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GLOBAL SOLUTION OF A TWO PHASE FREE BOUNDARY PROBLEM

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

1. Introduction

In this paper, we consider a two phase free boundary problem with a kinetic condition at the moving boundary. Let $\alpha, \beta : R^2 \to R$ be two functions satisfying conditions to be given later. The following problems $(P_b)(0 < b < 1)$ and (P_0) will be studied: find a triplet (u_1, u_2, s) satisfying the diffusion equations in the two phases separated by a moving boundary located at $x = s(t)$, a kinetic condition and some appropriate conditions as given below:

Problem (P_b) $(0 < b < 1)$

$$(1.1) \qquad \begin{cases} u_{1xx} - u_{1t} &= 0 \qquad 0 < x < s(t),\ t > 0 \\ \gamma u_{2xx} - u_{2t} &= 0 \qquad s(t) < x < 1,\ t > 0, \end{cases}$$

$$(1.2) \qquad s'(t) = \beta(u_1(s(t),t), u_{2x}(s(t),t)),$$

$$(1.3) \qquad s(0) = b,$$

$$(1.4) \qquad u_1(0,t) = f(t),\ u_{2x}(1,t) = 0,$$

$$(1.5) \qquad u_1(x,0) = h(x)\ (0 \le x \le b),\ u_2(x,0) = H(x)\ (b \le x \le 1),$$

$$(1.6) \qquad u_{1x}(s(t),t) = \alpha(u_1(s(t),t),\ u_{2x}(s(t),t)),\ u_2(s(t),t) = 0,$$

and

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Problem (P_0) with the conditions (1.3), (1.5) replaced by

$$(1.3)' \qquad s(0) = 0,$$

$$(1.5)' \qquad u_2(x,0) = H(x),\ 0 \le x \le 1.$$

The model of transport phenomena in polymers in [1] is a special case of our problem with $s'(t) = \theta^{-1}|u_1(s(t),t)|^n$ ($\theta > 0$: rexalation time) and there are $\beta_1, \gamma > 0,\ \alpha_1 \ge 0$ such that:

$$u_{1x}(s(t),t) = \beta_1 u_{2x}(s(t),t) - \{\alpha_1 + u_1(s(t),t) - \beta_1\gamma^{-1}u_2(s(t),t)\}.$$

We note that in our model there is a discontinuity for the two phases at the boundary $x = s(t)$. In [5], we find a variant of our problem with a kinetic condition $s'(t) = \beta(u_1(s(t),t))$ and with the condition $u_2(s(t),t) = 0$ replaced by $u_1(s(t),t) = u_2(s(t),t)$ (i.e. we get a continuity for the two phases across the curve $x = s(t)$). This was the object of our earlier paper [4]. In the present paper we are concerned with a local existence result for (P_b) $(0 \le b < 1)$ and we give sufficient conditions for the existence of a global solution of (P_b) $(0 \le b < 1)$. We shall give a sufficient condition for the disappearance of phase in finite time.

For the solution we shall use the method of integral equations. We will make repeated use of the maximum principle. First, we define the concept of a solution. For $\sigma > 0$, we say that the problem (P_b) $(0 < b < 1)$ or (P_0) has a solution $(u_1(x,t), u_2(x,t), s(t))$ on $(0,\sigma)$ if:

(a) u_{1xx}, u_{1t} are continuous for $0 < x < s(t),\ t \in (0,\sigma)$,

(b) u_{2xx}, u_{2t} are continuous for $s(t) < x < 1,\ t \in (0,\sigma)$,

(c) u_1, u_{1x} are continuous for $0 \le x \le s(t),\ t \in (0,\sigma)$,

(d) u_2, u_{2x} are continuous for $s(t) \le x \le 1,\ t \in (0,\sigma)$,

(e) u_1, u_2, s satisfied Problem (P_b) $(0 \le b < 1)$.

Following are the main results of this paper:

Theorem 1.

(i) *Let* $0 < b < 1$. *Suppose* $\gamma > 0, h \in C^1[0,b],\ H \in C^1[b,1],\ f \in C^1[0,T]$ $(T > 0)$, *that* $\alpha, \beta : R^2 \to R$ *are locally Lipschitzian, and that* $h(0) = f(0),\ H(b) = 0,\ h'(b) = \alpha(h(b), H'(b))$. *Then there exists* $0 < \sigma \le T$ *such that Problem* (P_b) *has a unique solution* (u_1, u_2, s) *on* $(0,\sigma)$.

(ii) *In the case* $s(0) = b = 0$ *we also assume that* $H \in C^1[0,1],\ H(0) = 0,\ f \in C^1[0,T]$, *that* α, β *are as above,* $\gamma > 0$ *and that* $\beta(f(0), H'(0)) > 0$. *Then there exists:* $0 < \sigma \le T$ *such that Problem* (P_0) *has a unique solution* (u_1, u_2, s) *on* $(0,\sigma)$.

Theorem 2. *Let* $0 < b < 1$, *let the assumptions in part* (i) *of Theorem 1 hold, and suppose in addition that*

(A) $f \in C^1[0,\infty),\ f(t) \ge 0$ *for every* $t \ge 0$,

(B) $h(x) \geq 0$ $(0 \leq x \leq b)$, $H(x) \leq 0$ $(b \leq x \leq 1)$,

(C) $\beta_0(u) > 0$ $\forall u \neq 0$, $\beta_0(0) = 0$ *where* $\beta_0(u) = \beta(u, v)$ $\forall v$,

(D) $\alpha(u, v) = \delta v + g(u)\beta_0(u)$ $(\delta > 0)$ *and there is an* $\alpha_0 \in R$ *such that if* $g(u) \geq 0$ *then* $u \leq \alpha_0$,

and either

(E) $\gamma \geq 1$ *and:*

$$\beta_0(u)|u + g(u) - \alpha_0^+ - k| \leq C|u| + C$$

for a $C > 0$, *with* $\alpha_0^+ = \max(\alpha_0, 0)$, $k = \max(\sup_{t\geq 0} f(t), \sup_{0\leq x\leq b} h(x))$

or both the two following conditions hold:

(E$_1$) *There are* $\bar{\alpha}_0, \alpha_1, k_0, p$ $(\alpha_1, k_0, p > 0)$ *such that*

* $g(u) < 0$ *then* $u > \bar{\alpha}_0$,

* $|u| \geq \alpha_1$ *then* $|g(u)|\beta_0(u) \geq k_0|u|^p$.

(E$_2$) *For an* $M > 4\max(|\bar{\alpha}_0|, 4\alpha_0^+, \alpha_1, 16k/3)$, $H(x)$ *satisfies the following inequalities:*

$$\sup_{b\leq x\leq 1}\{|H(x)| + |H'(x)|\} \leq \min\Big\{\frac{M}{4M_0}, \frac{k_0}{\delta M_0}\Big(\frac{M}{4}\Big)^p\Big\}$$

where $M_0 = 4 + \gamma^{-1}\sup_{|u|\leq M}\beta_0(u)$.

Then (P_b) has a global solution on $[0, T^*)$ *with* $T^* = \infty$ *or such that* $\lim_{t\uparrow T^*} s(t) = 1$.

