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SOME RESULTS ON REDUCTION PRINCIPLES, BIFURCATION AND HOPF BIFURCATION OF EQUATIONS CONCERNING LIPSCHITZ CONTINUOUS MAPPINGS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. Some reduction principles of equations independing or depending on a parameter and concerning Lipschitz continuous mappings are introduced and then are applied to consider the existence of nontrivial solutions and nontrivial periodic solutions with a small norm and the existence of bifurcation and Hopf bifurcation points of equations concerning Lipschitz continuous mappings in Banach spaces, investigating the definition and nonvanishing of the topological degree of mappings or the existence of regular nonzero solutions of algebraic equations in a finite-dimensional space. Some well-known results of other authors are generalized.

Introduction

Throughout of this paper, by X, Y we denote real or complex Banach spaces with the dual X^* and Y^*, respectively. Without misunderstanding, the same symbols $||\cdot||$, $\langle .,. \rangle$ stand for the norm in X, Y, X^*, Y^* and the paring between elements of X, X^* and of Y, Y^*, respectively. They should be understood in the concrete context. Let $\mathbf{R}^n$ stand for the n-dimensional Euclidean space. It is customary to simplify the notation for $\mathbf{R}^1$ by dropping the superscript, $\mathbf{R}^1 = \mathbf{R}$. We also use the same symbol $|\cdot|$ to indicate the norm of $\mathbf{R}^n$ for all $n = 1, 2, \dots$. Let D be an open bounded subset in X with the closure $\overline{D}$. It is well-known that there are many problems in physics, biochemistry, mechanics and specially, in applied mathematics which can be formulated as operator equations of the form

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$$F(x) = 0, \quad x \in \overline{D}, \tag{1}$$

where F is, in general, a nonlinear mapping from $\overline{D}$ into Y. If we can describe the solution set of this equation in some neighborhood of a known solution we can solve our problems. We assume that by some manner we already know a solution x_0 of (1) i.e. $F(x_0) = 0$. Without loss of generality we take $x_0 = 0$ and hence the set D is supposed to be an open bounded neighborhood of the origin in X. In the case when the mapping F is continuously Fréchet (or Hadamar) differentiable at $x_0 = 0$ and its derivative $F_x(0)$ is a one-to-one mapping from X onto Y, using the Implicit Function Theorem, one can show that $x_0 = 0$ is an isolated solution of this equation. In the case the derivative $F_x(0)$ is not one-to-one or, in general, the mapping F is not differentiable, some methods, for instance, Lyapunov-Schmidt procedure (see, for example, [16]), Center manifold method (see, for example, [3], [12], [15]), alternative method (see, for example, [4]) etc. are used to describe the solution set of the equation (1) in a neighborhood of a given solution. The main idea of these methods is as follows: If X and Y are finite dimensional spaces, one reduces the above equation to two equations on lower dimensional spaces which are simpler to be solved and better geometric insight. If the spaces X and Y are infinite dimensional Banach spaces, one reduces this equation to two equations, one equation is in an infinite space which can be easily solved by some well-known methods as the Implicit Function Theorem, the Banach Contraction Principle, the topological methods etc., the other equation is in a finite dimensional space. The reduction also forms a qualitative simplification. It then follows that if we have described the solution set of the reduced equations, then we can also describe the solution set of the original equation. These methods have been studied in many different settings and by many authors.

The purpose of this paper is to describe some reduction principles for equations independing or depending on a parameter and concerning Lipschitz continuous mappings and to use these principles to consider the existence of nontrivial solutions of stationary equations, nontrivial periodic solutions of dynamic systems, the existence of bifurcation and Hopf bifurcation points of equation depending on a parameter. The plan of the paper is as follows. In Section 1 we introduce the reduction principles to describe the solution set of the equation

$$T(u) + H(u) + K(u) = 0, \quad u \in \overline{D},$$

in a neighborhood of the solution $x_0 = 0$, the periodic solution set of the dynamic system

$$\dot{u} + T(u) + H(u) + K(u) = 0, \quad u \in \overline{D},$$

with T being a linear continuous mapping and H, K being Lipschitz continuous mappings. Further, we consider the cases of equations depending on a parameter. First, we investigate equation of the form

$$T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times \overline{D},$$

where Λ is an open subset of a normed space, for any $\lambda \in \Lambda, T, L(\lambda, .)$ are linear continuous mappings and $H(\lambda, .), K(\lambda, .)$ are Lipschitz continuous mappings with $H(\lambda, 0) = K(\lambda, 0) = 0$.

This problem leads to bifurcation problems. We describe the solution set of this equation in a neighborhood $(\overline{\lambda}, 0)$, where $\overline{\lambda} \in \Lambda$ satisfies some sufficient conditions to be imposed later. In the end of this section we introduce the reduction principle for dynamic systems depending on a parameter of the form

$$\dot{u} + T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times \overline{D},$$

with T, L, H, K as above. We want to describe the periodic solution set of this dynamic system in a neighborhood of a given solution $(\overline{\lambda}, 0)$ with $\overline{\lambda}$ satisfying some sufficient conditions below. This problem leads to Hopf bifurcation problems.

Section 2 is devoted to the existence of nontrivial solutions of equation independing on a parameter. In this section we assume that the linear part of considered equations is a Fredholm mapping with nullity n and index zero. Using the reduction principles introduced in Section 1, we reduce each of these equations to two equations, one equation is in a infinite dimensional space which is easily solved by the Banach Contraction Principle, the other equation is in a finite dimensional Euclidean space defined by the null space of the linear part. The solving of the last one can be reduced to the problem of finding conditions on the definition and nonvanishing of the topological degree of the mappings or on the existence of regular nonzero solutions of algebraic equations in a finite dimensional Euclidean space.

Section 3 is devoted to bifurcation problems concerning Lipschitz continuous mappings. We assume that $\overline{\lambda} \in \Lambda$ is a characteristic value of the pair (T, L) (i.e. $T(v) + L(\overline{\lambda}, v) = 0$ for some $v \neq 0$) such that the mapping

$T + L(\overline{\lambda}, .)$ is Fredholm with nullity p and index zero. Using the result obtained in Section 1, we reduce the bifurcation problem of the equation

$$T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times \overline{D},$$

to the bifurcation equation in a finite dimensional Euclidean space. To solve this bifurcation equation, we find some sufficient conditions on the definition and nonvanishing of the topological degree of mappings or, on the existence of regular nonzero solutions of algebraic equations in a finite dimensional Euclidean space. The results in this section generalize some well-known results obtained by McLeod and Sattinger [11], Buchner, Marsden and Schecter [1] and by Crandall and Rabinowits [5] in the case of simple characteristic values.

In Section 4, we consider the existence of Hopf bifurcation points of periodic solutions of the equation

$$\dot{u} + T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u) = 0$$

with T, L, H, K as above. It is well-known that in the year 1942, Hopf [7] proved the existence of the bifurcation of periodic solutions of the equation $\dot{u} = F(\lambda, u)$ at a critical $\overline{\lambda}$ under the conditions:

a) The eigenvalue $\sigma(\lambda)$ of $F_x(\lambda, 0)$ crosses the imaginary axis for critical $\lambda = \overline{\lambda}$ with $\operatorname{Re} \sigma'(\overline{\lambda}) \neq 0$, where σ' denotes the derivative of σ with respect to λ.

b) The purely imaginary eigenvalue $\sigma(\overline{\lambda}) = \pm i\mu_0$ is simple.

c) $F_x(\overline{\lambda}, 0)$ has no eigenvalue of the form $\pm k\mu_0, k = 0, 2, ...$

There are several generalizations of Hopf's result (see the papers of Ize [9], Chafee [2], Schmidt [13], Kielhöfer [10, etc.). In [10], Kielhöfer investigated the Hopf bifurcation of the equation

$$\dot{u} + Au + B(\lambda)u = F(\lambda, u)$$

in a Hilbert space with mappings B, F depending analytically in λ and $B(0) = 0$. The mapping A is assumed to have a purely imaginary eigenvalue $\pm i\mu_0$ with multiplicity $r \geq 1$. Then, he studied the bifurcation of periodic solutions at $\lambda = 0$, using the method of Lyapunov-Schmidt for evolution equations, following Iudovich [8]. This reduces the above problem to the bifurcation equation in $\mathbf{R}^{2r}$ of the form

$$Dv + \widetilde{B}(\lambda)v + G(\mu), v) = 0.$$

The parameter μ corresponds to the unknown period of the bifurcating solution. The vector v belongs to $\mathbf{R}^{2r-1}$ and the linear operators D and $\widetilde{B}(\lambda)$ as well as the nonlinear operator $G(\mu, \lambda, .)$ map $\mathbf{R}^{2r-1}$ into $\mathbf{R}^{2r}$. He then found some necessary and sufficient conditions for the Hopf bifurcation of the above system, showing that the positive number of branches which bifurcate at $\lambda = 0$ depends on the number of nontrivial solutions of four algebraic equations in $\mathbf{R}^{2r}$.

Let $\overline{\lambda} \in \Lambda$ be such that the mapping $T + L(\overline{\lambda}, .)$ is Fredholm and has $\pm i\beta_0$, $\beta_0 \neq 0$ as eigenvalue with multiplicity $p \geq 0$. We use Theorem 4 in Section 1 to reduce the above equation to the bifurcation equation in $\mathbf{R}^{2p}$. We shall find some sufficient conditions on the existence of the Hopf bifurcation, by studying the definition and nonvanishing of the topological degree of four mappings in $\mathbf{R}^{2p}$, or the existence of regular nonzero solutions of four algebraic equations in $\mathbf{R}^{2p}$. Our four algebraic equations in $\mathbf{R}^{2p}$ are different from the ones defined by Keilhöfer in [10]. At the end of this section we consider the special case when $p = 1$. Our results generalize Hopf's bifurcation theorem. The results in this section are also true when the mapping $T + L(\overline{\lambda}, .)$ has $\pm in\beta_0$ (for a finite number of $n = 0, 2, ...$) as eigenvalues with a finite multiplicity.

1. The reduction principles

In what follows we shall describe the solution set of the equation (1) in a neighborhood of a given solution with F of several forms. We first consider the equation

$$T(u) + H(u) + K(u) = 0, \quad u \in \overline{D}, \tag{2}$$

where T is a linear continuous mapping from X into Y, H and K are nonlinear mappings from $\overline{D}$ into Y with $H(0) = K(0) = 0$. Further we make the following hypotheses on these mappings and the spaces X, Y.

Hypothesis 1. There exist two finite dimensional spaces $X_0 \subset X$ and $Y_0 \subset Y$ with dim X_0= dim $Y_0 = n$ and two continuous projections $P_X : X \to X_0$ and $P_Y : Y \to Y_0$ such that

$$TP_X(x) = P_Y T(x) \quad \text{for all} \quad x \in X,$$

and if we set $Q_X = I - P_X, Q_Y = I - P_Y$

$$X_1 = Q_X(X), \quad Y_1 = Q_Y(Y)$$

with I denoting the identity mapping, then the following hold

(i) Ker $T = \{x \in X \ / \ T(x) = 0\} \subset X_0$,
(ii) the linear problem

$$T(v) = f, \quad f \in Y_1$$

has a unique solution $v = \Pi(f) \in X_1$, where the operator $\Pi : Y_1 \to X_1$ is continuous.

