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INVARIANCE OF THE GLOBAL MONODROMIES IN FAMILIES OF POLYNOMIALS OF TWO COMPLEX VARIABLES

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. We consider global monodromy fibrations defined by a family of polynomials of two complex variables. The main result gives certain sufficient conditions for the conjugacy of global monodromies.

1. Introduction

1.1. Let $f : \mathbf{C}^n \longrightarrow \mathbf{C}$ be a polynomial function. It is well-known that there exists a finite set $A_f \subset \mathbf{C}$ called the *bifurcation set* of f such that the restriction:

$$f : \mathbf{C}^n \setminus f^{-1}(A_f) \longrightarrow \mathbf{C} \setminus A_f$$

is a locally trivial C^∞-fibration (see, for example, [P], [T], [V]). This fibration allows us to introduce the *global monodromy* fibration which, for

$$r > \max\{|t| \mid \quad t \in A_f\} \quad \text{and} \quad S^1_r := \{t \in \mathbf{C} \mid \quad |t| = r\},$$

is the restriction

$$f : \{z \in \mathbf{C}^n \quad | \quad |f(z)| = r\} \longrightarrow S^1_r.$$

Fix $t_0 \in S^1_r$. The geometric monodromy associated with the path $s \longrightarrow t_0 e^{2\pi i s}$, $s \in [0,1]$, is a diffeomorphism of $f^{-1}(t_0)$ onto itself which induces an isomorphism

$$h : H_{n-1}(f^{-1}(t_0), \mathbf{Z}) \longrightarrow H_{n-1}(f^{-1}(t_0), \mathbf{Z})$$

that will be called the global monodromy of f.

We will give sufficient conditions for a family of polynomials of two

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variables $f_\alpha(x,y)$, $\alpha \in [0,1]$, such that the global monodromies of f_0 and f_1 are conjugate.

1.2. Let us recall some facts on the topology of polynomials of two variables. We say that a value $t_0 \in \mathbf{C}$ is *regular at infinity* if there exist a small $\delta > 0$ and a compact $K \subset \mathbf{C}^2$ such that the restriction

$$f : f^{-1}(D_\delta) \setminus K \longrightarrow D_\delta, \qquad D_\delta := \{t| \quad |t - t_0| < \delta\},$$

is a trivial C^∞-fibration [N]. If t_0 is not regular at infinity, it is called a *critical value at infinity* of f. If we denote by C_f (resp., $A_{f,\infty}$) the set of critical values (resp., the set of critical values at infinity) of f, then $A_f = C_f \cup A_{f,\infty}$ (see, for example, [HL]).

Let d be the degree of $f(x,y)$ and $f_d(x,y)$ be the homogeneous part of degree d of f. In $\mathbf{C}P^2$ we consider the family of curves

$$\overline{V_t} = \{(x:y:z) \mid \quad z^d f(\frac{x}{z}, \frac{y}{z}) - tz^d = 0\}.$$

We see that $\overline{V_t}$ is the compactification of $V_t = f^{-1}(t)$. For any t, the curves $\overline{V_t}$ intersect the line $z = 0$ at the points of $\{(x:y:z) \mid f_d(x,y) = 0, z = 0\} = \{A_1, \ldots, A_s\}$. Let $\mu_{\overline{V_t}}(A_i)$ be the Milnor number of $\overline{V_t}$ at A_i. For $t_0 \in \mathbf{C}$ put

$$\lambda(t_0) = \sum_{i=1}^{s} [\mu_{\overline{V}_{t_0}}(A_i) - \mu_{\overline{V}_t}(A_i)]$$

for t general enough.

It is proved in [HL] that $t_0 \in A_{f,\infty}$ if and only if $\lambda(t_0) > 0$. For every polynomial f, let

$$\lambda(f) = \sum_{t \in A_{f,\infty}} \lambda(t).$$

The total Milnor number of a polynomial f denoted by $\mu(f)$ is defined by

$$\mu(f) := \dim_{\mathbf{C}} \frac{\mathbf{C}[x,y]}{(f_x, f_y)} \cdot$$

Also, we put $\sigma(f) := \# A_{f,\infty}$, and $d(f) := \deg f(x,y)$.

In this note, we always suppose that all fibers of polynomials are reducible. In particular, this implies that the polynomials are primitive ([A],[S]).