In the latter case if we make the following additional assumption

(E$_3$) $\sup_{t\geq 0}|f'(t)| + \sup_{0\leq x\leq b}|h'(x)| + \sup_{|u|,|v|\leq M}|\alpha(u, v)| \leq M/16$

and

$$f(t) \geq M/8 \quad \forall t > 0,$$

then

$$T^* \leq (1 - b)\{\min_{M/16\leq|u|\leq M}\beta_0(u)\}^{-1}.$$

Remarks

1. If $g : \mathbb{R} \to \mathbb{R}$ satisfies $\limsup_{u\to+\infty} g(u) < 0$, $\liminf_{u\to-\infty} g(u) > 0$ and $\beta_0(u) = C|u|^n (n \geq 1)$, then the conditions (D), (E_1) hold. For example, we can take $g(u) = \alpha_0 - u$ (see [1]) or $g(u) = -u|u|^\rho (\rho > 0), g(u) = -u(1+|u|)^{-1}$.

2. If either

$$\sup_{u\in R}\beta_0(u) < \infty, \ |u + g(u)| \leq C(|u| + 1)$$

or

$$|\beta_0(u)| \le C(|u|+1),\ |u+g(u)| \text{ is bounded}$$

then condition (E) holds.

3. Condition (E_2) is satisfied if H is sufficiently small in $C^1[b,1]$. If $\sup\limits_{u\in R} \beta_0(u) < \infty$ then for every $H \in C^1[b,1], h \in C^1[0,b], f \in C^1[0,\infty)$ with $\sup\limits_{t\ge 0} f(t) < \infty$, we can always choose M large enough so that (E_2) hold.

4. In the case $b=0$, the method of this paper can be applied to one phase problems.

The remainder of the paper is devoted to the proofs of Theorem 1 and Theorem 2.

2. Proof of theorem 1

For $0 \le \tau < t$, we put

$$\begin{aligned}
K(x,t;\xi,\tau) &= \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\Big(-\frac{(x-\xi)^2}{4(t-\tau)}\Big),\\
G_1(x,t;\xi,\tau) &= K(x,t;\xi,\tau) - K(-x,t;\xi,\tau),\\
G_2(x,t;\xi,\tau) &= K(x,\gamma t;\xi,\gamma\tau) + K(2-x,\gamma t;\xi,\gamma\tau),\\
N_2(x,t;\xi,\tau) &= K(x,\gamma t;\xi,\gamma\tau) - K(2-x,\gamma t;\xi,\gamma\tau).
\end{aligned}$$

These are Green's functions which will be used. Consider first the case $s(0)=b>0$ $(0<b<1)$. Integrating the identities

$$\begin{aligned}
(G_1 u_{1\xi} - G_{1\xi} u_1)_\xi - (G_1 u_1)_\tau &= 0,\\
\gamma(G_2 u_{2\xi} - G_{2\xi} u_2)_\xi - (G_2 u_2)_\tau &= 0
\end{aligned}$$

over the regions $\{(\xi,\tau) : 0 \le \xi \le s(\tau), \varepsilon \le \tau \le t-\varepsilon\}$, $\{(\xi,\tau) : s(\tau) \le \xi \le 1, \varepsilon \le \tau \le t-\varepsilon\}$, respectively and letting $\varepsilon \downarrow 0$, we get after some rearrangements

$$
\begin{aligned}
(2.1) \qquad u_1(x,t) = & \int_0^b G_1(x,t;\xi,0)h(\xi)d\xi + \int_0^t f(\tau)G_{1\xi}(x,t;0,\tau)d\tau \\
& + \int_0^t G_1(x,t;s(\tau),\tau)u_1(s(\tau),\tau)s'(\tau)d\tau \\
& + \int_0^t G_1(x,t;s(\tau),\tau)\alpha(u_1(s(\tau),\tau),u_{2x}(s(\tau),\tau))d\tau \\
& - \int_0^t G_{1\xi}(x,t;s(\tau),\tau)u_1(s(\tau),\tau)d\tau, \quad 0<x<s(t)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.2) \qquad u_2(x,t) = & \int_b^1 G_2(x,t;\xi,0)H(\xi)d\xi - \\
& -\gamma \int_0^t G_2(x,t;s(\tau),\tau)u_{2\xi}(s(\tau),\tau)d\tau, \quad s(t)<x<1.
\end{aligned}
$$

Letting $x \uparrow s(t)$ in (2.1), differentiating (2.2) with respect to x and letting $x \downarrow s(t)$ in the result thus obtained, we get

$$
\begin{aligned}
(2.3) \qquad w(t) = & 2\int_0^b G_1(s(t),t;\xi,0)h(\xi)d\xi + 2\int_0^t f(\tau)G_{1\xi}(s(t),t;0,\tau)d\tau \\
& + 2\int_0^t G_1(s(t),t;s(\tau),\tau)\{w(\tau)s'(\tau) + \alpha(w(\tau),v(\tau))\}d\tau \\
& - 2\int_0^t G_{1\xi}(s(t),t;s(\tau),\tau)w(\tau)d\tau,
\end{aligned}
$$

$$
\begin{aligned}
(2.4) \qquad v(t) = & 2\int_b^1 N_2(s(t),t;\xi,0)H'(\xi)d\xi \\
& - 2\gamma\int_0^t G_{2x}(s(t),t;s(\tau),\tau)v(\tau)d\tau,
\end{aligned}
$$

where

$$w(t) = u_1(s(t),t),\ v(t) = u_{2x}(s(t),t),\ s'(t) = \beta(w(t), v(t)).$$

We shall consider the system of integral equations (2.3), (2.4) in w, v which can be shown to be equivalent to the original problem (P_b) $(0 < b < 1)$. For a local existence of (2.3), (2.4) we consider the operator T of $C[0,\sigma] \times C[0,\sigma]$ into itself, defined by

$$T(w,v) = (w_1, v_1),$$

where

$$\begin{aligned} w_1(t) = {} & 2\int_0^b G_1(s(t),t;\xi,0)h(\xi)d\xi + 2\int_0^t f(\tau)G_{1\xi}(s(t),t;0,\tau)d\tau \\ & + 2\int_0^t G_1(s(t),t;s(\tau),\tau)\{w(\tau)s'(\tau) + \alpha(w(\tau),v(\tau))\}d\tau \\ & - 2\int_0^t G_{1\xi}(s(t),t;s(\tau),\tau)w(\tau)d\tau, \end{aligned}$$

$$\begin{aligned} v_1(t) = {} & 2\int_b^1 N_2(s(t),t;\xi,0)H'(\xi)d\xi - \\ & - 2\gamma\int_0^t G_{2x}(s(t),t;s(\tau),\tau)v(\tau)d\tau. \end{aligned} \tag{2.5}$$

We put

$$|z|_\sigma = \sup_{0\le t\le\sigma} |z(t)|,\ \|(z_1,z_2)\|_\sigma = |z_1|_\sigma + |z_2|_\sigma\ \forall z, z_1, z_2 \in C[0,\sigma]$$

and denote by $B_\sigma(M)$ the closed ball with center $(h(b), H'(b))$ and radius M in $C[0,\sigma] \times C[0,\sigma]$. By the conditions of the theorem,there exist constants k_0', M_1 such that

$$\begin{aligned} |\beta(z_1,z_2) - \beta(\bar z_1,\bar z_2)| &\le k_0'(|z_1 - \bar z_1)| + |z_2 - \bar z_2)|), \\ |\alpha(z_1,z_2) - \alpha(\bar z_1,\bar z_2)| &\le k_0'(|z_1 - \bar z_1)| + |z_2 - \bar z_2)|) \end{aligned} \tag{2.6}$$

for every $z_1, z_2, \bar z_1, \bar z_2$ such that

$$|z_1 - h(b)| + |z_2 - H'(b)| \le M_1,\ |\bar{z}_1 - h(b)| + |\bar{z}_2 - H'(b)| \le M_1.$$

Now, we want to prove $T(B_\sigma(M_1)) \subset B_\sigma(M_1)$ for small σ. But this follows from:

$$|w_1(t) - h(b)| + |v_1(t) - H'(b)| \le C\sigma^{1/2} + C(\sigma),\ 0 < t < \sigma, \tag{2.7}$$

where $C(\sigma) \downarrow 0$ as $\sigma \downarrow 0$ and $(w, v) \in B_\sigma(M_1)$. Hence, it is sufficient to prove (2.7).

