin{aligned}\sigma_1 &= \frac{1}{(t-1)!}(dz_2 \wedge dw_2 + \ldots + dz_m \wedge dw_m)^{t-1}, \\ \sigma_2 &= \frac{1}{t!}(dz_2 \wedge dw_2 + \ldots + dz_m \wedge dw_m)^t.\end{aligned}$$

Then $\frac{1}{t!}\sigma^t = dz_1 \wedge dw_1 \wedge \sigma_1 + \sigma_2$ is a complex separable form and σ_1, σ_2 are forms induced from a complex symplectic form on $H^{m-1} \equiv \mathrm{span}_H\{e_2, \ldots$ $\ldots, e_m\}$, where $e_1, e_2, \ldots, e_m$ is the canonical basis of H^m. Let $\xi \in G(\frac{1}{t!}\sigma^t)$. It follows from Lemma 2.2 that ξ is isotropic and by Lemma 2.1

$$\xi = (a_1\varepsilon_1 + b_1 f_1) \wedge (a_2\varepsilon_2 + b_2 f_2) \wedge f_3 \wedge \ldots \wedge f_{2t}$$

where $\varepsilon_1, \varepsilon_2 \in \mathrm{span}_H\{e_1\}$ and $f_1, f_2, \dots, f_{2t} \in \mathrm{span}_H\{e_2, \dots, e_m\}$ are complex orthonomal vectors, and a_α, $b_\alpha \in C, |a_\alpha|^2 + |b_\alpha|^2 = 1$, $\alpha = 1, 2$.

We have

$$\begin{aligned}|\frac{1}{t!}\sigma^t(\xi)| &= |a_1a_2 dz_1 \wedge dw_1(\varepsilon_1 \wedge \varepsilon_2)\sigma_1(f_3 \wedge \dots \wedge f_{2t})\\ &\quad + b_1b_2\sigma_2(f_1 \wedge f_2 \wedge \dots \wedge f_{2t})|\\ &\leq (|a_1a_2| + |b_1b_2|).\max(\|dz_1 \wedge dw_1\|^*.\|\sigma_1\|^*, \|\sigma_2\|^*)\\ &\leq (|a_1|^2 + |b_1|^2)(|a_2|^2 + |b_2|^2) = 1.\end{aligned}$$

Since $\xi \in G(\frac{1}{t!}\sigma^t)$, the above inequalities become equalities. We consider ξ in the following cases:

Case $|a_1a_2| = 0$. Then $a_1, a_2 = 0$ and

$$\xi = b_1b_2 f_1 \wedge f_2 \wedge \dots \wedge f_{2t},$$

where $f_1 \wedge f_2 \wedge \dots \wedge f_{2t} \in G(\sigma_2)$.

By the induction hypothesis on n we have $\mathrm{span}_R\{f_1, f_2, \dots, f_{2t}\} \subset W$, where W is a t-dimensional quaternionic subspace of $H^{m-1} \equiv \mathrm{span}_H\{e_2, \dots, e_m\} \subset H^m$, therefore $\mathrm{span}_R\xi \subset W$.

Case $|a_1a_2| = 1$. Then $b_1, b_2 = 0$ and

$$\xi = a_1a_2\varepsilon_1 \wedge \varepsilon_2 \wedge f_3 \wedge \dots \wedge f_{2t},$$

where $\varepsilon_1 \wedge \varepsilon_2 \in G(dz_1 \wedge dw_1)$ and $f_3 \wedge \dots \wedge f_{2t} \in G(\sigma_1)$. By the induction hypothesis on s we have $\mathrm{span}_R\{f_3, \dots, f_{2t}\} \subset V$, where V is a $(t-1)$-dimensional quaternionic subspace of $H^{m-1} \equiv \mathrm{span}_H\{e_2, \dots \dots, e_m\} \subset H^m$. On the other hand, since $\mathrm{span}_R\{\varepsilon_1, \varepsilon_2\} \subset \mathrm{span}_H\{e_1\}$, we have $\mathrm{span}_R\xi \subset \mathrm{span}_H\{e_1\} \oplus V$, here $\mathrm{span}_H\{e_1\} \oplus V$ is a t-dimensional quaternionic subspace of H^m.