1.3. In the next section we will prove the following

Theorem. *Let $f_\alpha(x,y)$ be a family of polynomials of two variables whose coefficients are smooth complex-valued functions of $\alpha \in I := [0,1]$. Suppose that the numbers $\mu(f_\alpha)$, $\lambda(f_\alpha)$, $\sigma(f_\alpha)$ and $d(f_\alpha)$ are independent of α. Then the global monodromies of the polynomials f_0 and f_1 are conjugate.*

The above result can be considered as a global analogue of the Lê-Ramanujam theorem [LR]. In [HZ] a stronger result is proved for families of M-tame polynomials of n variables ($n \neq 3$). Note that for M-tame polynomials f, $\lambda(f) = \sigma(f) = 0$.

2. Proof of Theorem 1.3

2.1. Let $f \in \mathbf{C}[x,y]$. Suppose that $A_f \subset D_r := \{t \in \mathbf{C} \mid \ |t| < r\}$. Let

$$\begin{aligned} S_r^1 &= \partial D_r, \\ B_R &= \{(x,y) \in \mathbf{C}^2 \mid \ \|(x,y)\| \le R\}, \\ \overset{\circ}{B}_R &= \{(x,y) \in \mathbf{C}^2 \mid \|(x,y)\| < R\}, \\ S_R^3 &= \partial B_R. \end{aligned}$$

First of all we show that there exists $R_0 \gg 1$ such that the fibrations

$$f : f^{-1}(S_r^1) \longrightarrow S_r^1, \tag{1}$$

$$f : f^{-1}(S_r^1) \cap \overset{\circ}{B}_{R_0} \longrightarrow S_r^1, \tag{2}$$

$$f : f^{-1}(S_r^1) \cap \overset{\circ}{B}_R \longrightarrow S_r^1, \tag{3}$$

are isomorphic for $R \ge R_0$. To this end we need the following.

Lemma. *For $r > \max\{|t| \mid \ t \in A_f\}$, there exists $R_0 \gg 1$ such that all fibres $f^{-1}(t)$, $t \in S_r^1$, are transversal to all spheres S_R^3 with $R \ge R_0$.*

Proof. We first recall some characterizations of the values of $A_{f,\infty}$.

Suppose $t_0 \in \mathbf{C}$. For $\delta > 0, R \gg 1$, put

$$\varphi_{\delta,t_0}(R) = \inf_{\|z\|=R,\ f(z)\in \overline{D}_\delta} \|\mathrm{grad} f(z)\| .$$

The Lojasiewicz number at infinity of the curve $f^{-1}(t_0)$ is defined by

$$\mathcal{L}_{\infty,t_0}(f) = \lim_{\delta\to 0} \lim_{R\to\infty} \frac{\ln \varphi_{\delta,t_0}(R)}{\ln R} .$$

It is proved in [H1, H3] that $t_0 \in A_{f,\infty}$ if and only if $\mathcal{L}_{\infty,t_0}(f) < 0$. In particular, $t_0 \in A_{f,\infty}$ if there is a sequence $\{z_n\} \subset \mathbf{C}^2$ such that $\|z_n\| \to \infty, \|\text{grad} f(z_n)\| \to 0$, and $f(z_n) \to t_0$ as $n \to \infty$.

For a polynimial f let

$$\text{grad} f = \Big(\overline{\frac{\partial f}{\partial x}}, \overline{\frac{\partial f}{\partial y}}\Big) \cdot$$

Assume for the contrary that there exist $z_n \in \mathbf{C}^2, \lambda_n \in \mathbf{C}$ such that $\|z_n\| \to \infty$ as $n \to \infty$, and $\text{grad} f(z_n) = \lambda_n z_n$. By a version at infinity of the Curve Selection Lemma, there exists a real meromorphic curve

$$\psi : (0, \epsilon] \longrightarrow \mathbf{C}^2$$
$$\tau \mapsto z(\tau)$$

such that $f(z(\tau)) \in S^1_r$, $\text{grad} f(z(\tau)) = \lambda(\tau) z(\tau)$, and $\|z(\tau)\| \to \infty$ as $\tau \to 0$. Since $|f(z(\tau))| = r$, $f(z(\tau)) = t_0 + a_1 \tau^\rho + \cdots$ for some $t_0 \in S^1_r$ and $\rho > 0$. We have

$$\frac{df(z(\tau))}{d\tau} = \Big\langle \frac{dz}{d\tau}, \text{grad} f(z(\tau)) \Big\rangle = \overline{\lambda}(\tau) \Big\langle \frac{dz}{d\tau}, z(\tau) \Big\rangle.$$