Note that

$$s(t) = b + tk_1(b) + \int_0^t \{\beta(w(\tau), v(\tau)) - \beta(w(0), v(0))\}d\tau,\ 0 < t < \sigma, \tag{2.8}$$

where

$$k_1(b) = \beta(h(b), H'(b)) = \beta(w(0), v(0)),\ (w, v) \in B_\sigma(M_1).$$

In view of (2.6), (2.8) we have

$$|s(t) - b| \le \sigma\{|k_1(b)| + k_0' M_1\},\quad 0 < t < \sigma. \tag{2.9}$$

By (2.5) we can find a constant C such that

$$\begin{aligned} |v_1(t) - H'(b)| &\le C\sigma^{1/2} + |H'(b)| \left|1 - \int_b^1 2N_2(s(t), t; \xi, 0)d\xi\right| \\ &+4 \int_b^1 K(s(t), \gamma t; \xi, 0)\,|H'(\xi) - H'(b)|d\xi,\quad 0 < t < \sigma. \end{aligned} \tag{2.10}$$

Substituting $\zeta = (s(t) - \xi)(2t^{1/2})^{-1}$ into the integrals of the right hand side of (2.10), we get

$$\left|1 - 2\int_b^1 N_2(s(t), t; \xi, 0)d\xi\right| \le\ C_1(\sigma),\quad 0 < t < \sigma, \tag{2.11}$$

where $C_1(\sigma) \downarrow 0$ as $\sigma \downarrow 0$.

To estimate the last integral in the right hand side of (2.10) we note that if

$$(s(t)-1)(2t^{1/4})^{-1} \le \zeta \le (s(t)-b)(2t^{1/2})^{-1} \tag{2.12}$$

then

$$b \le \xi \le s(t) - (s(t)-1)t^{1/4}.$$

By (2.9) the above inequalities give

$$b \le \xi \le C_2(\sigma) + b, \quad C_2(\sigma) \downarrow 0 \quad as \quad \sigma \downarrow 0 \tag{2.13}$$

whenever (2.12) holds.

Accordingly, we have

$$\int_b^1 K(s(t),\gamma t;\xi,0)|H'(\xi)-H'(b)|d\xi \tag{2.14}$$

$$\le \Big\{ \int_{\varphi_1(t)}^{\varphi_0(t)} + \int_{\varphi_2(t)}^{\varphi_1(t)} \Big\} e^{-\zeta^2/\gamma} |H'(s(t)-2\zeta t^{1/2}) - H'(b)| d\zeta$$

$$\le C \sup_{b\le\xi\le C_2(\sigma)+b} |H'(\xi)-H'(b)| + C \sup_{b\le\xi\le 1} |H'(\xi)| \int_{\varphi_2(t)}^{\varphi_1(t)} e^{-\zeta^2/\gamma} d\zeta,$$

where

$$\varphi_0(t) = \frac{s(t)-b}{2\sqrt{t}}, \quad \varphi_1(t) = \frac{s(t)-1}{2\sqrt[4]{t}}, \quad \varphi_2(t) = \frac{s(t)-1}{2\sqrt{t}}.$$

In view of (2.13), (2.14) and since $\lim_{t\to 0} \varphi_i(t) = -\infty,\ i = 1,2$, we get

$$\int_b^1 K(s(t),\gamma t;\xi,0)|H'(\xi)-H'(b)|d\xi \le C_3(\sigma), \quad 0 < t < \sigma, \tag{2.15}$$

where $C_3(\sigma) \downarrow 0$ as $\sigma \downarrow 0$.

Combining (2.10), (2.11), (2.15) we get

$$|v_1(t) - H'(b)| \le C\sigma^{1/2} + |H'(b)|C_1(\sigma) + 4C_3(\sigma). \tag{2.16}$$

By the same estimates as for (2.16) we get

$$|w_1(t) - h'(b)| \leq C\sigma^{1/2} + C_4(\sigma), \quad 0 < t < \sigma, \tag{2.17}$$

where $C_4(\sigma) \downarrow 0$ as $\sigma \downarrow 0$.

In view of (2.16), (2.17) we obtain (2.7). Hence, for $\sigma > 0$ sufficient small we get $(w_1, v_1) \in B_\sigma(M_1)$, i.e. $T(B_\sigma(M_1)) \subset B_\sigma(M_1)$. By (2.6) and using similar estimates as in [3], it can be shown that T is a contraction on $B_\sigma(M_1)$ for $\sigma > 0$ small enough. Hence T has a unique fixed point in $B_\sigma(M_1)$ which is the solution of the system (2.3), (2.4). This completes the proof for the case $0 < b < 1$.

Consider next the case $b = 0$. We define k_2 in the same way as for k_0' in (2.6) with $h(b)$ replaced by $f(0)$. We also use the notations $M_1, B_\sigma(M)$ with similar meanings as before, with $(h(b), H'(b))$ replaced by $(f(0), H'(0))$. Letting $0 < b_1 < 1$ in the following arguments, we consider M, σ such that:

$$0 < M < \min\{M_1, k_1(4M_1k_2)^{-1}\}, \quad 0 < \sigma < 2b_1(3k_1)^{-1} \tag{2.18}$$

where $k_1 = \beta(f(0), H'(0)) > 0$.

For $(w_0, v_0) \in B_\sigma(M)$, we define $T_1(w_0, v_0) = (w_2, v_2)$. Here $w_2(t) = u_0(s(t), t)$, where $u_0(x, t)$ being the solution of the system

$$\begin{cases} s(t) = \int\limits_0^t \beta(w_0(\tau), v_0(\tau))d\tau \\ u_{0xx} - u_{0t} = 0, \quad 0 < x < s(t), \ 0 < t < \sigma \\ u_0(0, t) = f(t), \ u_{ox}(s(t), t) = \alpha(w_0(t), v_0(t)) \end{cases} \tag{2.19}$$

and

$$\begin{aligned} v_2(t) = 2 \int\limits_0^1 N_2(s(t), t; \xi, 0)H'(\xi)d\xi - \\ - 2\gamma \int\limits_0^t G_{2x}(s(t), t; s(\tau), \tau)v_0(\tau)d\tau. \end{aligned} \tag{2.20}$$

Note that

$$s(t) = tk_1 + \int\limits_0^t \{\beta(w_0(\tau), v_0(\tau)) - \beta(f(0), H'(0))\}d\tau. \tag{2.21}$$

Similarly as for (2.9), we can use (2.18),(2.21) and the Lipschitzian condition on β to get

$$0 < k_1/2 \leq s'(t) \leq 3k_1/2,$$

$$0 < tk_1/2 \leq s(t) \leq 3tk_1/2 \leq 3\sigma k_1/2 < b_1 < 1.$$

Hence (w_1, v_2) is well-defined (see Appendix 1). In the same way as for (2.16), we obtain the same estimate for $|v_2(t) - H'(0)|$. Therefore, if σ is small enough, then

$$|v_2(t) - H'(0)| \leq M/2 \quad 0 < t < \sigma. \tag{2.22}$$

To estimate $|w - f(0))|_\sigma$ we put

$$u^{\pm}(x,t) = |f - f(0)|_\sigma + x(\varepsilon + |\alpha(w_0, v_0)|) \pm (u_0(x,t) - f(0)) \tag{2.23}$$

for $\varepsilon > 0,\ 0 < x < s(t),\ 0 < t < \sigma$.