Hypothesis 2. There exists a real number $a > 1$, a constant $k_1 > 0$ and a real increasing continuous function $\rho : R \to R$ with $\lim\limits_{\delta \to 0} \rho(\delta) = 0$ such that

(i) $H(tu) = t^a H(u)$ holds for all $t \in [0,1], u \in \overline{D}$,

(ii) $||Q_Y H(u) - Q_Y H(v)|| \leq k_1 ||u - v||$ holds for all $u, v \in \overline{D}$,

(iii) $|\alpha|^{-a} ||K(\alpha u)|| \to 0$ as $\alpha \to 0$ uniformly in $u \in \overline{D}$,

(iv) $||Q_Y K(u) - Q_Y K(v)|| \leq \rho(||u - v||)||u - v||$ holds for all $u, v \in \overline{D}$.

Now, by choosing $D' \subset D$ smaller if necessary we may suppose that $D = D(0, r)$, the open ball with the center at the origin and the radius $r > 0$ in X. Also, we assume that $D_0 = P_X(D)$ and $D_1 = Q_X(D)$ are open balls in X_0 and X_1, respectively, say $D_1 = D_1(0, r_1)$. Further, let $\{v^1, ..., v^n\}$ be a basis of the space X_0. Let

$$U_1 = \Big\{ x = (x_1, ..., x_n) \in R^n \ / \ \sum_{j=1}^{n} x_j v^j \in D_0 \Big\}.$$

Without loss of generality we also assume that $U_1 = U(0, r_0)$, the open ball with the center at the zero in $\mathbf{R}^n$ and the radius $r_0 > 0$.

Theorem 1. *Under Hypotheses 1 and 2 there exists a number $t_0 \in (0,1]$ such that for any $x \in t_0 U_1$ one can find a unique $\psi(x) \in t_0 D_1$ with the following properties:*

(i) $\sum\limits_{j=1}^{n} x_j v^j + \psi(x), \quad x = (x_1, ..., x_n)$, *is a solution of the equation*

$$Q_Y\big(T(u) + H(u) + K(u)\big) = 0.$$

(ii) *There exists a constant $k_2 > 0$ such that for any $x^1, x^2 \in t_0 U_1$ we have*

$$||\psi(x^1) - \psi(x^2)|| \leq k_2 |x^1 - x^2|$$

(iii) $||\psi(|\alpha| x)|| = o(|\alpha|)$ *as $\alpha \to 0$ uniformly in $x \in U_1, \psi(0) = 0$.*

(iv) *If* $\overline{x} \in t_0U_1$, $\overline{x} = (\overline{x}_1, ..., \overline{x}_n)$ *is a solution of the equation*

$$P_Y(T(\sum_{j=1}^{n} x_jv^j + \psi(x)) + H(\sum_{j=1}^{n} x_jv^j + \psi(x)) + K(\sum_{j=1}^{n} x_jv^j + \psi(x)) = 0,$$

then $\sum_{j=1}^{n} \overline{x}_jv^j + \psi(\overline{x})$ *is a solution of the equation* (2).

(v) *If* $u \in t_0\overline{D}$ *is a solution of the equation* (2), *then there exists a unique* $\overline{x} \in t_0U_1$ *such that*

$$P_X(u) = \sum_{j=1}^{n} \overline{x}_jv^j$$

and

$$Q_X(u) = \psi(\overline{x}).$$

Proof. Let U_1, D_1 be as mentioned above. For any $t \in (0, 1]$ we set $U(t) = tU_1$ and $D(t) = tD_1$ and define the mapping $G : U_1 \times D_1 \to X_1$ by

$$G(x, w) = \Pi Q_Y\Big(H\Big(\sum_{j=1}^{n} x_jv^j + w\Big) + K\Big(\sum_{j=1}^{n} x_jv^j + w\Big)\Big),$$

$$(x, w) \in U_1 \times D_1.$$

Now, for $x \in U(t), w^1, w^2 \in D(t)$ we can write $x = tx', w^i = tw^i$, with $x' \in U_1, w^i \in D_1, i = 1, 2$. Therefore, we have

$$\begin{aligned} ||G(x, w^1) - G(x, w^2)|| &\leq ||\Pi Q_Y||\{k_1|t|^{a-1}||w^1 - w^2|| \\ &\quad + \rho(t||w^1 - w^2||)||w^1 - w^2||\} \\ &\leq ||\Pi Q_Y||(k_1|t|^{a-t} + \rho(2r_1t)||w^1 - w^2|| \\ &= C_1(t)||w^1 - w^2||, \end{aligned}$$

where

$$C_1(t) = ||\Pi Q_Y||(k_1|t|^{a-1} + \rho(2r_1t)).$$

And for $w \in D(t), w = tw'$, we have

$$\begin{aligned} ||G(x, w)|| &\leq ||\Pi Q_Y||(k_1|t|^a(r_0 + r_1) + t\rho(2r_1t)r_1) \\ &= C_2(t).t, \end{aligned}$$

where
$$C_2(t) = ||\Pi Q_Y||(k_1|t|^{a-1}(r_0 + r_1) + \rho(2r_1 t)r_1).$$
Since $C_1(t), C_2(t) \to 0$ as $t \to 0$, we conclude that there exists a number $t_0 \in (0,1]$ with $0 \leq C_1(t_0) < 1$ and $0 < C_2(t_0) < r_1$. It then follows that for any $x \in U(t_0)$ the mapping $G(x, .)$ is a contraction mapping and maps $D(t_0)$ into itself. Applying the Banach Contraction Principle, we deduce that there exists a unique point $\psi(x) \in D(t_0)$ satisfying
$$\psi(x) = G(x, \psi(x)).$$
This implies
$$Q_Y\Big(T(\psi(x)) + H\Big(\sum_{j=1}^{n} x_j v^j + \psi(x)\Big) + K\Big(\sum_{j=1}^{n} x_j v^j + \psi(x)\Big)\Big) = 0.$$
Together with the fact $T\Big(\sum_{j=1}^{n} x_j v^j\Big) = 0$ we obtain the proof of (i).

Now, let $x^1, x^2 \in t_0 U_1, x^i = t_0 x^i, i = 1, 2$. We have
$$\begin{aligned} ||\psi(x^1) - \psi(x^2)|| &= ||G(x^1, \psi(x^1)) - G(x^2, \psi(x^2))|| \\ &\leq ||\Pi Q_Y||\{k_1|t_0|^{a-1}(|x^1 - x^2| + ||\psi(x^1) - \psi(x^2)|| \\ &\quad + \rho(2(r_0 + r_1)t_0)(|x^1 - x^2| + ||\psi(x^1) - \psi(x^2)||\}. \end{aligned}$$
It follows
$$||\psi(x^1) - \psi(x^2)|| \leq \frac{C_3(t_0)}{1 - C_3(t_0)} ||x^1 - x^2||, \tag{3}$$
with
$$C_3(t_0) = ||\Pi Q_Y||(k_1|t_0|^{a-1} + \rho(2(r_0 + r_1)t_0)).$$
By choosing $t_0' < t_0$ if necessary we also assume that $C_3(t_0) < 1$. Therefore, to complete the proof of (ii) it remains to set $k_2 = \dfrac{C_3(t_0)}{1 - C_3(t_0)}$.

Now, we prove (iii). Since $H(0) = K(0) = 0$, we then have $G(0, 0) = 0$, and hence $\psi(0) = 0$. Therefore, it implies from (3) that for $x \in U(t_0)$ and $\alpha \in (-1, 1)$ we have
$$\Big|\Big|\frac{\psi(|\alpha| x)}{\alpha}\Big|\Big| \leq k_2.$$

Further, we can see that

$$\begin{aligned}\Big|\Big|\frac{\psi(|\alpha|x)}{\alpha}\Big|\Big| &\le ||\Pi Q_Y||(k_1|\alpha|^{a-1}\Big(|x|+\Big|\Big|\frac{\psi(|\alpha|x)}{\alpha}\Big|\Big|\Big)\\ &\quad+\rho\Big(|\alpha|\Big(|x|+\Big|\Big|\frac{\psi(|\alpha|x)}{\alpha}\Big|\Big|\Big)\Big(|x|+\Big|\Big|\frac{\psi(|\alpha|x)}{|\alpha|}\Big|\Big|\Big)\Big)\\ &\le ||\Pi Q_Y||(k_1|\alpha|^{a-1}(r_0+k_2)+\rho(|\alpha|(r_0+k_2)(r_0+k_2))\end{aligned}$$

holds for all $\alpha \in (-1,1)$ small enough. It follows

$$\lim_{|\alpha|\to 0}||\frac{\psi(|\alpha|x)}{|\alpha|}||=0,$$

or $||\psi(|\alpha|x|)||=o(|\alpha|)$ as $|\alpha|\to 0$. This proves (iii).

(iv) is obvious. Finally, we need only prove (v). Let $u \in t_0\overline{D}$ be a solution of the equation (2). We have $P_X(u)\in t_0\overline{D}_0$. It follows that there exists $\overline{x}\in t_0U_1$ such that $P_X(u)=\sum\limits_{j=1}^{n}\overline{x}_j v^j$ and $Q_X(u)$ is a fixed point of the mapping $G(\overline{x},.)$. Since $\psi(\overline{x})$ is a unique fixed point of $G(\overline{x},.)$, we conclude $Q_X(u)=\psi(\overline{x})$.

This completes the proof of the theorem.

Next, we consider the dynamic system of the form

$$\dot{u}+T(u)+H(u)+K(u)=0, \quad u\in\overline{D}, \tag{4}$$

where T, H and K satisfy the same hypotheses as above. The Banach space X is continuously embedded in Y. By a solution of (4) we mean a continuously differentiable mapping $x: R\to X$ such that the following properties hold

(a) $x(t)\in X$ for all $t\in I$,

(b) $\dot{x}(t)+T(x(t))+H(x(t))+K(x(t))=0$ for all $t\in R$.

Let $\mathcal{X}=C_{2\pi}(R,X)$ be the space of all 2π-periodic continuous differentiable mappings $u: R\to X$. Let $\mathcal{Y}=C_0([0,2\pi],Y)$ be the space of all 2π-periodic continuous mappings $h:[0,2\pi]\to Y$. One can easily verify that $\mathcal{X}$ and $\mathcal{X}^*=C_{2\pi}(R,X^*)$, $\mathcal{Y}$ and $\mathcal{Y}^*=C_0([0,2\pi],Y^*)$ is a pair of dual spaces, respectively. The inner products between $\mathcal{X}$ and $\mathcal{X}^*$, $\mathcal{Y}$ and $\mathcal{Y}^*$ are denoted by the same symbol $\langle,\rangle$ and defined by

$$\langle u,v\rangle = \int_0^{2\pi}(u(t),v(t))dt$$

for $(u, v) \in \mathcal{X} \times \mathcal{X}^*$ or $(u, v) \in \mathcal{Y} \times \mathcal{Y}^*$. The norm in $\mathcal{X}(\mathcal{Y})$ is defined by the sup-norm $||u|| = \sup_{t \in R} ||u(t)||$. It is clear that $u_0 = 0$ is a solution of (4). Now, to given $\epsilon > 0$ we want to find a solution u_ϵ of (4) with $0 < ||u_\epsilon|| < \epsilon$ and u_ϵ is a T_ϵ-periodic mapping with $|T_\epsilon - 2\pi| < \epsilon$. We say that u_ϵ is a nontrivial periodic solution with small norm. Setting $t = (1+\rho)\tau$ in the equation (4) we obtain the equation

$$\dot{u} + (1+\rho)\{T(u) + H(u) + K(u)\} = 0, \tag{5}$$

$$(\rho, u) \in I_1 \times \overline{\mathcal{D}}, I_1 = (-1, 1),$$

and $\mathcal{D} = \{u \in C_{2\pi}(R, X) \ / \ u(t) \in D \text{ for all } t \in R\}$.