Then

$$\frac{1}{\overline{\lambda}(\tau)} \frac{df(z(\tau))}{d\tau} + \frac{1}{\lambda(\tau)} \overline{\frac{df(z(\tau))}{d\tau}} = \frac{d}{d\tau} \|z(\tau)\|^2.$$

It follows that

$$|\lambda(\tau)| \leq 2 \frac{\Big|\dfrac{df(z(\tau))}{d\tau}\Big|}{\dfrac{d\|z(\tau)\|^2}{d\tau}} \cdot$$

Let $\|z(\tau)\| = b_1 \tau^\beta + \cdots$, $\beta < 0$, then $|\lambda(\tau)| \leq c \dfrac{|\tau|^{\rho-1}}{|\tau|^{2\beta-1}} = c|\tau|^{\rho-2\beta}$ for some $c > 0$. We have

$$\|\text{grad} f(z(\tau))\| = |\lambda(\tau)|.\|z(\tau)\| \leq c|\tau|^{\rho-\beta}.$$

Since $\rho > 0$ and $\beta < 0$, $\|\text{grad} f(z(\tau))\| \to 0$ as $\tau \to 0$. Thus, according to the result mentioned above, $t_0 \in A_{f,\infty}$, which is a contradiction and the lemma is proved.

Now, using this lemma we can construct a vector field tangent to $f^{-1}(S^1_r)$ and pointing to the infinity. In fact, there exists a smooth vector field $v(z)$ such that

(i) $\langle v(z), \mathrm{grad} f(z)\rangle \;=\; 0,$
(ii) $\langle v(z), z\rangle \;>\; 0.$
(By the Lemma, we can construct such a vector field locally, then extend it over $f^{-1}(S^1_r)$ by a smooth partition of unity). Put

$$w(z) = \frac{v(z)}{2\langle v(z), \overline{z}\rangle}(\|z\|^4 + 1).$$

This vector field is completely integrable, and let $p_{z_0}(\tau)$ be its integral curve with $p_{z_0}(0) = z_0$. By condition (i), if $z_0 \in f^{-1}(t) \cap \overset{\circ}{B}_{R_0}$, then $p_{z_0}(\tau) \in f^{-1}(t)$. Moreover,

$$\begin{aligned}
\frac{d\|p_{z_0}(\tau)\|^2}{d\tau} &= \langle \frac{dp_{z_0}(\tau)}{d\tau}, p_{z_0}(\tau)\rangle + \langle p_{z_0}(\tau), \frac{dp_{z_0}(\tau)}{d\tau}\rangle \\
&= 2\mathrm{Re}\langle \frac{dp_{z_0}(\tau)}{d\tau}, p_{z_0}(\tau)\rangle \\
&= 2\mathrm{Re}\langle w(p_{z_0}(\tau)), p_{z_0}(\tau)\rangle \\
&= \|p_{z_0}(\tau)\|^4 + 1.
\end{aligned}$$

Hence

$$\arctan\|p_{z_0}(\tau)\|^2 - \arctan\|z_0\|^2 \;=\; \tau,$$

or

$$\|p_{z_0}(\tau)\|^2 \;=\; \tan(\tau + \arctan\|z_0\|^2).$$

Let $\tau_0 = \dfrac{\pi}{2} - \arctan R_0^2$. Then $p_{z_0}(\tau_0) \to \infty$ as $\|z_0\| \to R_0$. Thus, the mapping

$$f^{-1}(S^1_r) \cap \overset{\circ}{B}_{R_0} \ni z_0 \mapsto p_{z_0}(\tau_0) \in f^{-1}(S^1_r)$$

is an isomorphism between two fibrations.

2.2. In this step of the proof, we show that the conditions

$$\begin{aligned}
\mu(f_\alpha) &= const,\\
\sigma(f_\alpha) &= const,\\
\lambda(f_\alpha) &= const,\\
d(f_\alpha) &= const
\end{aligned}$$

imply the existence of a number $r > 0$ such that $A_{f_\alpha} \subset D_r$ for all $\alpha \in I$.