Then we have

$$u^{\pm}_{xx} - u^{\pm}_t = 0,\ u^{\pm}(0,t) \geq 0,\ u^{\pm}_x(s(t),t) > 0 \ \ (0 < x < s(t),\ 0 < t < \sigma).$$

Hence, by the maximum principle $u^{\pm}(x,t) \geq 0$ for every $0 \leq x \leq s(t),\ 0 \leq t < \sigma$. Letting $\varepsilon \downarrow 0$ in (2.23), we have therefore

$$|u_0(x,t) - f(0)| \leq |f - f(0)|_\sigma + x|\alpha(w_0, v_0)|_\sigma. \tag{2.24}$$

We note that $0 \leq x \leq s(t) \leq 3k_1\sigma/2$ for every $0 \leq t \leq \sigma$. Hence, in (2.24), for σ sufficiently small, we have

$$|u_0(x,t) - f(0)| \leq M/2,\ 0 < x < s(t),\ 0 < t < \sigma.$$

Since $w_2(t) = u_0(s(t),t)$ the above inequality gives

$$|w_2 - f(0)|_\sigma \leq M/2. \tag{2.25}$$

By (2.22), (2.25), we have

$$T_1(B_\sigma(M)) \subset B_\sigma(M).$$

Now, we prove T_1 is a contraction on $B_\sigma(M)$. For $(\bar{w}_0, \bar{v}_0)$ in $B_\sigma(M)$, we define $\bar{s}(t), \bar{u}_0(x,t), \bar{v}_2(t), \bar{w}_2(t)$ as in (2.19), (2.20). Standard estimates (see e.g. Friedman [3]) give

$$|v_2 - \bar{v}_2|_\sigma \leq C\sigma^{1/2}\|(w_0 - \bar{w}_0, v_0 - \bar{v}_0)\|_\sigma \tag{2.26}$$

for a $C > 0$ independent from σ.

In order to get the necessary estimates for $|w_2 - \bar{w}_2|_\sigma$ we put

$$U(x,t) = \bar{u}_0(x,t) - u_0(x + s(t) - \bar{s}(t), t),$$
$$\psi(t) = \max(\bar{s}(t) - s(t), 0).$$

We note that $U(\bar{s}(t),t) = \bar{w}_2(t) - w_2(t)$ and that

$$\begin{aligned} U_{xx} - U_t &= u_{0x}(x + s(t) - \bar{s}(t), t)(s'(t) - \bar{s}'(t)) \equiv F(x,t), \\ U(\psi(t),t) &= \bar{u}_0(\psi(t),t) - u_0(\psi(t) + s(t) - \bar{s}(t), t), \\ U_x(s(t),t) &= \alpha(\bar{w}_0(t), \bar{v}_0(t) - \alpha(w_0(t), v_0(t)) \end{aligned} \tag{2.27}$$

for every (x,t) in the set

$$D_\sigma = \{(x,t) : \ \psi(t) < x < \bar{s}(t), \ 0 < t < \sigma \ \}.$$

By Appendix 1 we have

$$\begin{cases} |u_{0x}(x,t)| \le C & 0 \le x \le s(t), \ 0 \le t \le \sigma \\ |\bar{u}_{0x}(x,t)| \le C & 0 \le x \le \bar{s}(t), \ 0 \le t \le \sigma, \end{cases} \tag{2.28}$$

where C is a generic constant independent from σ.

Hence, by (2.6) (with k_0' replaced by k_2) and (2.28) we have

$$|F(x,t)| \le C|s'(t) - \bar{s}'(t)| \le Ck_2 \|(w_0 - \bar{w}_0, v_0 - \bar{v}_0)\|_\sigma \ \ \forall (x,t) \in D_\sigma, \tag{2.29}$$

$$\begin{aligned} |\bar{u}_0(x,t) - u_0(y,t)| &= \Big| \int_0^x \bar{u}_{0x}(\xi,t) d\xi - \int_0^y u_{0y}(\xi,t) d\xi \Big| \\ &\le C(|x| + |y|), \quad 0 < x < \bar{s}(t), \ 0 < y < s(t). \end{aligned} \tag{2.30}$$

From (2.27), (2.30) we get

$$|U(\psi(t),t)| \le C|s(t) - \bar{s}(t)| \le \frac{3}{2} Ctk_2 \|(w_0 - \bar{w}_0, v_0 - \bar{v}_0)\|_\sigma. \tag{2.31}$$

On the other hand, from (2.6), (2.19) we get

$$|U_x(\bar{s}(t),t)| \le k_2 \|(w_0 - \bar{w}_0, v_0 - \bar{v}_0)\|_\sigma. \tag{2.32}$$

Consider the function

$$U^{\pm}(x,t) = e^d(1 - e^{-x}) \sup_{(x,t) \in D_\sigma} |F(x,t)| + \sup_{0 \le t \le \sigma} |U(\psi(t),t)|$$

$$+x(\varepsilon + \sup_{0\le t\le\sigma} |U_x(\bar{s}(t),t)|) \pm U(x,t), \tag{2.33}$$

where $\varepsilon > 0,\ d = \frac{3}{2}\sigma k_1 \ge \bar{s}(t) \ge x \ge 0,\ 0 \le t \le \sigma$.

We have

$$U^{\pm}_{xx} - U^{\pm}_t \le 0,\ U^{\pm}(\psi(t),t) \ge 0,\ U^{\pm}_x(\bar{s}(t),t) > 0, \quad \forall (x,t) \in D_\sigma. \tag{2.34}$$

By (2.34) the maximum principle gives

$$U^{\pm}(\bar{s}(t),t) \ge 0 \quad \forall t \in [0,\sigma].$$

Letting $\varepsilon \downarrow 0$ in (2.33) gives

$$\begin{aligned} |U(\bar{s}(t),t)| \le C\bar{s}(t)\Big\{ \sup_{(x,t)\in D_\sigma} |F(x,t)| + \sup_{0\le t\le\sigma} |U_x(\bar{s}(t),t)|\Big\} \\ + \sup_{0\le t\le\sigma} |U(\psi(t),t)|. \end{aligned} \tag{2.35}$$

In view of (2.29), (2.31), (2.30), (2.35) we get

$$|(w_2 - \bar{w}_2)(t)| \le C\sigma \|(w_0 - \bar{w}_0, v_0 - \bar{v}_0)\|_\sigma \quad 0 \le t \le \sigma. \tag{2.36}$$

By (2.26), (2.36) T_1 is a contraction on $B_\sigma(M)$ provided

$$0 < \sigma < \min\left\{\frac{1}{4C}, \frac{1}{4C^2}, \frac{2b_1}{3k_1}\right\}.$$

Hence, T_1 has a unique fixed point in $B_\sigma(M)$. This completes the proof of Theorem 1.