Now we make the following assumptions on these mappings and the spaces $\mathcal{X}, \mathcal{Y}$.

Hypothesis 3. There exist two finite dimensional spaces $\mathcal{X}_0 \subset \mathcal{X}$ and $\mathcal{Y}_0 \subset \mathcal{Y}$ with $\dim \mathcal{X}_0 = \dim \mathcal{Y}_0 = n$ and two continuous projections $P_\mathcal{X} : \mathcal{X} \to \mathcal{X}_0$ and $P_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y}_0$ such that

$$\Big(\frac{d}{dt} + T\Big) P_\mathcal{X}(x) = P_\mathcal{Y}\Big(\frac{d}{dt} + T\Big)(x) \quad \text{for all} \quad x \in X,$$

and if we set

$$Q_\mathcal{X} = I - P_x, \quad Q_\mathcal{Y} = I - P_\mathcal{Y}$$
$$\mathcal{X}_1 = Q_\mathcal{X}(\mathcal{X}), \quad \mathcal{Y}_1 = Q_\mathcal{Y}(\mathcal{Y}),$$

then the following hold

(i) Ker $\big(\frac{d}{dt} + T\big) = \Big\{x \in \mathcal{X} \ / \ \frac{dx}{dt} + T(x) = 0\Big\} \subset \mathcal{X}_0$,
(ii) the linear problem

$$\frac{dv}{dt} + T(v) = f, \quad f \in \mathcal{Y}_1$$

has a unique solution $v = \Pi(f) \in \mathcal{X}_1$, where the operator $\Pi : \mathcal{Y}_1 \to \mathcal{X}_1$ is continuous.

Hypothesis 4. Hypothesis 2 is satisfied with D replaced by $\mathcal{D}$ and the norm $||.||$ is replaced by the sup-norm.

Further, as before we assume $\mathcal{D} = \mathcal{D}(0, r), \mathcal{D}_1 = Q_\mathcal{X}(\mathcal{D}) = \mathcal{D}_1(0, r_1)$. Let $\{\phi^1, ..., \phi^n\}$ be a basis of the space $\mathcal{X}_0$. We also assume

$$U_1 = \Big\{x = (x_1, ..., x_n) \in R^n \ / \ \sum_{j=1}^n x_j \phi^j \in \mathcal{D}_0\Big\} = U_1(0, r_0).$$

We have

Theorem 2. *Under Hypotheses 3 and 4 there exists a number* $t_0 \in (0,1]$ *such that for any* $\alpha \in t_0 I_1, x \in t_0 U_1$ *one can find a unique* $\psi(\alpha,x) \in t_0\mathcal{D}_1$ *with the following properties*

(i) $(\alpha, \sum_{j=1}^{n} x_j\phi^j + \psi(\alpha,x)), x = (x_1,...,x_n)$, *is a solution of the equation*

$$Q_{\mathcal{Y}}(\dot{u} + (1+\rho)\{T(u) + H(u) + K(u)\}) = 0.$$

(ii) *There exists a constant* $k_3 > 0$ *independent of* α *such that for* $x^1, x^2 \in t_0U_1$ $\alpha \in t_0I_1$ *we have*

$$||\psi(\alpha,x^1) - \psi(\alpha,x^2)|| \leq k_3|x^1 - x^2|.$$

(iii) $||\psi(|\alpha|^d, |\alpha|x)|| = o(|\alpha|)$ *as* $\alpha \to 0$ *uniformly in* $x \in t_0U_1$ *for* $d = a, a-1$ *and* $\psi(\alpha,0) = 0$ *for all* $\alpha \in t_0I_1$.

(iv) *If* $\overline{x} \in t_0U_1$, $\overline{x} = (\overline{x}_1,...,\overline{x}_n)$, *is a solution of the equation*

$$P_{\mathcal{Y}}\Big(\frac{d\big(\sum_{j=1}^{n} x_j\phi^j + \psi(\alpha,x)\big)}{dt}\Big) + (1+\alpha)\Big\{T\Big(\sum_{j=1}^{n} x^j\phi^j + \psi(\alpha,x)\Big) +$$

$$H\Big(\sum_{j=1}^{n} x_j\phi^j + \psi(\alpha,x)\Big) + K\Big(\sum_{j=1}^{n} x_j\phi^j + \psi(\alpha,x)\Big)\Big\} = 0$$

with $\alpha \in t_0I_1$, *then* $\Big(\alpha, \sum_{j=1}^{n} \overline{x}_j\phi^j + \psi(\alpha,\overline{x})\Big)$ *is a solution of the equation* (5).

(v) *If* $(\alpha,\overline{u}) \in t_0I_1 \times t_0\mathcal{D}$ *is a solution of the equation* (5), *then there exists a unique* $\overline{x} \in t_0U_1$ *such that*

$$P_{\mathcal{X}}(\overline{u}) = \sum_{j=1}^{n} \overline{x}_j\phi^j$$

and

$$Q_{\mathcal{X}}(\overline{u}) = \psi(\alpha,\overline{x}).$$

Proof. The proof of this theorem proceeds exactly as the one of Theorem 1 with the mapping $G : U_1 \times D_1 \to X_1$ replaced by the mapping $\mathcal{G} : I_1 \times U_1 \times \mathcal{D}_1 \to \mathcal{X}_1$ defined by

$$\mathcal{G}(\alpha, x, w) = \Pi Q_{\mathcal{Y}}(\alpha T\Big(\sum_{j=1}^{n} x_j \phi^j + w\Big) + H\Big(\sum_{j=1}^{n} x_j \phi^j + w\Big) + K\Big(\sum_{j=1}^{n} x_j \phi^j + w)\Big).$$

Further, we consider the following equation depending on a parameter

$$T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times \overline{D}, \tag{6}$$

where $\overline{D}$ is as above, Λ is an open subset of a normed space. For any fixed $\lambda \in \Lambda, T, L(\lambda, .)$ are continuous linear mappings from X into Y, $H(\lambda, .), K(\lambda, .)$ are nonlinear mappings from $\overline{D}$ into Y and $H(\lambda, 0) = K(\lambda, 0) = 0$.

Let $\overline{\lambda} \in \Lambda$ be given. We make the following hypotheses on these mappings and spaces.

Hypothesis 5. Hypothesis 1 is satisfied with T replaced by $T + L(\overline{\lambda}, .)$ everywhere.

Hypothesis 6. There exists a real number b such that $\alpha L(\overline{\lambda}, v) = L(\alpha^b \overline{\lambda}, v)$ holds for all $\alpha \in [0, 1], v \in \overline{D}$.

Hypothesis 7. There exist a real number $a > 1$, a constant k_1 and a continuous increasing real function $\rho : R \to R$ with $\lim_{\delta \to 0} \rho(\delta) = 0$ such that

(i) $H(\lambda, tu) = t^a H(\lambda, u)$ holds for all $t \in [0, 1], u \in \overline{D}, \lambda \in \Lambda$,

(ii) $\Big|\Big|Q_Y H\Big(\dfrac{\overline{\lambda}}{(1+\alpha)^b}, u\Big) - Q_Y H\Big(\dfrac{\overline{\lambda}}{(1+\alpha)^b}, v\Big)\Big|\Big| \le k_1 ||u - v||$ holds for all $\alpha \in \Big(-\dfrac{1}{2}, \dfrac{1}{2}\Big), u, v \in \overline{D}$,

(iii) $\Big|\Big|\ |\alpha|^{-a} K\Big(\dfrac{\overline{\lambda}}{(1+\alpha)^b}, \alpha u\Big)\Big|\Big| \to 0$ as $\alpha \to 0$ uniformly in $u \in \overline{D}$,

(iv) $\Big|\Big|Q_Y K\Big(\dfrac{\overline{\lambda}}{(1+\alpha)^b}, u\Big) - Q_Y K\Big(\dfrac{\overline{\lambda}}{(1+\alpha)^b}, v\Big)\Big|\Big| \le \rho(||u - v||)||u - v||$

holds for all $\alpha \in (-\dfrac{1}{2}, \dfrac{1}{2}), u, v \in \overline{D}$, where b is from Hypothesis 6.

Let us take $I_1 \subset \Big(-\frac{1}{2}, \frac{1}{2}\Big)$ such that $\frac{\overline{\lambda}}{1+\alpha} \in \Lambda$ holds for all $\alpha \in I_1$. Let D_0, D_1, U_1, $\{v^1, ..., v^n\}$ be as above. We have

Theorem 3. *Under Hypotheses 5, 6 and 7 there exists a point $t_0 \in (0, 1]$ such that for any $\alpha \in t_0 I_1, x \in t_0 U_1$ one can find a unique $\psi(\alpha, x) \in t_0 D_1$ with the following properties*

(i) $\sum\limits_{j=1}^{n} x_j v^j + \psi(\alpha, x)$, $x = (x_1, ..., x_n)$, *is a solution of the equation*

$$Q_Y\Big(T(u) + L\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, u\Big) + H\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, u\Big) + K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, u\Big)\Big) = 0,$$

$$(\alpha, u) \in I \times D.$$

(ii) *There exists a constant $k_2 > 0$ such that for any $\alpha \in t_0 I_1, x^1, x^2 \in t_0 U_1$ we have*

$$||\psi(\alpha, x^1) - \psi(\alpha, x^2)|| \leq k|x^1 - x^2|$$

(iii) $||\psi(|\alpha|^{a-1}, |\alpha| x)|| = 0(|\alpha|)$ *as* $|\alpha| \to 0$ *uniformly in* $x \in t_0 U_1$, $\psi(\alpha, 0) = 0$ *for any* $\alpha \in t_0 I_1$.

(iv) *If* $\overline{x} \in t_0 U_1, \overline{x} = (\overline{x}_1, ..., \overline{x}_n)$, *is a solution of the equation*

$$P_Y\Big(T\Big(\sum_{j=1}^{n} x_j v^j + \psi(\alpha, x)\Big) + L\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j v^j + \psi(\alpha, x)\Big) +$$

$$H\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j v^j + \psi(\alpha, x)\Big) + K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j v^j + \psi(\alpha, x)\Big)\Big) = 0,$$

$$\alpha \in t_0 I_1,$$

then $\sum\limits_{j=1}^{n} \overline{x}_j v^j + \psi(\alpha, \overline{x})$ *is a solution of the equation* (6) *with* $\lambda = \frac{\overline{\lambda}}{(1+\alpha)^b}$.

(v) *If* $\overline{u} \in t_0 \overline{D}$ *is a solution of the equation* (6) *with* $\lambda = \frac{\overline{\lambda}}{(1+\alpha)^b}, \alpha \in t_0 I_1$, *then there exists a point* $\overline{x} \in t_0 U_1$ *such that* $P_X(\overline{u}) = \sum\limits_{j=1}^{n} \overline{x}_j v^j$ *and* $Q_X(\overline{u}) = \psi(\alpha, \overline{x})$.