We will show that there exists r such that $C_{f_\alpha} \subset D_r$. Let Σ_{f_α} be the set of all critical points of f_α. It is enough to show that there exists

$R \gg 1$ such that $\Sigma_{f_\alpha} \subset B_R$ for all $\alpha \in I$. In fact, if we choose R_0 such that $\Sigma_{f_0} \subset \overset{\circ}{B}_{R_0}$ and put

$$\varphi_{0,R} := \frac{\mathrm{grad} f_0}{\|\mathrm{grad} f_0\|} : S^3_R \longrightarrow S^3_1,$$

then $\mu(f_0)$ is the degree of $\varphi_{0,R} : \mu(f_0) = d(\varphi_{0,R}),\ R > R_0$. Consider the mapping

$$\varphi_{\alpha,R} := \frac{\mathrm{grad} f_\alpha}{\|\mathrm{grad} f_\alpha\|} : S^3_R \longrightarrow S^3_1.$$

Then, for all sufficiently small α, $d(\varphi_{\alpha,R}) = d(\varphi_{0,R}) = \mu(f_0)$. Suppose that there exists $z(\alpha) \in \Sigma_{f_\alpha}$ such that $\|z(\alpha)\| \to \infty$ as $\alpha \to 0$. Take α_1 sufficiently small with $\|z(\alpha_1)\| > R$ and $R_1 > \|z(\alpha_1)\|$. We have

$$d(\varphi_{\alpha_1,R_1}) > d(\varphi_{\alpha_1,R}) = d(\varphi_{0,R}) = \mu(f_0).$$

On the other hand, $\mu(f_{\alpha_1}) \geq d(\varphi_{\alpha_1,R_1})$. These inequalities give a contradiction to the condition $\mu(f_\alpha) = const$.

We now show that there exists r such that $A_{f_\alpha,\infty} \subset D_r$, for all $\alpha \in I$. Without loss of generality, we can suppose that for $t \in \mathbf{C}$, and α sufficiently close to 0, the map

$$\pi_{\alpha,t} := \pi|_{f_\alpha^{-1}(t)} : f_\alpha^{-1}(t) \longrightarrow \mathbf{C}$$
$$(x,y) \mapsto x$$

is proper. Considering $f_\alpha(x,y) - t$ as a polynomial in $\mathbf{C}[x,t][y]$, we put $\Delta(\alpha,x,t) = \mathrm{disc}_y(f_\alpha(x,y) - t)$. Let

$$\Delta(\alpha,x,t) = q_0(\alpha,t)x^{m(\alpha)} + q_1(\alpha,t)x^{m(\alpha)-1} + \cdots .$$

We first claim that the degree $m(\alpha)$ in x of $\Delta(\alpha,x,t)$ is constant. In fact, $m(\alpha)$ can be computed in terms of $d(f_\alpha), \mu(f_\alpha), \lambda(f_\alpha)$ as follows.

For generic systems of coodinates, $\pi_{\alpha,t}$ has only simple critical points and the number of these points is exactly equal to $m(\alpha)$. Let

$$(x_1(\alpha,t), y_1(\alpha,t)), \ldots, (x_{m(\alpha)}(\alpha,t), y_{m(\alpha)}(\alpha,t))$$

be critical points of $\pi_{\alpha,t}$. In the plane of x's, choose $x_0 \neq x_i(\alpha,t), i = 1,\ldots,m(\alpha)$. We connect x_0 with $x_i(\alpha,t)$ by paths T_i such that each T_i has no points of self-intersection, and that $T_i \cap T_j = \{x_0\}$ $(i \neq j)$. Put

$$O_{\alpha,t} = \pi^{-1}_{\alpha,t}\Big(\bigcup_{i=1}^{m(\alpha)} T_i\Big) .$$

Then, $O_{\alpha,t}$ is a deformation retract of $f_\alpha^{-1}(t)$ (see [H2]). Hence,

$$\chi(f_\alpha^{-1}(t)) = \chi(O_{\alpha,t}) \ .$$

The set $O_{\alpha,t}$ can be identified with an $1-$dimensional graph of $d(f_\alpha) + m(\alpha)$ vertices and $2m(\alpha)$ edges. Thus

$$\chi(O_{\alpha,t}) = d(f_\alpha) + m(\alpha) - 2m(\alpha) = d(f_\alpha) - m(\alpha).$$

Since f_α is primitive, by [B] we have

$$\chi(f_\alpha^{-1}(t)) = 1 - \mu(f_\alpha) - \lambda(f_\alpha).$$

These equalities imply

$$m(\alpha) = d(f_\alpha) + \mu(f_\alpha) + \lambda(f_\alpha) - 1 = \mathit{const}.$$