3. Proof of theorem 2

We first consider the case where condition (E) is satisfied. We define

$$\begin{cases} A = \{T > 0 :\ (P_b) \text{ has a unique solution on } (0,\sigma),\ \forall \sigma \in (o,T) \\ T^* = \sup A. \end{cases} \tag{3.1}$$

If $T^* = \infty$, then (P_b) has a unique global solution. If $T^* < \infty$ then we have to prove that $\lim_{t\uparrow T^*} s(t) = 1$. Suppose for the contrary that

$$\lim_{t\uparrow T^*} s(t) = b^* < 1.$$

From assumption (C) we get $s'(t) = \beta_0(u_1(s(t),t) \geq 0$. Hence

$$0 < b \leq s(t) \leq b^* < 1 \quad (0 \leq t < T^*). \tag{3.2}$$

Let σ be any number such that $0 < \sigma < T^*$. Consider $u_2(x,t)$ in the region

$$D_{2\sigma} = \{(x,t) : s(t) < x < 1, \quad 0 < t < \sigma\}.$$

Since

$$u_2(x,0) = H(x) \leq 0, \quad u_{2x}(1,t) = 0,$$

the strong maximum principle for parabolic equations (see [2]) implies that u_2 attains a maximum $u_2(s(t),t) = 0$ on the curve $x = s(t)$, $0 \leq t \leq \sigma$. Hence we have

$$v(t) = u_{2x}(s(t),t) \leq 0 \quad \forall t \in [0,\sigma] \subset [0,T^*).$$

Since σ is any number in $(0,T^*)$ we get

$$v(t) \leq 0 \quad 0 \leq t < T^*. \tag{3.3}$$

Now we put

$$\begin{aligned}
&k = \max\{\sup_{t\geq 0} f(t), \sup_{0\leq x\leq b} h(x)\},\\
&z(x,t) = u_1(x,t) - k,\\
&C_1 = \{(x,0) : \ 0 \leq x \leq b\},\\
&C_2 = \{(0,t) : \ 0 \leq t \leq \sigma\},\\
&C_3 = \{(s(t),t) : \ 0 \leq t \leq \sigma\}.
\end{aligned}$$

Then

$$z(x,t) \leq 0 \quad \text{on} \quad C_1 \cup C_2. \tag{3.4}$$

From assumption (D) of the theorem we get

$$z_x(s(t),t) = \delta u_{2x}(s(t),t) + g(k + z(s(t),t))s'(t). \tag{3.5}$$

We note that $z_{xx} - z_t = 0$ on

$$D_{1\sigma} = \{(x,t) : 0 < x < s(t),\ 0 < t < \sigma\}.$$

By the maximum principle, z attains a maximum on $C_1 \cup C_2 \cup C_3$. If a maximum is attained on $C_1 \cup C_2$ then by (3.4) we have $z(x,t) \le 0$ for every $(x,t) \in D_{1\sigma}$. If z attains a maximum on C_3 at $(s(t_0), t_0)$ $(0 < t_0 \le \sigma)$, say, then we will have

$$z_x(s(t_0), t_0) \ge 0. \tag{3.6}$$

From (3.3)-(3.6) we get $g(k + z(s(t_0), t_0))s'(t_0) \ge 0$. Hence, by assumptions (C), (D) of the theorem we have

$$\text{either} \ \ z(s(t_0), t_0) = 0 \text{ or } \ z(s(t_0), t_0) + k \le \alpha_0$$

and thus we conclude that

$$z(x,t) \le \max\{\alpha_0, 0\} = \alpha_0^+ \ \ \forall (x,t) \in \bar{D}_{1\sigma}.$$

In either case

$$z(x,t) \le \alpha_0^+ \quad \forall (x,t) \in D_{1\sigma}$$

and we have $z(s(t),t) \le \alpha_0^+$. Since σ is any number in $(0, T^*)$, it follows that:

$$z(s(t),t) \le \alpha_0^+ \ \ 0 \le t < T^*. \tag{3.7}$$

By (2.2), (1.6) we get for $x \downarrow s(t)$

$$\gamma \int_0^t G_2(s(t), t; s(\tau), \tau) v(\tau) d\tau = \int_b^1 G_2(s(t), t; \xi, 0) H(\xi) d\xi. \tag{3.8}$$

Since $G_2(s(t), t; s(\tau), \tau) > 0,\ v(\tau) \le 0$, using (3.8) we have

$$\gamma \int_0^t G_2(s(t), t; s(\tau), \tau) |v(\tau)| d\tau \le \sup_{b \le x \le 1} |H(x)|. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} K(-s(t), t; s(\tau), \tau) &\le K(s(t), t; s(\tau), \tau) \\ &\le \gamma^{1/2} G_2(s(t), t; s(\tau), \tau) \end{aligned} \tag{3.10}$$

the latter inequality following from $\gamma \geq 1$.

In view of (3.9), (3.10) we have

$$\int_0^t G_1(s(t),t;s(\tau),\tau)|v(\tau)|d\tau \leq \gamma^{-1/2} \sup_{b\leq x\leq 1} |H(x)|. \tag{3.11}$$

Using Green's function as in the beginning of Sec. 2 we have

$$\begin{aligned} z(s(t),t) - \alpha_0^+ &= 2\int_0^b (h(\xi) - k - \alpha_0^+)G_1(s(t),t;\xi,0)d\xi \\ &+ 2\int_0^t G_1(s(t),t;s(\tau),\tau)\delta v(\tau)d\tau \\ &+ 2\int_0^t G_1(s(t),t;s(\tau),\tau)s'(\tau)(w(\tau) + g(w(\tau)) - k - \alpha_0^+)d\tau \\ &+ 2\int_0^t G_{1\xi}(s(t),t;0,\tau)(f(\tau) - k - \alpha_0^+)d\tau \\ &- 2\int_0^t G_{1\xi}(s(t),t;s(\tau),\tau)(z(s(\tau),\tau) - \alpha_0^+)d\tau. \end{aligned} \tag{3.12}$$

Recall that $w(t) = u_1(s(t),t),\ v(t) = u_{2x}(s(t),t)$.

Noting that $G_{1\xi}(s(t),t;s(\tau),\tau) \geq 0$, we have by (3.7)

$$\int_0^t (z(s(\tau),\tau) - \alpha_0^+)G_{1\xi}(s(t),t;s(\tau),\tau)d\tau \leq 0. \tag{3.13}$$

In view of (3.7), (3.11)-(3.13) we deduce that

$$
\begin{aligned}
&\alpha_0^+ \geq z(s(t),t) \geq \alpha_0^+ - 2 \sup_{0\leq\xi\leq b} |h(\xi) - k - \alpha_0^+| \\
&\qquad - 2\sup_{t\geq 0} |f(t) - k - \alpha_0^+| - 2\delta\gamma^{-1/2} \sup_{b\leq x\leq 1} |H(x)| \\
&\qquad + 2\int_0^1 G_1(s(t),t;s(\tau),\tau)s'(\tau)(w(\tau)+g(w(\tau)) - k - \alpha_0^+)d\tau.
\end{aligned}
\tag{3.14}
$$

From (3.14) and (E) we get

$$
|z(s(t),t)| \leq M_2 + M_2 \int_0^t G_1(s(t),t;s(\tau),\tau)(1+|z(s(\tau),\tau)|)d\tau
$$

for an $M_2 > 0$. By Gronwall's inequality the above inequality gives

$$
|z(s(t),t)| \leq M_3 \quad \text{for a } M_3 > 0,\ 0 \leq t < T^*. \tag{3.15}
$$

From (2.4), (3.15) we deduce

$$
|u_{2x}(s(t),t)| = |v(t)| \leq M_4 \quad \text{for a } M_4 > 0. \tag{3.16}
$$

In view of (3.15), (3.16) the maximum principle gives

$$
\begin{aligned}
&|u_1(x,t)| \leq M_4 + k && 0 \leq x \leq s(t),\ 0 \leq t < T^*, \\
&|u_2(x,t)| \leq \max\{M_4, \sup_{b\leq x\leq 1} H'(x)\}, && s(t) \leq x \leq 1,\ 0 \leq t < T^*.
\end{aligned}
\tag{3.17}
$$

By (3.12), (3.17), and the local existence theorem we can extend the solution (u_1,u_2,s) of (P_b) on the interval $[0,T^*+\delta_1]$ with $\delta_1 > 0$ sufficiently small. Hence $T^*+\delta \in A$, $T^*+\delta_1 \leq \sup A = T^*$. This contradiction completes the proof of the first part.