Proof. The proof of this theorem proceeds exactly as the one of Theorem 1 with the mapping $G : U_1 \times D_1 \to X_1$ replaced by the mapping $\overline{G}$: $I_1 \times U_1 \times D_1 \to X_1$ defined by

$$\overline{G} = (\alpha, x, w) = \Pi Q_Y \Big(\alpha\Big(T\sum_{j=1}^{n} x_j v^j + w\Big) +$$

$$(1+\alpha)^b H\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j v^j + w\Big) + (1+\alpha)^b K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j v^j + w\Big)\Big).$$

Next, we consider the following dynamic system depending on a parameter

$$\dot{u} + T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u) = 0, (\lambda, u) \in \Lambda \times \overline{D}, \tag{7}$$

where $\Lambda, \overline{D}$ are as above, T, L, H and K are as in the equation (6). Let $\overline{\lambda} \in \Lambda$ be given. To any $\epsilon > 0$ we want to seek a solution $(\lambda_\epsilon, u_\epsilon)$ of (7) with $|\lambda_\epsilon - \overline{\lambda}| < \epsilon, 0 < ||u_\epsilon|| < \epsilon$ and u_ϵ is a T_ϵ-periodic mapping with $|T_\epsilon - 2\pi| < \epsilon$. As before, setting $t = (t+\rho)\tau$ in the equation (7) we obtain

$$\dot{u} + (1+\rho)\{T(u) + L(\lambda, u) + H(\lambda, u) + K(\lambda, u)\} = 0,$$

$$(\rho, \lambda, u) \in I_1 \times \Lambda \times \mathcal{D}. \tag{8}$$

Now, let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be defined as above and $\overline{\lambda} \in \Lambda$ be given. We make the following hypotheses on these mappings and spaces.

Hypothesis 8. Hypothesis 3 is satisfied with the mapping T replaced by the mapping $T + L(\overline{\lambda}, .)$ everywhere.

Hypothesis 9. Hypothesis 6 is satisfied with $\overline{D}$ replaced by $\overline{\mathcal{D}}$.

Hypothesis 10. Hypothesis 7 is satisfied with $\overline{D}$ replaced by $\overline{\mathcal{D}}$ and the norm $||.||$ is replaced by the sup-norm.

We have

Theorem 4. *Under Hypotheses 8, 9 and 10 there exists a point $t_0 \in (0,1]$ such that for any $\rho \in t_0 I_1$, $\alpha \in t_0 I_1$, $x \in t_0 U_1$ one can find a unique $\psi(\rho, \alpha, x) \in t_0 \mathcal{D}_1$ with the following properties*

(i) $\sum\limits_{j=1}^{n} x_j \phi^j + \psi(\rho, \alpha, x)$, $x = (x_1, ..., x_n)$, *is a solution of the equation*

$$Q_{\mathcal{Y}}\Big\{\dot{u} + (1+\rho)\Big\{T(u) + K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, u\Big) +$$

$$H\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, u\Big) + K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, u\Big)\Big\}\Big\} = 0.$$

(ii) *There exists a constant* $k_2 > 0$ *such that for any* ρ, $\alpha \in t_0 I_1$, x^1, $x^2 \in t_0 U_1$ *we have*

$$||\psi(\rho, \alpha, x^1) - \psi(\rho, \alpha, x^2)|| \leq k_2 |x^1 - x^2|.$$

(iii) $||\psi(|\alpha|^c, |\alpha|^d, |\alpha| x)|| = o(|\alpha|)$ *as* $|\alpha| \to 0$ *for all* $c, d = a, a-1$, $\psi(\rho, \alpha, 0) = 0$ *for all* ρ, $\alpha \in t_0 I_1$.

(iv) *If* $\overline{x} \in t_0 U_1$ *is a solution of the equation*

$$P_{\mathcal{Y}}\Big\{ \frac{d\Big(\sum_{j=1}^{n} x_j \phi^j + \psi(\rho, \alpha, x)\Big)}{d\tau} + (1+\rho)\Big(T \sum_{j=1}^{n} x_j \phi^j + \psi(\rho, \alpha, x)\Big) +$$

$$L\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} \overline{x}_j \phi^j + \psi(\rho, \alpha, x)\Big) + H\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j \phi^j + \psi(\rho, \alpha, x)\Big)$$

$$+ K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j \phi^j + \psi(\rho, \alpha, x)\Big)\Big\} = 0$$

then $\big(\rho, \alpha, \sum_{j=1}^{n} x_j \phi^j + \psi(\rho, \alpha, x)\big)$ *satisfies the equation (8) with* $\lambda = \dfrac{\overline{\lambda}}{(1+\alpha)^b}$.

(v) *If* $\overline{u} \in t_0 \mathcal{D}$ *is a solution of the equation (8) with* $\rho \in t_0 I_1, \lambda = \dfrac{\overline{\lambda}}{(1+\alpha)^b}, \alpha \in t_0 I_1$, *then there exists a point* $\overline{x} \in t_0 U_1$ *such that* $P_{\mathcal{X}}(\overline{u}) = \sum_{j=1}^{n} \overline{x}_j \phi^j$ *and* $Q_{\mathcal{X}}(\overline{u}) = \psi(\rho, \alpha, \overline{x})$.

Proof. The proof of this theorem proceeds exactly as the one of Theorem 1 with the mapping $G : U_1 \times D_1 \to X_1$ replaced by the mapping $\overline{\mathcal{G}} : I_1 \times I_1 \times U_1 \times \mathcal{D}_1 \to \mathcal{X}_1$ defined by

$$\overline{\mathcal{G}}(\rho, \alpha, x, w) = \Pi Q_{\mathcal{Y}}\Big(\frac{\alpha - \rho}{1+\rho} \frac{d(\sum_{j=1}^{n} x_j \phi^j + w)}{d\tau} + \alpha T\Big(\sum_{j=1}^{n} x_j \phi^j + w\Big)$$

$$+ (1+\alpha)^b H\Big(\frac{\overline{\lambda}}{(1+\alpha)^b}, \sum_{j=1}^{n} x_j \phi^j + w\Big) + (1+\alpha)^b K\Big(\frac{\overline{\lambda}}{(1+\alpha)^b} \sum_{j=1}^{n} x_j \phi^j + w\Big)\Big)$$

$$(\rho, \alpha, x, w) \in I_1 \times I_1 \times U_1 \times \mathcal{D}_1.$$

2. The existence of nontrivial solutions

First, we consider the existence of nontrivial solutions of the equation (2) in a neighborhood of the origin. We assume that the mapping T is Fredholm with nullity p and index zero. By T^* we denote the adjoint mapping of the mapping T. Let

$$\begin{aligned}\ker T &= [v^1, ..., v^p],\\ \ker T^* &= [\psi^1, ..., \psi^p].\end{aligned}$$

By the Haln-Banach Theorem there exist p functionals on X and p elements in Y such that $\langle v^j, f^k\rangle \ = \delta_{jk},\ \langle \psi^n, z^m\rangle \ = \delta_{nm},\ j,k,m,n = 1, ..., p$. We set

$$\begin{aligned}X_0 &= [v^1, ..., v^p],\\ X_1 &= \{y \in X \ / \ \langle y, f^k\rangle \ = 0, k = 1, ..., p\}\\ Y_0 &= [z^1, ..., z^p]\\ Y_1 &= \{y \in Y \ / \ \langle y, \psi^k\rangle \ = 0, k = 1, ..., p\}.\end{aligned}$$

The projectors $P_X : X \to X_0$, $P_Y : Y \to Y_0$ are defined by

$$P_X(x) = \sum_{k=1}^{p} \langle x, f^k\rangle v^k, \quad P_Y(y) = \sum_{k=1}^{p} \langle y, \psi^k\rangle z^k.$$

Q_X and Q_Y are defined as in Hypothesis 1. One can easily verify that Hypothesis 1 is satisfied. In what follows we assume that the mappings H and K satisfy Hypothesis 2. Therefore, the mapping H can be considered to be defined in whole X. Indeed, for any $x \in X$ there exists a number $\alpha > 0$ such that $\alpha x \in D$. We put $H(x) = \dfrac{1}{\alpha^a} H(\alpha, x)$. Further, we define the mapping $A : \mathbf{R}^p \to \mathbf{R}^p, A = (A_1, ..., A_p)$, by

$$A_k(x) = \ \Big\langle H\Big(\sum_{j=1}^{p} x_j v^j\Big), \psi^k\Big\rangle,\ x = (x_1, ..., x_p) \in \mathbf{R}^p.$$

We impose the following hypothesis on this mapping

Hypothesis 11. There exists an open bounded subset $\Omega \subset \mathbf{R}^p$ such that the topological degree, deg $(A, \Omega, 0)$, of the mapping A with respect to Ω and the origin is defined and different from zero.

We have

Theorem 5. *Let H and K satisfy Hypothesis 2 and let Hypothesis 11 be satisfied. Let t_0, ψ exist by Theorem 1. Then there exists a neighborhood $I_0 \subset t_0 I_1$ such that for any $\alpha \in I_0$ one can find $x(\alpha) \in \Omega$ such that $u(\alpha) = \sum_{j=1}^{p} |\alpha| x_j(\alpha) v^j + \psi(|\alpha| x(\alpha))$, $x(\alpha) = (x_1(\alpha), ..., x_p(\alpha))$, is a solution of the equation* (2). *Moreover, $u(\alpha) \neq 0$ for $\alpha \neq 0$ if $x(\alpha) \neq 0$.*

Proof. Since Ω is bounded, we can take a neighborhood I_2 of zero in R, $I_2 \subset t_0 I_1$ such that $\alpha\Omega \subset t_0 U_1$ holds for all $\alpha \in I_2$. We define the mapping $E : I_2 \times \Omega \to R^p$, $E = (E_1, ..., E_p)$, by

$$E_k(\alpha, x) = \begin{cases} \Big\langle H\Big(\sum_{j=1}^{p} x_j v^j + \dfrac{\psi(|\alpha|x)}{|\alpha}\Big) + \\[2ex] \quad +|\alpha|^{-a} K\Big(\sum_{j=1}^{p} |\alpha| x_j v^j + \dfrac{\psi(\alpha|x)}{|\alpha|}\Big), \psi^k\Big\rangle \quad \text{for } \alpha \neq 0, \\[2ex] \Big\langle H\Big(\sum_{j=1}^{p} x_j v^j\Big), \psi^k\Big\rangle, \text{ for } \alpha = 0. \end{cases}$$

Using the assertion (iii) of Theorem 1 and the continuity of the mappings H and K we conclude that E is a continuous mapping. We claim that there exists a neighborhood $I_0 \subset I_2$ such that $E(t, \alpha, x) \neq 0$ for all $\alpha \in I_0, t \in [0, 1]$ and $x \in \partial\Omega$. Indeed, by contrary we assume that this claim is not true. It then follows that for any n there exist $\alpha_n \in I_n \subset \dfrac{1}{n} I_2$, $t_n \in [0, 1]$ and $x_n \in \partial\Omega$ such that $E(t_n\alpha_n, x_n) = 0$. Since the topological degree deg $(A, \Omega, 0)$ is defined, we deduce that $t_n\alpha_n \neq 0$. Without loss of generality we may assume $t_n \to \bar{t}, x_n \to \overline{x} \in \partial\Omega$, $\alpha_n \to 0$. Hence, $||\psi(|t_n\alpha_n|x_n)|| \to 0$ as $n \to +\infty$. Letting $n \to +\infty$ we obtain $E(0, \overline{x}) = 0$, or $A(\overline{x}) = 0$ with $\overline{x} \in \partial\Omega$ and we have a contradiction. This proves the claim. Consequently, for any $\alpha \in I_0$, the topological degree $\deg(E(\alpha, .), \Omega, 0)$ is defined and

$$\deg(E(\alpha, .), \Omega, 0) = \deg(A, \Omega, 0) \neq 0.$$

Therefore, for any $\alpha \in I_0$ there exists $x(\alpha) \in \Omega$ such that $E(\alpha, x(\alpha)) = 0$. Multiplying both sides of this equality with $|\alpha|^a$, $\alpha \neq 0$, we obtain

$$\Big\langle H\Big(\sum_{j=1}^{p} |\alpha| x_j v^j + \psi(|\alpha| x(\alpha))\Big) + K\Big(\sum_{j=1}^{p} |\alpha| x_j v^j + \psi(|\alpha| x(\alpha))\Big), \psi^k\Big\rangle = 0.$$

Together with the fact $\left\langle T\Big(\sum_{j=1}^{p}|\alpha|x_jv^j+\psi(|\alpha|x(\alpha))\Big),\psi^k\right\rangle=0$ we conclude

$$P_Y\Big(T\Big(\sum_{j=1}^{p}|\alpha|x_jv^j+\psi(|\alpha|x(\alpha))\Big)+H\Big(\sum_{j=1}^{p}|\alpha|x_jv^j+\psi(|\alpha|x(\alpha))\Big)+$$

$$K\Big(\sum_{j=1}^{p}|\alpha|x_jv^j+\psi(|\alpha|x(\alpha))\Big)\Big)=0.$$

Applying the assertion (iv) of Theorem 1, we verify that $\sum_{j=1}^{p}|\alpha|x_jv^j+\psi(|\alpha|x(\alpha))$ is a solution of the equation (2).