This means that for any $\alpha \in I, q_0(\alpha,t)$ is the non-zero polynomial in t (possibly of degree 0). Since the coefficients of $q_0(\alpha,t)$ are smooth complex-valued functions of $\alpha \in I$,

$$\#\{t \mid q_0(\alpha,t) = 0\} \geq \#\{t \mid q_0(0,t) = 0\}.$$

Here, we have a strict inequality iff there exists $t(\alpha) \in \mathbf{C}$ such that $q_0(\alpha, t(\alpha)) = 0, t(\alpha) \to \infty$ as $\alpha \to 0$. On the other hand, according to a result of [H2],

$$A_{f_\alpha,\infty} = \{t(\alpha) \in \mathbf{C} \ |q_0(t(\alpha)) = 0\}.$$

It follows from the condition $\sigma(\alpha) = \mathit{const}$ that there exists $r > 0$ such that $A_{f_\alpha,\infty} \subset D_r$.

Now, we can repeat the proof of [L].

2.3. Lemma. *Suppose that r is chosen as in 2.2. Then there exist $R_0 \gg 1$ and $\alpha_0 > 0$ such that for any $\alpha \in [0,\alpha_0]$, the maps*

$$f_\alpha : f_\alpha^{-1}(S_r^1) \cap B_{R_0} \longrightarrow S_r^1 \tag{4}$$

are C^∞ locally trivial fibrations. Moreover, the fibrations defined by f_0 and f_{α_0} are differentiably isomorphic.

Proof. By Lemma 2.1, there exists $R_0 \gg 1$ such that for all $t \in S_r^1$, the fibres $f_0^{-1}(t)$ are transversal to $S_{R_0}^3$. We claim that there exists $\alpha_0 > 0$ such that if $\alpha \in [0,\alpha_0]$, all fibres $f_\alpha^{-1}(t)$ with $t \in S_r^1$ are transversal to $S_{R_0}^3$. In fact, if it is not so, there exist $z(\alpha) \in S^3_R$, $\lambda(\alpha) \in \mathbf{C}$, $t(\alpha) = f_\alpha(z(\alpha)) \in S^1_r$, and $\mathrm{grad} f_\alpha(z(\alpha)) = \lambda(\alpha) z(\alpha)$. By the Curve Selection Lemma, we may assume that $z(\alpha)$, $\lambda(\alpha)$, $t(\alpha)$ are real analytic functions of α. Letting $\alpha \to 0$, we see that there exist $z(0) \in S^3_{R_0}$, $\lambda(0) \in \mathbf{C}$, $t_0 \in S^1_r$ such that $\mathrm{grad} f_0(z(0)) = \lambda(0) z(0)$, $f_0(z(0)) = t_0$. This means that the fibre $f_0^{-1}(t_0)$ is not transversal to $S^3_{R_0}$. Hence we obtain a contradiction.

Now, suppose R_0, α_0 as above. Let $I_1 := [0, \alpha_0]$. Consider the map

$$\Phi : \mathbf{C}^2 \times I_1 \longrightarrow \mathbf{C} \times I_1 : \quad (x, y, \alpha) \mapsto (f_\alpha(x, y), \alpha) \ .$$

Let

$$\Sigma_0 = \Phi^{-1}(S^1_r \times \{0\}) \cap (B_{R_0} \times \{0\}),$$

$$\Sigma_1 = \Phi^{-1}(S^1_r \times \{\alpha_0\}) \cap (B_{R_0} \times \{\alpha_0\}),$$

$$\varphi_0 : \Sigma_0 \underset{(x,y,0) \mapsto f_0(x,y)}{\longrightarrow} S^1_r,$$

$$\varphi_1 : \Sigma_1 \underset{(x,y,\alpha_0) \mapsto f_{\alpha_0}(x,y)}{\longrightarrow} S^1_r.$$