Now, we consider the case assumptions (E_i) hold, $i = 1,2$. In this case, if we put

$$
\begin{cases}
A_M = \{T > 0 : \ (P_b) \text{ has a unique solution on } (0,\sigma) \\
\qquad\qquad \text{such that } \ \|(w,v)\|_\sigma < M \ \ \forall\sigma \in (0,T)\} \\
T_1^* = \sup A_M
\end{cases}
\tag{3.18}
$$

then (3.3), (3.7), (3.9) also hold for every $t \in [0, T^*)$. If $T_1^* = \infty$, then (P_b) has a unique global solution. If $T^* < \infty$ we have to prove that $\lim_{t \uparrow T_1^*} s(t) = 1$. As for the first part, we suppose by contradiction that $\lim_{t \uparrow T_1^*} s(t) = b_1^* < 1$. Then

$$0 < b \le s(t) \le b_1^* < 1 \qquad \forall t \in [0, T_1^*). \tag{3.19}$$

In view of (2.4), (3.3) we have

$$\begin{aligned} 0 \ge v(t) \ge 2 \int_b^1 H'(\xi) N_2(s(t), t; \xi, 0) d\xi \\ - 2\gamma \int_0^t v(\tau) K_x(s(t), \gamma t; s(\tau), \gamma\tau) d\tau, \end{aligned} \tag{3.20}$$

where K was defined at the beginning of the Section 2 and the inequality $K_x(2 - s(t), \gamma t;\, s(\tau), \gamma\tau) \ge 0$ has been used. Noting that $s'(t) = \beta_0(w(t))$ we have by (3.18)

$$|K_x(s(t), \gamma t; s(\tau), \gamma\tau)| \le (2\gamma)^{-1} G_2(s(t), t; s(\tau), \tau) \sup_{|u| \le M} \beta_0(u) \tag{3.21}$$

for every t, τ such that $0 < \tau < t < T_1^*$.

By (3.9), (3.21) the inequalities in (3.20) give

$$|v(t)| \le (4 + \gamma^{-1} \sup_{|u| \le M} \beta_0(u)) \sup_{b \le x \le 1} \{|H(x)| + |H'(x)|\}, \quad 0 < t < T_1^*. \tag{3.22}$$

Consider $u_1(x,t)$ on $D_{1\sigma}$ (as defined in the first part of the proof with σ being any number in $(0, T_1^*)$). If $u_1(x,t)$ has a minimum on $C_1 \cup C_2$ (see Fig. 1).

Then by assumptions (A), (B) and (3.7) we have

$$0 \le u_1(x,t) \le k + \alpha_0^+ \qquad \forall (x,t) \in D_{1\sigma}. \tag{3.23}$$

If $u_1(x,t)$ has a minimum on C_3 at $(s(t_1), t_1)$, $0 < t_1 \le \sigma$, say, then $u_{1x}(s(t_1), t_1) \le 0$. Hence, by (3.3), (D) and (1.6) we get

$$\delta v(t_1) \le -g(w(t_1)) s'(t_1). \tag{3.24}$$

Thus

$$g(w(t_1)) < 0$$

or

$$|w(t_1)| < \alpha_1$$

Fig. 1

or

$$\delta|v(t_1)| \geq |g(w(t_1))|s'(t_1) \geq k_0|w(t_1)|^p.$$

In any case, by the assumptions on the function g we have in view of (3.22), (3.7)

$$|w(t_1)| \leq \max\{k + \alpha_0^+, |\bar{\alpha}_0|, \alpha_1, M_5\} \tag{3.25}$$

with

$$M_5 = \{\delta k_0^{-1}(4 + \gamma^{-1} \sup_{|u|\leq M} \beta_0(u)) \sup_{b\leq x\leq 1} (|H(x)| + |H'(x)|)\}^{1/p}.$$

From (3.7), (3.23), (3.25) we get

$$|w(t)| \leq \max\{k + \alpha_0^+, |\bar{\alpha}_0|, \alpha_1, M_5\}, \quad 0 \leq t < \sigma < T_1^*. \tag{3.26}$$

Since σ is any number in $(0, T_1^*)$, (3.26) holds for every t in $(0, T_1^*)$. Now, from assumption (E_2) we get in view of (3.26)

$$|w(t)| \leq M/4 \qquad \forall t \in [0, T_1^*) \tag{3.27}$$

and in view of (3.22),

$$|v(t)| \leq M/4 \qquad \forall t \in [0, T_1^*). \tag{3.28}$$

By virtue of (3.27), (3.28) we then have

$$|w(t)| + |v(t)| \leq M/2 < M \qquad \forall t \in [0, T_1^*). \tag{3.29}$$

Now, just as for the first part of the proof, we can extend the solution (u_1, u_2, s) on the interval $[0, T_1^* + \delta_2)$ with $\delta_2 > 0$ sufficiently small, which is a contradiction. This completes the proof of the second part.

To get the last result of Theorem 2 we note that

$$u_1(s(t), t) = f(t) + \int_0^{s(t)} u_{1x}(\xi, t) d\xi \qquad 0 \le t < T_1^*. \tag{3.30}$$

By the estimates derived in Appendix 1 we have

$$|u_{1x}(x, t)| \le \sup_{t \ge 0} |f'(t)| + \sup_{0 \le t \le \sigma} |\alpha(w(t), v(t))| + \sup_{0 \le x \le b} |h'(x)| \tag{3.31}$$

for every $(x, t) \in \bar{D}_{1\sigma}$ $(0 < \sigma < T_1^*)$.

In view of (3.29), (3.30), (3.31) the assumption (E_3) gives

$$|u_1(s(t), t)| \ge M/8 - Ms(t)/16 \ge M/16 \qquad 0 \le t < \sigma.$$

Since σ is any number in $(0, T_1^*)$, the above inequalities hold for every $t \in [0, T_1^*)$. This gives

$$1 \ge s(t) = b + \int_0^t \beta_0(u_1(s(\tau), \tau)) d\tau \ge \min_{M \ge |u| \ge M/16} \beta_0(u) t + b$$

for every $t \in [0, T_1^*)$. Hence we have

$$T_1^* \le (1 - b) \{ \min_{M \ge |u| \ge M/16} \beta_0(u) \}^{-1}.$$

This completes the proof of the Theorem 2.

Appendix 1

The purpose of this appendix is to establish two propositions necessary for the proof of Theorem 1.