Now, we assume that $\sum_{j=1}^{p}|\alpha|x_jv^j+\psi(|\alpha|x(\alpha))=0$ for $\alpha\neq 0$. It then follows $\sum_{j=1}^{p}x_jv^j=\dfrac{-\psi(|\alpha|x(\alpha))}{|\alpha|}\in X_0\cap X_1=\{0\}$ and hence $x_j(\alpha)=0$ for all $j=1,...,p$, or $x(\alpha)=0$. This completes the proof of the theorem.

Remark 1. If Ω does not contain zero, then $x(\alpha)\neq 0$ for all $\alpha\in I_0,\alpha\neq 0$.

Next, we prove some sufficient conditions to show the existence of nontrivial periodic solutions of small norm of the equation (4). We assume that the linear mapping T is Fredholm and has $\pm i\beta_0$ as eigenvalues with multiplicity p. For the sake of simplicity of notation we also suppose that T has no eigenvalue of the form $\pm ni\beta_0, n=0,2,...$ (the following theorem are also valid for the case when the mapping T has a finite number of eigenvalues of the form $\pm ni\beta_0$). Without loss of generality we set $\beta_0=1$. Let

$$\mathrm{Ker}(T+iI)=[v^1,...,v^p],$$
$$\mathrm{Ker}(T+iI)^*=[\gamma^1,...,\gamma^p].$$

It then follows

$$\mathrm{Ker}(\frac{d}{dt}+T)=[\phi^1,...,\phi^{2p}],$$
$$\mathrm{Ker}(\frac{d}{dt}+T)^*=[\psi^1,...,\psi^{2p}],$$

with

$$\phi^{2k-1}(t)=\mathrm{Re}(e^{it}v^k),\quad \phi^{2k}(t)=\mathrm{Im}(e^{it}v^k),$$
$$\psi^{2k-1}(t)=\mathrm{Re}(e^{it}\gamma^k),\quad \psi^{2k}(t)=\mathrm{Im}(e^{it}\gamma^k),\quad k=1,...,p.$$

Let $\mathcal{X}, \mathcal{Y}$ be defined as in the equation (4). By the Hahn Banach Theorem one can find $2p$ functionals $f^1, ..., f^{2p}$ on $\mathcal{X}$ and $2p$ elements $z^1, ..., z^{2p}$ on $\mathcal{Y}$ with $\langle \phi^j, f^k \rangle = \delta_{jk}$; $\langle z^j, \psi^k \rangle = \delta_{jk}$, $j, k = 1, ..., 2p$. We put

$$\mathcal{X}_0 = [\phi^1, ..., \phi^{2p}], \mathcal{X}_1 = \{x \in \mathcal{X} \;/\; \langle x, f^k \rangle = 0, k = 1, ..., 2p\},$$
$$\mathcal{Y}_0 = [z^1, ..., z^{2p}], \mathcal{Y}_1 = \{x \in \mathcal{Y} \;/\; \langle y, \psi^k \rangle = 0, k = 1, ..., 2p\}.$$

The projection $P_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_0, P_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_0$ are defined by

$$P_{\mathcal{X}}(x) = \sum_{k=1}^{2p} \langle x, f^k \rangle \phi^k; \quad P_{\mathcal{Y}}(y) = \sum_{k=1}^{2p} \langle y, \psi^k \rangle z^i,$$

and the projection $Q_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}_1$; $Q_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_1$ are defined as in Hypothesis 3. It then easily verifies that Hypothesis 3 is satisfied.

Further in the sequel for $\sigma = 1$ or $\sigma = -1$ we define the following mappings $A : \mathbf{R}^{2p} \to \mathbf{R}^{2p}$, $B^\sigma : \mathbf{R}^{2p} \to \mathbf{R}^{2p}$, $A = (A_1, ..., A_{2p})$, $B^\sigma = (B^\sigma_1, ..., B^\sigma_{2p})$, by

$$A_k(x) = \Big\langle H\Big(\sum_{j=1}^{2p} x_j \phi^j\Big), \psi^k \Big\rangle,$$

$$B^\sigma_k(x) = \sigma \Big\langle T\Big(\sum_{j=1}^{2p} x_j \phi^j\Big), \psi^k \Big\rangle + A_k(x), \quad k = 1, ..., 2p.$$

We make the following hypotheses on these mappings.

Hypothesis 12. There exists a point $x^* \in \mathbf{R}^{2p}$ and a neighborhood U^* of x^* in $\mathbf{R}^{2p}$ such that the topological degree, $\deg(A, U^*, 0)$, of the mapping A with respect to U^* and zero is defined and different from zero.

Hypothesis 13. Hypothesis 12 is satisfied with x^*, U^* and A replaced by $x^{*\sigma}, U^{*\sigma}$ and B^σ, respectively.

We have

Theorem 6. *Let Hypothesis 4 be satisfied. Let t_0, ψ be from Theorem 2. In addition, assume that either* (a) *Hypothesis 12 is satisfied or* (b) *Hypothesis 13 is satisfied. Then there exists a neighborhood $I_0 \subset t_0 I_1$ such that for any $\alpha \in I_0$ one can find $x^\sigma(\alpha) = (x^\sigma_i(\alpha), ..., x^\sigma_{2p}(\alpha)) \in U^{*\sigma}$ $(U^{*\sigma} = U^*$ for the case* (a)) *for which $(\rho^\sigma(\alpha), u^\sigma(\alpha))$ satisfies the equation* (5), *where*

$$\rho^\sigma(\alpha) = \sigma |\alpha|^a,$$

$$u^\sigma(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^\sigma(\alpha)\phi^j + \psi(|\alpha|^a x^\sigma(\alpha)) \quad \textit{for the case } (\mathrm{a})$$

and

$$u^\sigma(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^\sigma(\alpha)\phi^j + \psi(|\alpha|^{a-1} x^\sigma(\alpha)) \quad \textit{for the case } (\mathrm{b}).$$

Moreover, $\rho^\sigma(\alpha), u^\sigma(\alpha) \to 0$, $u^\sigma(\alpha) \neq 0$ *if* $x^\sigma(\alpha) \neq 0$ *and the mapping* $\overline{u}^\sigma(\alpha)(t) = u^\sigma(\alpha)(\dfrac{t}{(1+\rho^\sigma(\alpha))})$ *is a* $(1+\rho^\sigma(\alpha))2\pi$*-periodic mapping.*

Proof. The proof of this theorem is similar to the one of Theorem 5 with Ω, E replaced by $U^{*\sigma}$ and F^σ respectively, where $F^\sigma : I_2 \times U_1^\sigma \to R^{2p}$, $\quad F^\sigma = (F_1^\sigma, ..., F_{2p}^\sigma)$ and

$$F_k^\sigma(\alpha, x) = \begin{cases} \Big\langle \sigma T\Big(\sum_{j=1}^{2p} |\alpha| x_j \phi^j + \psi(|\alpha|^a, |\alpha| x)\Big) + \\ +(1+\sigma|\alpha|^a|)H\Big(\sum_{j=1}^{2p} \alpha x_j \phi^j + \dfrac{\psi(|\alpha|^a, |\alpha| x)}{\alpha}\Big) \\ +(1+\sigma|\alpha|^a|)|\alpha|^{-a} K\Big(\sum_{j=1}^{2p} |\alpha| x_j \phi^j \\ +\psi(|\alpha|^a, |\alpha| x)\Big), \psi^k\Big\rangle \text{ for } \alpha \neq 0, \\ A_k(x), \text{ for } \alpha = 0, \end{cases}$$

for the case (a) and

$$F_k^\sigma(\alpha, x) = \begin{cases} \Big\langle \sigma T\Big(\sum_{j=1}^{2p} x_j \phi^j + \dfrac{\psi(|\alpha|^{a-1}, |\alpha| x)}{\alpha}\Big) + \\ +(1+\sigma|\alpha|^{a-1})H\Big(\sum_{j=1}^{2p} \alpha x_j \phi^j + \dfrac{\psi(|\alpha|^{a-1}, |\alpha| x)}{\alpha}\Big) \\ +(1+\sigma|\alpha|^{a-1})|\alpha|^{-a} K\Big(\sum_{j=1}^{2p} |\alpha| x_j \phi^j \\ +\psi(|\alpha|^{a-1}, |\alpha| x)\Big), \psi^k\Big\rangle \quad \text{for } \alpha \neq 0, \\ B_k^\sigma(x), \text{ for } \alpha = 0 \end{cases}$$

for the case (b).

This completes the proof of the theorem.

Remark 2. If $U^{*\sigma}$ does not contain zero, then $x^\sigma(\alpha) \neq 0$ for all $\alpha \neq 0$. It follows $u^\alpha(x) \neq 0$ for all $\alpha \neq 0$.

3. Bifurcation problem

In this section we consider bifurcation points of the equation (6) with T, L, H and K given as above. Any point $(\lambda, 0)$ is called a trivial solution. A point $(\overline{\lambda}, 0)$ is called a bifurcation point if for any given $\epsilon > 0$ one can find $(\lambda_\epsilon, u_\epsilon)$ satisfying the equation (6) with $|\lambda_\epsilon - \overline{\lambda}|_A < \epsilon$ and $0 < ||u_\epsilon|| < \epsilon$. In what follows we shall find sufficient conditions for the existence of such a point.

Now, let $\overline{\lambda} \in \Lambda$ be such that the mapping $T + L(\overline{\lambda}, .)$ is Fredholm with nullity p and index zero. Let

$$\begin{aligned}
&\text{Ker } (T + L(\overline{\lambda}, .)) = [v^1, ..., v^p], \\
&\text{Ker } (T + L(\overline{\lambda}, .))^* = [\psi^1, ...\psi^p].
\end{aligned}$$

Further, let X_j, Y_j, $j = 0, 1$, P_X, P_Y, Q_X, Q_Y be defined as in Section 2. It follows that Hypothesis 5 is satisfied.