Since $\mathrm{rank}\, \Phi = 2$ over $\Phi^{-1}(S^1_r \times I_1)$, by Lemma 2.1, Σ_0 (resp. Σ_1) is a compact manifold with boundary. Furthermore the map φ_0 (resp. φ_1) has no critical point in the interior of Σ_0 (resp. Σ_1), and its restriction to the boundary has maximal rank. Thus, by a version of the Ehresmann lemma for the case of manifolds with boundary, φ_0 and φ_1 are locally trivial fibrations. To see that these fibrations are isomorphic we suppose that Ω and V are open neighborhoods of I_1 and S^1_r, respectively. By the choice of R_0 the restriction

$$\Phi : \Phi^{-1}(V \times \Omega) \cap (S^3_{R_0} \times \Omega) \longrightarrow V \times \Omega$$

is a submersion over $V \times \Omega$. Let v be a vector field on $V \times \Omega$ defined by $v(t, \alpha) = (0, \alpha_0)$. Then we can construct in $\Phi^{-1}(V \times \Omega) \cap (S^3_{R_0} \times \Omega)$ a vector field v_1 which is tangent to $S^3_{R_0} \times \Omega$ such that for every $z \in \Phi^{-1}(V \times \Omega) \cap (S^3_{R_0} \times \Omega)$,

$$D_z \Phi . v_1(z) = v(\Phi(z)) = (0, \alpha_0) \ .$$

Let $z_i \in \Phi^{-1}(V \times \Omega) \cap (S^3_{R_0} \times \Omega)$. There exist a neighborhood U_i of z_i in $\Phi^{-1}(V \times \Omega)$ and a diffeomorphism

$$\theta_i : U_i \longrightarrow [U_i \cap (S^3_{R_0} \times \Omega)] \times (R_1, R_2) \ , \quad 0 < R_1 < R_0 < R_2,$$

such that for every $R \in (R_1, R_2)$,

$$\theta_i(U_i \cap (S^3_R \times \Omega)) = [U_i \cap (S^3_{R_0} \times \Omega)] \times \{R\} ,$$

and $\Phi\theta_i^{-1}$ has maximum rank on $[U_i \cap (S^3_{R_0} \times \Omega)] \times \{R\}$. This is possible because for R sufficiently close to R_0, the restriction of Φ to $\Phi^{-1}(V \times \Omega) \cap (S^3_R \times \Omega)$ induces a submersion over $V \times \Omega$. Thus, we can define a vector field w_i on U_i such that for every $z \in U_i \cap (S^3_{R_0} \times \Omega)$, $w_i(z) = v_1(z)$ and for every $z \in U_i \cap (S^3_R \times \Omega)$, $R \in (R_1, R_2)$, the following hold.

(i) $w_i(z)$ is tangent to $(S^3_R \times \Omega)$,

(ii) $D_z\Phi.w_i(z) = (0, \alpha_0)$.

Let $i_1, i_2, \dots, i_n$ be indices such that $(U_{i_j})_{1 \le j \le n}$ is a covering of $\Phi^{-1}(S^1_r \times I_1) \cap (S^3_{R_0} \times I_1)$. Let $\overline{U}$ be a compact neighborhood of $\Phi^{-1}(S^1_r \times I_1) \cap (S^3_{R_0} \times I_1)$ contained in $\cup_{i=1}^n U_{i_j}$. In $U_2 := \Phi^{-1}(V \times \Omega) \cap (B_{R_0} \times \Omega) \setminus \overline{U}$ we consider a vector field v_2 such that for every $z \in U_2$, $D_z\Phi.v_2(z) = (0, \alpha_0)$. This is possible because Φ induces a submersion of U_2 on $V \times \Omega$.

Let $\{\overline{\psi}_{i_1}, \cdots, \overline{\psi}_{i_n}, \overline{\psi}_2\}$ be a partition of unity associated with $U_{i_1}, \cdots U_{i_n}, U_2$. Then the vector field w defined by

$$w = \sum_{j=1}^{n} \overline{\psi}_{i_j} w_{i_j} + \overline{\psi}_2.v_2$$

is differentiable with compact support. For every $z \in \Phi^{-1}(S^1_r \times I_1) \cap (S^3_{R_0} \times I_1)$ we have

(i) $w(z)$ is tangent to $S^3_{R_0} \times I_1$,

(ii) $D_z\Phi.w(z) = (0, \alpha_0)$.

Moreover, for every $z \in \Phi^{-1}(S^1_r \times I_1) \cap (\mathring{B}_{R_0} \times I_1)$, $D_z\Phi.w(z) = (0, \alpha_0)$.

This vector field is completely integrable and, if $p_z : \mathbf{R} \longrightarrow \Phi^{-1}(V \times \Omega) \cap (B_{R_0} \times \Omega)$ is an integral curve with $p_z(0) = z$, then for $z \in \Sigma_0$ we have $p_z(1) \in \Sigma_1$.