Consider the system

(A.1)
$$\begin{cases} u_{xx} - u_1 = 0 & (x, t) \in D = \{(x, t) : \ 0 < x < \rho(t), \ 0 < t < T\} \\ u(x, 0) = h(x) & 0 \le x \le \rho(0) = b \\ u(0, t) = f_1(t), \quad u_x(\rho(t), t) = f_2(t), & 0 < t < T. \end{cases}$$

Proposition A. *Let $\theta, T > 0$, $\rho, f_1 \in C^1[0,T]$, $f_2 \in C[0,T]$, $h \in C^1[0,b]$, $h(0) = f_1(0)$, $h'(b) = f_2(0)$ and $0 < \theta < \rho(t)$ for $0 \le t < T$. Then* (A.1) *has a unique solution $u(x,t)$ on D such that $u, u_x \in C(\bar{D})$, $u_t, u_{xx} \in C(D)$. Moreover*

$$\begin{cases} |u(x,t)| & \le |f_1|_T + x|f_2|_T + |h|_b \\ |u_x(x,t)| & \le (d-x)|f_1'|_T + |f_2|_T + |h'|_b \quad \forall (x,t) \in D, \end{cases} \tag{A.2}$$

where $|f|_\sigma = \sup\limits_{0 \le t \le \sigma} |f(t)|$ for $f \in C[0,\sigma]$ and $d = |\rho|_T$.

Proof. We first study the case $f_2 \in C^1[0,T]$, $h \in C^2[0,T]$. Consider the system

$$\begin{cases} \tilde{v}_{xx} - \tilde{v}_t = 0 \quad \forall (x,t) \in D \\ \tilde{v}(x,0) = h''(x) \quad 0 \le x \le b \\ \tilde{v}(0,t) = f_1'(t),\ \tilde{v}_x(\rho(t),t) = f_2'(t) - \rho'(t)\tilde{v}(\rho(t),t). \end{cases} \tag{A.3}$$

In the same way as for (2.1), (2.3) we can transform (A.3) into the following equations

$$\begin{aligned} \tilde{v}(x,t) = & \int_0^b G_1(x,t;\xi,0)h''(\xi)d\xi + \int_0^t f'(t)G_{1\xi}(x,t;0,\tau)d\tau \\ & + \int_0^t G_1(x,t;\rho(\tau),\tau)\left\{\tilde{w}(\tau)\rho'(\tau) + f_2'(\tau) - \tilde{w}(\tau)\right\}d\tau \\ & - \int_0^t G_{1\xi}(x,t;\rho(\tau),\tau)\tilde{w}(\tau)d\tau, \end{aligned} \tag{A.4}$$

$$\begin{aligned} \tilde{w}(t) = 2 & \int_0^b G_1(\rho(t),t;\xi,0)h''(\xi)d\xi + 2\int_0^t f'(t)G_{1\xi}(\rho(t),t;0,\tau)d\tau + \\ & +2\int_0^t G_1(\rho(t),t;\rho(\tau),\tau)\left\{\tilde{w}(\tau)\rho'(\tau) + f_2'(\tau) - \tilde{w}(\tau)\right\}d\tau \\ & -2\int_0^t G_{1\xi}(\rho(t),t;\rho(\tau),\tau)\tilde{w}(\tau)d\tau, \end{aligned} \tag{A.5}$$

where $\tilde{w}(t) = \tilde{v}(\rho(t), t)$ and G_1 is as in (2.1).

(A.5) is a linear Volterra integral equation of second kind. Hence there is a unique $\tilde{w} \in C[0,T]$ satisfying (A.5). From (A.4) we get the solution $\tilde{v} \in C_*(\bar{D} \setminus \{(0,0),(b,0)\})$ (the space of all bounded continuous functions on $\bar{D} \setminus \{(0,0),(b,0)\}$). Putting

$$u(x,t) = h(x) + \int_0^t \tilde{v}(x,\tau)d\tau$$

we get the solution of (A.1). Integrating (A.4) with respect to t, we get an integral equation in u (of the same form as (2.1)). By differentiating with respect to x the equation thus obtained we can prove that $u_x \in C(\bar{D})$. To prove (A.2) we use the same argument as for (2.35) with the functions

$$\begin{aligned} w_1^{\pm}(x,t) &= |f_1|_T + x(\varepsilon_1 + |f_2|_T) + |h|_b \pm u(x,t), \\ w_2^{\pm}(x,t) &= (d-x)(\varepsilon_1 + |f_1'|_T) + |h'|_b + |f_2|_T \pm u_x(x,t) \end{aligned}$$

to get $w_1^{\pm} \geq 0$ for every $(x,t) \in \bar{D}$. Letting $\varepsilon_1 \downarrow 0$ in the inequalities thus obtained we get

$$\text{(A.6)} \qquad \begin{cases} |f_1|_T + x|f_2|_T + |h|_b \pm u(x,t) \geq 0 \\ (d-x)|f_1'|_T + |f_2|_T + |h'|_b \pm u_x(x,t) \geq 0. \end{cases}$$

This completes the proof in the case $f_2 \in C^1[0,T], h \in C^2[0,b]$.

Now, for $f_2 \in C[0,T], h \in C^1[0,b]$, we choose sequences $\{f_{2n}\} \subset C[0,T]$, $\{h_n\} \subset C^2[0,b]$, $h_n(0) = f_1(0)$, $h_n'(b) = f_{2n}(0)$ such that

$$\text{(A.7)} \qquad \begin{cases} f_{2n} \to f_2 & \text{in } C[0,T] \\ h_n \to h & \text{in } C^1[0,b]. \end{cases}$$

By the first part of the proof the system

$$\begin{cases} u_{nxx} - u_{nt} = 0 & (x,t) \in D \\ u_n(x,0) = h_n(x) & 0 \leq x \leq b \\ u_n(0,t) = f_1(t), \quad u_{nx}(\rho(t),t) = f_{2n}(t) \end{cases}$$

has a unique solution u_n satisfying $u_n, u_{nx} \in C(\bar{D})$. Using the same argument as for (A.6), we have

$$\begin{aligned} |(u_n - u_m)(x,t)| &\leq x|f_{2n} - f_{2m}|_T + |h_n - h_m|_b, \\ |(u_{nx} - u_{mx})(x,t)| &\leq |f_{2n} - f_{2m}|_T + |h_n' - h_m'|_b. \end{aligned}$$

In view of (2.7), the above inequalities show that $\{u_n\}$, $\{u_{nx}\}$ are Cauchy sequences in $C(\bar{D})$. Hence there is u_0 with $u_0, u_{0x} \in C(\bar{D})$ and $u_n \to u_0$, $u_{nx} \to u_{0x}$ in $C(\bar{D})$. It can be shown that u_0 satisfies (A.1). This completes the proof of the proposition.