Case $0 < |a_1a_2| < 1$. Then

$$\xi = (a_1\varepsilon_1 + b_1f_1) \wedge (a_2\varepsilon_2 + b_2f_2) \wedge f_3 \wedge \dots \wedge f_{2t}$$

where $\varepsilon_1 \wedge \varepsilon_2 \in G(dz_1 \wedge dw_1)$, $f_3 \wedge \dots \wedge f_{2t} \in G(\sigma_1)$ and $f_1 \wedge f_2 \wedge \dots \wedge f_{2t} \in G(\sigma_2)$.

According to the second case we have $\mathrm{span}_R\{f_3, \dots, f_{2t}\} \subset V$, where V is a $(t-1)$-dimensional quaternionic subspace of H^{m-1}. Therefore

$$\mathrm{span}_R\{f_3, \dots, f_{2t}, if_3, \dots, if_{2t}\} = V.$$

According to the first case we have $\operatorname{span}_R\{f_1, f_2, \dots, f_{2t}\} \subset W$, where W is a t-dimensional quaternionic subspace of H^{m-1}. Therefore

$$\operatorname{span}_R\{f_1, f_2, \dots, f_{2t}, if_1, if_2, \dots, if_{2t}\} = W.$$

Let $W = V \oplus U$ be the orthogonal decomposition of W (with respect to the quaternionic inner product $\langle ., . \rangle_H$ and therefore also with respect to the real inner product $\langle ., . \rangle_R$), where U is a 1-dimensional quaternionic subspace of H^{m-1}. Then from the orthogonal decomposition (with respect to the real inner product) $W = V \oplus \operatorname{span}_R\{f_1, f_2, if_1, if_2\}$ it follows that $U = \operatorname{span}_R\{f_1, f_2, if_1, if_2\}$.

Consider a quaternionic orthonormal basis $e'_\alpha (\alpha = 2, \dots, m)$ in H^{m-1} such that $e'_2 \in U$ and let dz'_α, $dw'_\alpha (\alpha = 2, \dots, m)$ be the complex basis dual to the complex orthonormal basis $e'_2, \dots, e'_m, je'_2, \dots, je'_m$. Then

$$dz_2 \wedge dw_2 + \dots + dz_m \wedge dw_m = dz'_2 \wedge dw'_2 + \dots + dz'_m \wedge dw'_m.$$

We have

$$\begin{aligned}
\sigma_2(f_1 \wedge \dots \wedge f_{2t}) = dz'_2 \wedge dw'_2 \wedge \Big[&\frac{1}{(t-1)!}(dz'_3 \wedge dw'_3 + \dots \\
&+ dz'_m \wedge dw'_m)^{t-1}\Big](f_1 \wedge \dots \wedge f_{2t}) \\
&+ \frac{1}{t!}(dz'_3 \wedge dw'_3 + \dots + dz'_m \wedge dw'_m)^t (f_1 \wedge \dots \wedge f_{2t}) = \\
&= dz'_2 \wedge dw'_2(f_1 \wedge f_2).\frac{1}{(t-1)!}(dz'_3 \wedge dw'_3 + \dots \\
&\quad + dz'_m \wedge dw'_m)^{t-1}(f_3 \wedge \dots \wedge f_{2t}) \\
&= dz'_2 \wedge dw'_2(f_1 \wedge f_2).\sigma_1(f_3 \wedge \dots \wedge f_{2t})
\end{aligned}$$

(the last equality happens because of $dz'_2(f_\alpha) = 0, dw'_2(f_\alpha) = 0$, for $\alpha = 3, \dots, 2t$).
Therefore

$$\begin{aligned}
\frac{1}{t!}\sigma^t(\xi) = a_1 a_2 dz_1 \wedge dw_1(\varepsilon_1 \wedge \varepsilon_2)\sigma_1(f_3 \wedge \dots \wedge f_{2t}) \\
+ b_1 b_2 dz'_2 \wedge dw'_2(f_1 \wedge f_2).\sigma_1(f_3 \wedge \dots \wedge f_{2t}).
\end{aligned}$$