Thus, we obtain a diffeomorphism Ψ from Σ_0 onto Σ_1 which makes the following diagram

$$\Sigma_0 @ > h >> \Sigma_1$$

$$@V\varphi_0 VV @V\varphi_1 VV$$

$$S_r^1 @ > id >> S_r^1$$

commutative. Thus, φ_0 and φ_1 are isomorphic. The proof of Lemma 2.3 is now complete.

2.4. Now we prove that the monodromies of f_0 and f_1 are conjugate.

First, we will show that their fibrations are of the same fibre homotopy. Indeed, by Lemma 2.3, the fibrations

$$f_0 : f_0^{-1}(S_r^1) \cap B_{R_0} \longrightarrow S_r^1 \tag{5}$$

and

$$f_{\alpha_0} : f_{\alpha_0}^{-1}(S_r^1) \cap B_{R_0} \longrightarrow S_r^1 \tag{6}$$

are isomorphic. By Lemma 2.1, there exists $R_1 \gg 1$ such that the fibration

$$f_{\alpha_0} : f_{\alpha_0}^{-1}(S_r^1) \cap B_{R_1} \longrightarrow S_r^1 \tag{7}$$

is isomorphic to the global monodromy fibration of f_1. If $R_1 \leq R_0$, everything is clear. Suppose that $R_0 < R_1$. The fibration (6) is contained in the fibration (7). Hence we have to prove that the inclusion of (6) in (7) is a fibre homotopy equivalence. To prove this, by a result of [D] it is sufficient to show that the inclusion of the fiber $F = f_{\alpha_0}^{-1}(t) \cap B_{R_0}$ of (6) in the fiber $\widetilde{F} = f_{\alpha_0}^{-1}(t) \cap B_{R_1}$ of (7) is a homotopy equivalence for every $t \in S_r^1$. We claim that the inclusion F in $\widetilde{F}$ gives an isomorphism η of homology groups $H_1(F)$ and $H_1(\widetilde{F})$. In fact, the function $\|z\|^2_{|_{\widetilde{F}}}$ is Morse and the index at each critical point is 0 or 1 [AF]. Thus $\widetilde{F}$ is obtained from F, up to homotopy type, by attaching cells of dimension ≤ 1. It follows that the group $H_1(\widetilde{F}, F)$ is free. In the sequence

$$0 \longrightarrow H_1(F) \longrightarrow H_1(\widetilde{F}) \longrightarrow H_1(\widetilde{F}, F) \longrightarrow 0,$$

we have by [B]

$$\operatorname{rank} H_1(F) = \mu(f_0) + \lambda(f_0) = \mu(f_{\alpha_0}) + \lambda(f_{\alpha_0}) = \operatorname{rank} H_1(\widetilde{F}).$$

Thus $H_1(\widetilde{F}, F) = 0$, and the inclusion of F in $\widetilde{F}$ is an isomorphism of homology groups. Since $\widetilde{F}$ is connected, the inclusion of F in $\widetilde{F}$ is a homotopy equivalence.

Put

$$X = f_{\alpha_0}^{-1}(S_r^1) \cap B_{R_0},$$

$$\widetilde{X} = f_{\alpha_0}^{-1}(S_r^1) \cap B_{R_1}.$$

Consider the Wang diagram

$$0@ >>> H_2(\widetilde{X})@ >>> H_1(\widetilde{F})@ > \widetilde{h} - id >> H_1(\widetilde{F})@ >>> H_1(\widetilde{X})@ >>> 0$$

$$@.@AAA@A\eta AA@A\eta AA@AAA$$

$$0@ >>> H_2(X)@ >>> H_1(F)@ > h - id >> H_1(F)@ >>> H_1(X)@ >>> 0,$$

where the vertical arrows are the inclusions and $h, \widetilde{h}$ are associated with the monodromy maps φ_1 and $\widetilde{\varphi}_1$. By the Five-Lemma, the inclusion $X \subset \widetilde{X}$ induces isomorphisms

$$H_2(X) \xrightarrow{\sim} H_2(\widetilde{X}) \ ,$$

$$H_1(X) \xrightarrow{\sim} H_1(\widetilde{X}) \ .$$

These imply

$$\eta(h - id)\eta^{-1} = \widetilde{h} - id \ .$$

Hence $\eta \circ h \circ \eta^{-1} = \widetilde{h}$. Thus, Theorem 1.4 is proved.

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