Now, we consider the system

$$\text{(A.8)} \qquad \begin{cases} u_{xx} - u_t = 0 & \forall (x,t) \in D_1 \\ u(0,t) = f_1(t), & u_x(\rho_1(t), t) = f_2(t) \quad \forall t \in (0,T], \end{cases}$$

where $\rho_1(0) = 0$ and

$$D_1 = \{(x,t) :\ 0 < x < \rho_1(t),\ 0 < t < T\}.$$

Proposition B. *Let* $\theta_1, T > 0,\ f_1 \in C^1[0,T],\ f_2 \in C[0,T],\ \rho_1 \in C^1[0,T]$, *and* $\rho_1(0) = 0, \rho_1'(t) \geq \theta_1 > 0, \forall t \in [0,T]$. *Then* (A.8) *has a unique solution* $u(x,t)$ *on* D_1 *such that* $u, u_x \in C(\bar{D}_1), u_{xx}, u_t \in C(D_1)$. *Moreover,*

$$\text{(A.9)} \qquad \begin{cases} |u(x,t)| \leq |f_1|_T + x|f_2|_T \\ |u_x(x,t)| \leq (d_1 - x)|f_1'|_T + |f_2|_T \end{cases}$$

with $(x,t) \in D_1, d_1 = |\rho_1|_T$. *Here the notation* $|f|_\sigma$ *is as in proposition A.*

Proof. We first study the case $f_2 \in C^1[0,T]$. Put

$$\rho_\varepsilon(t) = \rho_1(t) + \varepsilon \quad (\varepsilon > 0),$$
$$H_\varepsilon(x) = ax^3/6 + bx^2/2 + cx + d$$

with

$$a = (1 + \rho_1'(0)\varepsilon)^{-1}(f_2'(0) - \rho_1'(0) f_1'(0)), \quad b = f_1'(0),$$
$$c = f_2(0) - a\varepsilon^2/2 - b\varepsilon, \quad d = f_1(0).$$

Consider the system

(A.10)
$$\begin{cases} u^\varepsilon_{xx} - u^\varepsilon_t = 0 \quad 0 < x < \rho_\varepsilon(t), \quad 0 < t < T \\ u^\varepsilon(0,t) = f_1(t),\ u^\varepsilon_x(\rho_\varepsilon(t), t) = f_2(t),\ u^\varepsilon(x,0) = H_\varepsilon(x)\ (0 \leq x \leq \varepsilon. \end{cases}$$

We have the compatibility conditions

$$H_\varepsilon(0) = f_1(0), \quad H_\varepsilon'(\varepsilon) = f_2(0), \quad H_\varepsilon''(0) = f_1'(0),$$
$$H_\varepsilon'''(\varepsilon) + \rho_1'(0) H''(\varepsilon) = f_2'(0).$$

Hence, by using the method of Green's function (see e.g. the proof of Proposition A) we can show that (A.10) has a unique solution u^ε on D_ε, where

$$D_\varepsilon = \{(x,t): \; 0 < x < \rho_\varepsilon(t), \; 0 < t < T\},$$

such that $u^\varepsilon, u^\varepsilon_x, u^\varepsilon_{xx} \in C(\bar{D}_\varepsilon)$ and

$$\text{(A.11)} \qquad \begin{cases} |u^\varepsilon(x,t)| & \le |f_1|_T + x|f_2|_T + |H_\varepsilon|_\varepsilon \\ |u^\varepsilon_x(x,t)| & \le (d_1 + \varepsilon - x)\,|f_1'|_T + |f_2|_T + |H_\varepsilon'|_\varepsilon . \end{cases}$$

By the integral equation for $u^\varepsilon_t(\rho_\varepsilon(t),t)$ (see (A.5)) we can use Gronwall's inequality to prove that there exists a $C > 0$ such that

$$(A.12) \qquad |u^\varepsilon_t(\rho_\varepsilon(t),t)| \le C \quad \forall t \in [0,T],$$

where C is a generic constant independent from ε.

In view of (A.12) we can use the maximum principle to get

$$(A.13) \qquad |u^\varepsilon_t(x,t)| \le \max\{C, |H_\varepsilon''|_\varepsilon, |f_1'|_T\} \;\; \forall (x,t) \in D_\varepsilon.$$

Now let $\bar\varepsilon > \varepsilon > 0$. In the same way as for (A.6) we have

$$\text{(A.14)} \qquad |u^\varepsilon(x,t) - u^{\bar\varepsilon}(x + \bar\varepsilon - \varepsilon, t)| \le \sup_{0 \le t \le T} |f_1(t) - u^{\bar\varepsilon}(\bar\varepsilon - \varepsilon, t)|$$

$$+ \sup_{0 \le x \le \varepsilon} |H_\varepsilon(x) - H_{\bar\varepsilon}(x + \bar\varepsilon - \varepsilon)| \;\; \forall (x,t) \in D_\varepsilon.$$

Since $f_1(t) = u^{\bar\varepsilon}(0,t)$ we have

$$(A.15) \qquad |f_1(t) - u^{\bar\varepsilon}(\bar\varepsilon - \varepsilon, t)| \le |\bar\varepsilon - \varepsilon| \sup_{0 \le x \le \rho_1(t)} |u^{\bar\varepsilon}_x(x,t)|.$$

From (A.11), (A.15), (A.14) we get

$$(A.16) \qquad |I(x,t)| \le C|\bar\varepsilon - \varepsilon| \quad \forall (x,t) \in D_\varepsilon,$$

where

$$I(x,t) = u^\varepsilon(x,t) - u^{\bar\varepsilon}(x + \bar\varepsilon - \varepsilon, t) \; .$$

Noting that $I_x(\rho^\varepsilon(t),t) = 0$ we have

$$I_x^2(x,t) = -2 \int\limits_x^{\rho_\varepsilon(t)} I_x(\xi,t) I_{xx}(\xi,t) d\xi.$$

Hence

$$(A.17)\qquad I_x^4(x,t) \le 4\Big(\int\limits_x^{\rho_\varepsilon(t)} I_x^2(\xi,t)d\xi\Big)\Big(\int\limits_x^{\rho_\varepsilon(t)} I_{xx}^2(\xi,t)d\xi\Big).$$

Similarly, we have

$$(A.18)\qquad \int\limits_x^{\rho_\varepsilon(t)} I_x^2(\xi,t)d\xi \le |I(0,t)I_x(0,t)| + \int\limits_x^{\rho_\varepsilon(t)} |I(\xi,t)I_{xx}(\xi,t)|d\xi.$$

In view of (A.13), (A16)-(A.18) one has

$$(A.19)\qquad I_x^4(x,t) \le C|\varepsilon - \bar{\varepsilon}| \quad \forall (x,t) \in D_\varepsilon.$$

From (A.16), (A.19), we infer that $\{u^\varepsilon\}$, $\{u_x^\varepsilon\}$ are Cauchy sequences in $C(\bar{D}_1)$. Letting $\varepsilon \downarrow 0$ we get the solution of (A.8). From (A.11) we get (A.9) as $\varepsilon \downarrow 0$. This completes the proof in the case $f_2 \in C^1[0,T]$.

Now, if $f_2 \in C[0,T]$, we choose a sequence $\{g_n\}$ in $C^1[0,T]$ such that $g_n \to f_2$ strong in $C[0,T]$. By the first part, the system of

$$u_{xx}^n - u_t^n = 0,\ u^n(0,t) = f_1(t),\ u_x^n(\rho_1(t),t) = g_n(t), \quad \forall (x,t) \in D_1$$

has a unique solution u^n satisfying $u^n, u_x^n \in C(\bar{D}_1)$. In the same way as for (A.11) we have for every $m,n \in N$

$$|(u^n - u^m)(x,t)| \leq x|g_n - g_m|_T,$$
$$|(u^n_x - u^m_x)(x,t)| \leq |g_n - g_m|_T \quad \forall (x,t) \in \bar{D}_1.$$

The above inequalities show that $\{u^n\}$, $\{u^n_x\}$ are Cauchy sequence in $C(\bar{D}_1)$. Using the same argument as for the proof of Proposition A, we complete the proof of Proposition B.

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