Hence

$$\begin{aligned}
1 &= |\frac{1}{t!}\sigma^t(\xi)| \\
&= |a_1 a_2 dz_1 \wedge dw_1(\varepsilon_1 \wedge \varepsilon_2) + b_1 b_2 dz'_2 \wedge dw'_2(f_1 \wedge f_2)| \\
&= |(dz_1 \wedge dw_1 + dz'_2 \wedge dw'_2)(\eta)|,
\end{aligned}$$

where $\eta = (a_1\varepsilon_1 + b_1 f_1) \wedge (a_2\varepsilon_2 + b_2 f_2) \in G_R(2, \mathrm{span}_H\{e_1, e_2'\})$. It follows that

$$\eta \in G(dz_1 \wedge dw_1 + dz_2' \wedge dw_2')$$

where $\omega = dz_1 \wedge dw_1 + dz_2' \wedge dw_2'$ is a complex symplectic form on $H^2 \cong \mathrm{span}_H\{e_1, e_2'\}$.

Applying the theorem for the case $s = 1$ with respect to the form ω we have $\mathrm{span}_R \eta \subset V_1$ where V_1 is a 1-dimensional quaternionic subspace of $\mathrm{span}_H\{e_1, e_2'\} \subset H^m$. Consequently,

$$\mathrm{span}_R \xi = \mathrm{span}_R \eta \oplus \mathrm{span}_R\{f_3 \wedge \ldots \wedge f_{2t}\} \subset V_1 \oplus V.$$

Here the orthogonal sum $V_1 \oplus V$ determines a t-dimensional quaternionic subspace of H^m. Thus, the theorem is proved for the case $s = t$ and $n = m$. The proof of Theorem 3.1 is now complete.

Remark. We know that the special Lagrangian form $Re(dz_1 \wedge \ldots \wedge dz_p)$ on a p-dimensional Hermitian space V have the set of maximal directions consisting of all special Lagrangian subspaces (see [HL1-Theorem III.1.10]). Next we see that if $V_0 \equiv H^s \times \{0\} \subset H^n$ then

$$\mathrm{Re}\frac{1}{s!}\sigma^s\big|_{V_0} = Re(dz_1 \wedge dw_1 \wedge \ldots \wedge dz_s \wedge dw_s)$$

is a special Lagrangian form on V_0. Since σ is $Sp(n)$-invariant it follows that $\mathrm{Re}\frac{1}{s!}\sigma^s\big|_V$ is also a special Lagrangian form on any s-dimensional quaternionic subspace V of H^n. Define

$$\mathrm{SLAG}\,(V) = \{\eta \in G_R(2s, V) \ \mid \ \ \mathrm{Re}\frac{1}{s!}\sigma^s(\eta) = 1\}.$$

The following theorem is an expansion for an arbitrary $s \leq n$ of a result given by Harvey and Bryant [HB, Theorem 2.38] for the case $s = 2$.

Theorem 3.2. *Let σ be the complex symplectic form on $H^n \cong C^{2n}$ mentioned above. Then $\mathrm{Re}\frac{1}{s!}\sigma^s$ is a calibration (form of comass 1) and*

$$G(Re\frac{1}{s!}\sigma^s) = \bigcup_{V \in G_H(s, H^n)} \mathrm{SLAG}(V)$$

where $G_H(s, H^n)$ is the set of all s-dimensional quaternionic subspace of H^n.

Proof. By the above remark it is sufficient to prove that if $\eta \in G_R(2s, H^n)$ satisfies $\mathrm{Re}\frac{1}{s!}\sigma^s(\eta) = 1$ then $\mathrm{span}_R\eta \subset V$ with respect to a s-dimensional quaternionic subspace $V \subset H^n$. But this follows from Proposition 2.3 and Theorem 3.1.

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