In addition, we assume that the mappings $L(\overline{\lambda}, .), H, K$ satisfy Hypotheses 6 and 7, respectively. We define the mappings $C, D^\sigma : R^p \to R^p$, $C = (C_1, ..., C_p)$, $D^\sigma = (D_1^\sigma, ..., D_p^\sigma)$, by

$$\begin{aligned}
C_k(x) &= \ \Big\langle H\Big(\sum_{j=1}^{p} x_j v^j\Big), \psi^k\Big\rangle, \\
D_k^\sigma(x) &= \sigma\ \Big\langle (T \sum_{j=1}^{p} x_j v^j), \psi^k\Big\rangle\ + C_k(x), \quad k = 1, ..., p,
\end{aligned}$$

and impose the hypotheses on these mapping as follows:

Hypothesis 14. There exists a point $x^* \in \mathbf{R}^p$ and an open bounded neighborhood U^* of x^* in $\mathbf{R}^p$ not containing the origin such that the topological degree, $\deg(C, U^*, 0)$, of C with respect to U^* and the origin is defined and different from zero.

Hypothesis 15. Hypotheses 14 is satisfied with x^*, U^*, C replaced by $x^{*\sigma}, U^{*\sigma}$ and D^σ, respectively.

The following theorem generalizes the result obtained by the second author (see [14, Theorem 1]), which is proved for the case the mappings

H and K are Lipschitz in both variables. This also extends some well-known results due to McLeod and Saffinger [11], Buchner, Marsden and Schecter [1], etc.

Theorem 7. *Let Hypotheses 6 and 7 be satisfied. Let t_0, ψ be from Theorem 3. In addition, assume that either* (a) *Hypothesis 14 is satisfied, or* (b) *Hypothesis 15 is satisfied. Then $(\overline{\lambda}, 0)$ is a bifurcation point of the equation* (6). *More precisely, there is a neighborhood $I_0 \subset t_0 I_1$ such that for any $\alpha \in I_0$ one can find $x^\sigma(\alpha) = (x_1^\sigma(\alpha), ..., x_p^\sigma(\alpha)) \in U^{*\sigma}$ ($U^{*\sigma} = U^*$ for* (a)*) for which $(\lambda^\sigma(\alpha), v^\sigma(\alpha))$ satisfies the equation* (6), *where*

$$\lambda^\sigma(\alpha) = \frac{\overline{\lambda}}{(1+\sigma|\alpha|^a)^b}\,,$$

and

$$v^\sigma(\alpha) = \sum_{j=1}^{p} |\alpha| x_j^\sigma(\alpha) v^j + \psi(|\alpha|^a, |\alpha| x^\sigma(\alpha))$$

for the case (a) *and*

$$\lambda^\sigma(\alpha) = \frac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1})^b}\,,$$

$$v^\sigma(\alpha) = \sum_{j=1}^{p} |\alpha| x_j^\sigma(\alpha) v^j + \psi(|\alpha|^{a-1}, |\alpha| x^\sigma(\alpha)), \quad \textit{for the case } \text{(b)},$$

$\lambda^\sigma(\alpha) \to \overline{\lambda}, v^\sigma(\alpha) \to 0$ *as* $\alpha \to 0$ *and* $v^\sigma(\alpha) \neq 0$ *for* $\alpha \neq 0$.

Proof. The proof of this theorem proceeds exactly as the one of Theorem 5 with Ω, E replaced by $U^{*\sigma}$ and M^σ respectively, where

$$M_k^\sigma(\alpha, x) = \begin{cases} \Big\langle \sigma T\Big(\sum_{j=1}^{p} |\alpha| x_j v^j + \psi(|\alpha|^a, |\alpha| x) \\ +(1+\sigma|\alpha|^a)^a\Big) H\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^a)^b}, \ \sum_{j=1}^{p} |\alpha| x_j v^j + \dfrac{\psi(\alpha|^a x)}{|\alpha|} \\ +(1+\sigma|\alpha|^a)|\alpha|^{-a} K\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^a)^b}, \ \sum_{j=1}^{p} |\alpha| x_j v^j \\ +\psi(|\alpha|^a, |\alpha| x)\Big), \psi^k \Big\rangle \quad \text{for } \alpha \neq 0, \\ C_k(x), \quad \text{for } \alpha = 0, \end{cases}$$

for the case (a)

$$M_k^\sigma(\alpha,x)=\begin{cases}\langle \sigma T\Big(\sum_{j=1}^{p}|\alpha|x_j v^j+\dfrac{\psi(|\alpha|^a,|\alpha|x)}{|\alpha|}\Big)\\ +(1+\sigma|\alpha|^a)H\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1})^b},\ \sum_{j=1}^{p}x_j v^j+\dfrac{\psi(\alpha|^{a-1}x)}{|\alpha|}\\ +(1+\sigma|\alpha|^{a-1})|\alpha|^{-a}K\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1})},\ \sum_{j=1}^{p}|\alpha|x_j v^j\\ +\psi(|\alpha|^{a-1},|\alpha|x)\Big),\psi^k\rangle \quad \text{for } \alpha\neq 0,\\ D_k^\sigma(x), \quad \text{for } \alpha=0,\end{cases}$$

for the case (b).

This completes the proof of the theorem.

Remark 3. If $x^{*j\sigma}, U^{*j\sigma}, j=1,2$, satisfy Hypothesis 14 or 15 with $U^{*1\sigma}\cap U^{*2\sigma}=\emptyset$, we then conclude that $(\lambda^{1\sigma}(\alpha),v^{1\sigma}(\alpha))\neq(\lambda^{2\sigma}(\alpha),v^{2\sigma}(\alpha))$ for all $\alpha\in I_0^1\cap I_0^2$ where $I_0^j,(\lambda^{j\sigma}(\alpha),v^{j\sigma}(\alpha))$ exist by Theorem 7.

Next, we consider the equation (6) in the case $\overline{\lambda}$ is a simple characteristic value of the pair (T,L), i.e. the case when $p=1$, and Ker $(T-L(\overline{\lambda},.))=[v^1]$, Ker $(T-L(\overline{\lambda},.))^*=[\psi^1]$. In addition we assume $\langle T(v^1),\psi^1\rangle\neq 0$. The following theorem is an extension of the result obtained by the second author (see [14, Theorem 7]) and the result obtained by Crandall and Rabinowitz in [5].

Theorem 8. *Let $\overline{\lambda}, v^1, \psi^1$ be as above. Let Hypotheses 6 and 7 be satisfied and t_0,ψ be from Theorem 3. Then $(\overline{\lambda},0)$ is a bifurcation point of the equation* (6)*. More precisely, there is a neighborhood $I_0\subset t_0I_1$ such that for any $\alpha\in I_0$ we can find $\beta^\sigma(\alpha)$ for which $(\lambda^\sigma(\alpha),v^\sigma(\alpha))$ with*

$$\lambda^\sigma(\alpha)=\frac{\overline{\lambda}}{(1+|\alpha|^{a-1}\beta^\sigma(\alpha))^b}\,,$$

and

$$v(\alpha)=|\alpha|v^1+\psi(|\alpha|^{a-1}\beta^\sigma(\alpha),|\alpha|)$$

satisfies the equation (6)*. $\lambda^\sigma(\alpha)\to 0, v(\alpha)\to 0$ as $\alpha\to 0, v(\alpha)\neq 0$ for $\alpha\neq 0$.*

Proof. We put $\overline{\beta}=\langle H(\overline{\lambda},v^1),\psi^1\rangle\,/\,\langle T(v^1),\psi^1\rangle$ and take an open bounded neighborhood U^* of $\overline{\beta}$ in R. We define the mapping $N^\sigma: t_0I_1\times U^*\to R$ by

$$N^\sigma(\alpha,\beta)=\begin{cases}\langle\sigma\beta T\Big(v^1+\dfrac{\psi(|\alpha|^{a-1}\beta|\alpha|)}{|\alpha|}\Big)\\ +(1+\sigma|\alpha|^{a-1}\beta)H\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1}\beta^\sigma(\alpha)^b},\\ v^1+\dfrac{\psi(|\alpha|^{a-1}\beta|\alpha|)}{|\alpha|}\Big)\\ +(1+|\alpha|^{a-1}\beta)|\alpha|^{-a}K\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1}\beta)^b},\beta|\alpha|v^1+\\ \psi(|\alpha|^{a-1}\beta|\alpha|)\Big),\psi^1\rangle,\ \text{for } \alpha\neq 0,\\ \langle\sigma\beta T(v^1)+H(\overline{\lambda},v^1),\psi^1\rangle \text{ for } \alpha=0.\end{cases}$$

By the same arguments as in the proof of Theorem 5 we conclude that N^σ is a continuous mapping and there exists a neighborhood $I_0\subset t_0I_1$ such that $N^\sigma(t\alpha,\beta)\neq 0$ for all $\alpha\in I_0; t\in[0,1]$ and $\beta\in\partial U^*$. Thus, the topological degree, $\deg(N^\sigma(t\alpha,.),U^*,0)$, of $N^\sigma(t\alpha,.)$ with respect to U^* and zero is defined and hence

$$\deg(N^\sigma(\alpha,.),U^*,0)=\deg(N^\sigma(0,.),U^*,0)$$

$$=\deg(\langle\sigma\beta T(v^1)+H(\overline{\lambda},v^1),\psi^1\rangle,U^*,0)\neq 0.$$

It then follows that for any $\alpha\in I_0$ there exists $\beta^\sigma(\alpha)\in U^*$ with $N^\sigma(\alpha,\beta^\sigma(\alpha))=0$. Multiplying both sides of this equality with $|\alpha|^a$ we obtain

$$\langle\sigma|\alpha|^{a-1}\beta^\sigma(\alpha)T(|\alpha|v^1+\psi(|\alpha|^{a-1}\beta(\alpha),|\alpha|))+$$

$$(1+\sigma|\alpha|^{a-1}(\alpha))H\Big(\frac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1}\beta^\sigma(\alpha))^b},\ |\alpha|v^1+\psi(|\alpha|^{a-1}\beta^\sigma(\alpha),\alpha)\Big)+$$

$$(1+\sigma|\alpha|^{a-1}(\alpha))K\Big(\frac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1}\beta^\sigma(\alpha))^b},\ (|\alpha|v^1+\psi(|\alpha|^{a-1}\beta^\sigma(\alpha),|\alpha|)\Big),\psi^1\rangle=0$$

Together with the fact

$$\langle T(|\alpha|v^1+\psi(|\alpha|^{a-1}\beta^\sigma(\alpha),|\alpha|))+L(\overline{\lambda},|\alpha|v^1+\psi(|\alpha|^{a-1}\beta^\sigma(\alpha),|\alpha|)),\psi^1\rangle=0$$

we obtain

$$P_Y(T(v^\sigma(\alpha))+L(\lambda^\sigma(\alpha),v^\sigma(\alpha))+H(\lambda^\sigma(\alpha),v^\sigma(\alpha))+K(\lambda^\sigma(\alpha),v^\sigma(\alpha)))=0$$

with

$$\lambda^\sigma(\alpha) = \frac{\overline{\lambda}}{(1+\sigma|\alpha|^{a-1}\beta^\sigma(\alpha))^b},$$

and

$$v^\sigma(\alpha) = |\alpha|v^1 + \psi(|\alpha|^{a-1}\beta^\sigma(\alpha), |\alpha|).$$

Consequently, to complete the proof of the theorem it remains to apply the assertion (iv) of Theorem 3.

4. Hopf bifurcation problem

In this section we consider the existence of Hopf bifurcation points of the equation (7) with $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ defined as in Section 1. Let $\overline{\lambda} \in \Lambda$ be such that the linear mapping $T+L(\overline{\lambda}, .)$ is Fredholm and has $\pm i\beta_0$ as eigenvalues with multiplicity $p \geq 1$. For the sake of simplicity of writting we only restrict our consideration to the case when $T + L(\overline{\lambda}, .)$ has no eigenvalue of the form $\pm ni\beta_0$ with $n = 0, 2, 3, ...$ Without loss of generality we also assume that $\beta_0 = 1$. Let

$$\text{Ker } (T + L(\overline{\lambda}, .) + iI) = [v^1, ..., v^p],$$

$$\text{Ker } (T + L(\overline{\lambda}, .) + iI)^* = [\gamma^1, ..., \gamma^p],$$

and

$$\text{Ker } \big(\frac{d}{d\tau} + T + L(\overline{\lambda}, .)\big) = [\phi^1, ..., \phi^{2p}],$$

$$\text{Ker } \big(\frac{d}{d\tau} + T + L(\overline{\lambda}, .)\big)^* = [\psi^1, ..., \gamma^{2p}],$$

with

$$\phi^{2k-1}(t) = \text{Re } (e^{it}v^k), \phi^{2k}(t) = \text{Im } (e^{it}v^k),$$

$$\psi^{2k-1}(t) = \text{Re } (e^{it}\gamma^k), \psi^{2k}(t) = \text{Im } (e^{it}\gamma^k).$$

Further, let $\mathcal{X}_k, \mathcal{Y}_k$, $k = 0, 1$, $P_\mathcal{X}$, $P_\mathcal{Y}$, $Q_\mathcal{X}$, $Q_\mathcal{Y}$ be the same as in Section 2. For $\sigma = 1$ or $\sigma = -1$ we define the following mappings $\mathcal{A}$, $\mathcal{B}^\sigma$, $\mathcal{C}^\sigma$, $\mathcal{D}^\sigma : R^{2p} \to R^{2p}$, $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_{2p})$, $\mathcal{B}^\sigma = (\mathcal{B}^\sigma_1, ..., \mathcal{B}^\sigma_{2p})$, $\mathcal{C}^\sigma = (\mathcal{C}^\sigma_1, ..., \mathcal{C}^\sigma_{2p})$, $\mathcal{D}^\sigma = (\mathcal{D}^\sigma_1, ..., \mathcal{D}^\sigma_{2p})$, by

$$\mathcal{A}_k(x) = \ \Big\langle H\Big(\sum_{j=1}^{2p} x_j\phi^j\Big), \psi^k\Big\rangle,$$

$$\mathcal{B}_k^\sigma(x) = -\sigma\Big\langle \frac{d\big(\sum_{j=1}^{2p} x_j\phi^j\big)}{d\epsilon}, \psi^k\Big\rangle + \mathcal{A}_k(x),$$

$$\mathcal{C}_k^\sigma(x) = \sigma\Big\langle T\Big(\sum_{j=1}^{2p} x_j\phi^j\Big), \psi^k\Big\rangle + \mathcal{A}_k(x),$$

$$\mathcal{D}_k^\sigma(x) = -\sigma\Big\langle L(\overline{\lambda}, \sum_{j=1}^{2p} x_j\phi^j), \psi^k\Big\rangle + \mathcal{A}_k(x),$$

$$k = 1, ..., 2p, \quad x = (x_1, ..., x_{2p})$$

and make the following hypotheses on there mappings.

Hypothesis 16. There exists an open subset Ω of $\mathbf{R}^{2p}$ with $0 \notin \overline{\Omega}$ such that the topological degree, $\deg(\mathcal{A}, \Omega, 0)$, of the mapping $\mathcal{A}$ with respect to Ω and the origin is defined and different from zero.

Hypothesis 17. Hypothesis 16 with Ω and $\mathcal{A}$ replaced by Ω_σ and $\mathcal{B}^\sigma$, respectively.

Hypothesis 18. Hypothesis 16 with Ω and $\mathcal{A}$ replaced by Ω_σ and $\mathcal{C}^\sigma$, respectively.

Hypothesis 19. Hypothesis 16 with Ω and $\mathcal{A}$ replaced by Ω_σ and $\mathcal{D}^\sigma$, respectively.

As in Section 2 we have seen that Hypothesis 8 is satisfied.

We have

Theorem 9. *Let Hypotheses 9 and 10 be satisfied and t_0, ψ exist by Theorem 4. In addition, assume that one of the following is satisfied:* (a) *Hypothesis 16;* (b) *Hypothesis 17;* (c) *Hypothesis 18;* (d) *Hypothesis 19. Then $(\overline{\lambda}, 0)$ is a Hopf bifurcation point of periodic solutions of the equation* (7)*. More precisely, there exists a neighborhood I_0 of zero in $R, I_0 \subset t_0 I_1$ such that for any $\alpha \in I_0$ one can find $x^\sigma(\alpha) = (x^\sigma(\alpha)_1, ..., x^\sigma(\alpha)_{2p})) \in \Omega_\sigma$ (for the case* (a) *$\Omega_\sigma = \Omega$) such that $(\rho^\sigma(\alpha), \lambda^\sigma(\alpha), v^\sigma(\alpha))$ satisfies the equation* (8) *where*

$$\rho^\sigma(\alpha) = \sigma|\alpha|^c,$$

$$\lambda^\sigma(\alpha) = \frac{\overline{\lambda}}{(1 + \sigma|\alpha|^d)^b},$$

and

$$v^\sigma(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j^\sigma(\alpha)\phi^j + \psi(\sigma|\alpha|^c, \sigma|\alpha|^d, |\alpha| x^\sigma(\alpha))$$

with $c = d = a$ *for the case* (a)*;* $c = a - 1, d = a$ *for the case* (b)*;* $c = a - 1, d = a - 1$ *for the case* (c) *and* $c = a, d = a - 1$ *for the case* (d)*. Moreover, we have* $\rho^\sigma(\alpha) \to 0$, $\lambda^\sigma(\alpha) \to \overline{\lambda}$, $v^\sigma(\alpha) \to 0$ *as* $\alpha \to 0$, $v(\alpha) \neq 0$ *for* $\alpha \neq 0$ *and the mapping* $u^\sigma(\alpha)(t) = v^\sigma(\alpha)\big(\frac{t}{1+\rho^\sigma(\alpha)}\big)$ *is* $(1+\rho^\sigma(\alpha))2\pi$ *- periodic.*

Proof. Applying the assertion (iv) of Theorem 4 we need to verify that $(\sigma|\alpha|^c, \sigma|\alpha|^d, v^\sigma(|\alpha|))$ as above satisfies the equation

$$P\mathcal{Y}\Big\{\frac{d\big(\sum_{j=1}^{2p} x_j\phi^j + \psi(\rho,\beta,x)\big)}{d\tau} + (1+\rho)\Big(T\Big(\sum_{j=1}^{2p} x_j\phi^j + \psi(\rho,\beta,x)\Big)+$$

$$L\Big(\frac{\overline{\lambda}}{(1+\beta)^b}, \sum x_j\phi^j + \psi(\rho,\beta,x)\Big) + H\Big(\frac{\overline{\lambda}}{(1+\beta)^b}, \sum_{j=1}^{2p} x_j\phi^j + \psi(\rho,\beta,x)\Big)$$

$$+K\Big(\frac{\overline{\lambda}}{(1+\beta)^b}, \sum_{j=1}^{2p} x_j\phi^j + \psi(\rho,\beta,x)\Big)\Big)\Big\} = 0, \tag{9}$$

for all α belonging to some neighborhood $I_0 \subset t_0 I_1$. Indeed, for $\sigma = 1$ or $\sigma = -1, c, d = a, a-1$, we define the mapping $S^{\sigma cd} : t_0I_1 \times t_0U_1 \to R^{2p}$, $S^{\sigma cd} = (S_1^{\sigma cd} ..., S_{2p}^{\sigma cd})$, by

$$S_k^{\sigma cd}(\alpha, x)$$
$$= \begin{cases} \Big\langle(\sigma|\alpha|^{d-a+1} - |\alpha|^{c-a+1})\dfrac{d\big(\sum_{j=1}^{2p} x_j\phi^j + \psi(\sigma|\alpha|^c, \sigma|\alpha|^d, |\alpha|x)\big)}{d\tau} \\ +(\sigma|\alpha|^{d-a-1} + |\alpha|^{(c+d)-a-1})T\Big(\sum_{j=1}^{2p} x_j\phi^j + \dfrac{\psi(\sigma|\alpha|^c, \sigma|\alpha|^d, |\alpha|x)}{|\alpha|}\Big) \\ +(1+\sigma|\alpha|^d)(1+\sigma|\alpha|^c)\Big(H\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^d)^b}\sum_{j=1}^{2p} x_j\phi^j \\ +\dfrac{\psi(\sigma|\alpha|^c, \sigma|\alpha|^d, |\alpha|x)}{|\alpha|}\Big) + |\alpha|^{-a}K\Big(\dfrac{\overline{\lambda}}{(1+\sigma|\alpha|^d)^b}, \sum_{j=1}^{2p} x_j\phi^j \\ +\psi(\sigma|\alpha|^c, \sigma|\alpha|^d, |\alpha|x)\Big)\Big), \psi^k\Big\rangle \text{ for } \alpha \neq 0, \\ \mathcal{F}_k^{\sigma cd}(x), \quad \text{for} \quad \alpha = 0, \end{cases}$$

where

$$\mathcal{F}_k^{\sigma aa}(x) = \mathcal{A}_k(x) \quad \text{(for the case (a))},$$
$$\mathcal{F}_k^{\sigma(a-1)a}(x) = \mathcal{B}_k^{\sigma}(x) \quad \text{(for the case (b))},$$
$$\mathcal{F}_k^{\sigma(a--1)(a-1)}(x) = \mathcal{C}_k^{\sigma}(x) \quad \text{(for the case (c))},$$
$$\mathcal{F}_k^{\sigma a(a-1)}(x) = \mathcal{D}_k^{\sigma}(x) \quad \text{(for the case (d))}, \quad k = 1, ..., 2p.$$

By the same arguments as in the proof of Theorem 5 we conclude that $S^{\sigma cd}$ is a continuous mapping and there exists a neighborhood $I_0 \subset t_0 I_1$ such that $S^{\sigma cd}(t\alpha, x) \neq 0$ for all $\alpha \in I_0, t \in [0,1]$ and $x \in \partial\Omega_\sigma$. Thus the topological degree, $\deg(S^{\sigma cd}, \Omega_\sigma, o)$ is defined for c, d corresponding to the case (a) $(c = d = a)$, the case (b) $(c = a-1, d = a)$, the case (c) $(c = a-1, d = a-1)$, the case (d) $(c = a, d = a-1)$ provided one of these case is satisfied. It then follows that for any fixed $\alpha \in I_0$ we have in any case

$$\deg(S^{\sigma cd}(\alpha, .), \Omega_\sigma, 0) = \deg(\mathcal{F}^{\sigma cd}, \Omega_\sigma, 0) \neq 0.$$

Therefore, for any $\alpha \in I_0$ there exists $x^\sigma(\alpha) \in \Omega_\sigma$ such that $S^{\sigma cd}(\alpha, x^\sigma(\alpha)) = 0$. Multiplying both sides of this equality with $|\alpha|^{-a}, \alpha \neq 0$, we obtain

$$\Big\langle \sigma(|\alpha|^d - |\alpha|^c)\frac{d(v^\sigma(\alpha))}{d\tau} + (\sigma|\alpha|^d + |\alpha|^{c+d})T(v^\sigma(\alpha))$$
$$(1+\sigma|\alpha|^d)(1+|\alpha|^c|(H(\lambda^\sigma(\alpha), v^\sigma(\alpha)) +$$
$$K(\lambda^\sigma(\alpha), v^\sigma(\alpha))), \psi^k \Big\rangle = 0, \quad k = 1, ..., 2p,$$

with

$$\lambda^\sigma(\alpha) = \frac{\overline{\lambda}}{(1+\sigma|\alpha|^d)^b},$$

and

$$v^\sigma(\alpha) = \sum_{j=1}^{2p} |\alpha| x_j \phi_j + \psi(\sigma|\alpha|^c, \sigma|\alpha|^d, |\alpha| x)).$$

Together with the fact

$$\Big\langle \frac{d(v^\sigma(\alpha))}{d\tau} + T(v^\sigma(\alpha)) + L(\overline{\lambda}, v^\sigma(\alpha)), \psi^k \Big\rangle = 0, \quad k = 1, ..., 2p,$$

we obtain

$$\Big\langle (1+\sigma|\alpha|^d)\frac{d(v^\sigma(\alpha))}{d\tau} + (1+\sigma|\alpha|^d)((1+\sigma|\alpha|^c)T(v^\sigma(\alpha))$$

$$+(1+\sigma|\alpha|^c)L(\overline{\lambda}, v^\sigma(\alpha))(1+\sigma|\alpha|^d)(1+\sigma|\alpha|^c)(H(\lambda^\sigma(\alpha), v^\sigma(\alpha))$$
$$+K(\lambda^\sigma(\alpha), v^\sigma(\alpha))), \psi^k\rangle \ = 0,$$

or

$$\langle\frac{v^\sigma(\alpha))}{d\epsilon} + (1+\sigma|\alpha|^c)\{T(v^\sigma(\alpha)) + L(\lambda^\sigma(\alpha), v^\sigma(\alpha))$$
$$+H(\lambda^\sigma(\alpha), v^\sigma(\alpha)) + K(\lambda^\sigma(\alpha))\}, \psi^k\rangle \ = 0, \quad k = 1, ..., 2p.$$

This implies

$$P_{\mathcal{Y}}\Big\{\frac{d(v^\sigma(\alpha))}{d\epsilon} + (1+\rho^\sigma(\alpha))(T(v^\sigma(\alpha) + L(\lambda^\sigma(\alpha), v^\sigma(\alpha))+$$
$$H(\lambda^\sigma(\alpha), v^\sigma(\alpha) + K(\lambda^\sigma(\alpha), v^\sigma(\alpha)))\Big\} = 0$$

with $\rho^\sigma(\alpha) = \sigma|\alpha|^c$.

This means that $(\rho^\sigma(\alpha), \lambda^\sigma(\alpha), v^\sigma(\alpha)), \alpha \in I_0$, satisfies the equation (9) in any case (a), (b), (c) and (d). It is obvious that $\rho^\sigma(\alpha) \to 0$, $\lambda^\sigma(\alpha) \to \overline{\lambda}$, $v^\sigma(\alpha) \to 0$ and $v^\sigma(\alpha) \neq 0$ if $\alpha \neq 0$. Consequently, to complete the proof of the theorem it remains to apply the assertion (iv) of Theorem 4.

Remark 4. As in Remark 3, if Ω^j_σ, $j = 1, 2$, satisfy one of Hypotheses 16-19 with $\Omega^1_\sigma \cap \Omega^2_\sigma = \emptyset$ then $(\rho^{1\sigma}(\alpha), \lambda^{1\sigma}(\alpha), v^{1\sigma}(\alpha)) \neq (\rho^{1\sigma}(\alpha), \lambda^{2\sigma}(\alpha), v^{2\sigma}(\alpha))$ for all $I^1_0 \cap I^2_0$, where $I^j_0, (\rho^{j\sigma}(\alpha), \lambda^{j\sigma}(\alpha), v^{j\sigma}(\alpha))$ exist by Theorem 9.

Remark 5. We assume that if there exists a point $\overline{x}^\sigma \in \Omega_\sigma$ with $\mathcal{F}^{\sigma cd}(\overline{x}^\sigma) = 0$ and

$$\beta = \det\Big(\frac{\partial \mathcal{F}^{\sigma cd}_k}{\partial x_j}(\overline{x}^\sigma)\Big)_{k,j=1,\dots,2p} \neq 0, \tag{10}$$

then one of Hypotheses 16-19 for (a) $c = d = a$, (b) $c = a-1$, $d = a$, (c) $c = a-1$, $d = a-1$, (d) $c = a$, $d = a-1$, respectively is satisfied. Indeed, the condition (10) implies that there exists a neighborhood Ω_σ of the point $\overline{x}^\sigma$ such that

$$\mathrm{Sign}\det\Big(\frac{\partial \mathcal{F}^{\sigma cd}_k}{\partial x_j}(x^\sigma)\Big)_{k,j=1,\dots,2p} = \mathrm{Sign}\beta,$$

and hence

$$\deg(\mathcal{F}^{\sigma cd}, \Omega_\sigma, c) = \begin{cases} 1 & \text{if} \quad \beta > 0, \\ -1 & \text{if} \quad \beta < 0. \end{cases}$$

To the conclusion of this section we consider the equation (2) in the case when $\Lambda = \mathbf{R}$ and $\overline{\lambda} \in \Lambda$ is such that the mapping $T + L(\overline{\lambda}, .)$ is Fredholm and has only $\pm i$ as a simple eigenvalues, i.e. $p = 1$. Let ϕ^j, ψ^j, $j = 1, 2$, $P_{\mathcal{Y}}$, $Q_{\mathcal{Y}}$ etc. be defined as above. In addition, we assume that the mapping $L(., u)$ for any fixed $u \in \mathcal{D}$, is a C^ℓ-mapping with $\ell \geq 2$ and the mapping $H(\overline{\lambda}, .)$ is an a-linear form with an odd number (for the sake of simplicity of writing, we take $a = 3$), the mappings H and K satisfy Hypothesis 10. In this case the equation (8) can be rewritten as

$$\dot{u} + (1+\rho)\Big\{T(u) + L(\overline{\lambda}, u) + \sum_{j=1}^{r} \frac{1}{j!} L_{\lambda\ldots\lambda}(\overline{\lambda}, 0)(\lambda - \overline{\lambda})^j(u) + \mathcal{O}(|\lambda - \overline{\lambda}|^{r+1}) + H(\lambda, u) + K(\lambda, u)\Big\} = 0, \tag{11}$$

where r is some integer less than ℓ; $L_{\lambda\ldots\lambda}(\overline{\lambda}, 0)$ ($\lambda\ldots\lambda - j$ times) denotes the j-th partial derivative of L with respect to λ. Setting $\alpha = \lambda - \overline{\lambda}$, we obtain

$$\dot{u} + (1+\rho)\Big\{T(u) + L(\overline{\lambda}, u) + \sum_{j=1}^{r} \frac{\alpha^j}{j!} L_{\lambda\ldots\lambda}(\overline{\lambda}, 0)(u) + \mathcal{O}(|\alpha|^{r+1}) + H(\overline{\lambda} + \alpha, u) + K(\overline{\lambda} + \alpha, u)\Big\} = 0.$$

By the same method as in the proof of Theorem 4, we conclude that there exists a point $t_0 \in (0, 1]$ such that for any $\rho, \alpha \in t_0 I_1, x \in t_0 U_1$ one can find a unique $\psi(\rho, \alpha, x) \in t_0 \mathcal{D}_1$ satisfying

$$\begin{aligned} Q_{\mathcal{Y}}\Big\{ &\frac{d\big(\sum_{j=1}^{2} x_j\phi^j + \psi(\rho,\alpha,x)\big)}{d\tau} + (1+\rho)\Big(T\Big(\sum_{j=1}^{2} x_j\phi^j + \psi(\rho,\alpha,x)\Big) + \\ &L\Big(\overline{\lambda}, \sum_{j=1}^{2} x_j\phi^j + \psi(\rho,\alpha,x)\Big) + \sum_{k=1}^{r} \frac{\alpha^k}{k!} L_{\lambda\ldots\lambda}\Big(\sum_{j=1}^{2} x_j\phi^j + \\ &\psi(\rho,\alpha,x)) + \mathcal{O}(|\alpha|^{r+1}\Big) + H\Big(\overline{\lambda} + \alpha, \sum_{j=1}^{2} x_j\phi^j + \psi(\rho,\alpha,x)\Big) \\ &+ K\Big(\overline{\lambda} + \alpha, \sum_{j=1}^{2} x_j\phi^j + \psi(\rho,\alpha,x)\Big)\Big)\Big\} = 0 \end{aligned} \tag{12}$$

with $\psi(\rho,\alpha,x)=0$ and $\dfrac{||\psi(z\rho,z\alpha,z(1,\beta_0))||}{|z|}\to 0$ as $z\to 0$, for any fixed $\beta_0\neq 0$.

Taking fixed $\beta_0\neq 0$ and setting $I_2=t_0I_1$ we define the mapping $h:I_2\times I_2\times I_2\times I_2\to R^2$, $h=(h_1,h_2)$, by

$$h_k(\rho,\alpha,z)$$
$$=\begin{cases}\Big\langle -\rho\dfrac{d(\phi^1+\beta\phi^2)}{d\tau}+(1+z\rho)\Big\{\displaystyle\sum_{j=1}^{r}\dfrac{\alpha^j z^{j-1}}{j!}L_{\lambda\ldots\lambda}(\overline{\lambda},0)(\phi^1+\beta_0\phi^2)\\ +zH(\overline{\lambda}+z\alpha,\phi^1+\beta\phi^2)+h.o.t\Big\},\psi^k\Big\rangle,\quad k=1,2,\ z\neq 0,\\ \Big\langle -\rho\dfrac{d(\phi^1+\beta_0\phi^2)}{d\tau}+\alpha L_\lambda(\overline{\lambda},\phi^1+\beta\phi^2),\psi^k\Big\rangle,\ z=0.\end{cases}$$

Further, put

$$\begin{aligned} a=&\ \langle H(\overline{\lambda},\phi^1),\psi^1\rangle,\\ b=&\ \langle H(\overline{\lambda},\phi^2),\psi^1\rangle,\\ \theta_j=&\ \langle L_{\lambda\ldots\lambda}(\overline{\lambda},\phi^1),\psi^1\rangle,\\ \eta_j=&\ \langle L_{\lambda\ldots\lambda}(\overline{\lambda},\phi^2),\psi^1\rangle.\end{aligned}$$

A simple calculation shows

$$h_1(\rho,\alpha,z)=\begin{cases}-\rho\beta_0+(1+2\rho)\Big\{\sum\limits_{j=1}^{r}\dfrac{\alpha^j z^{j-1}}{j!}(\theta_j+\beta_0\eta_j)+\\ z(a-\beta_0b+\beta_0^2a-b\beta_0^3)+h.o.t\Big\},\quad \text{for}\quad z\neq 0,\\ -\rho\beta_0+\alpha(\theta_1+\beta_0\eta_1),\quad \text{for}\quad z=0,\end{cases}$$

$$h_2(\rho,\alpha,z)=\begin{cases}\rho\beta_0+(1+2\rho)\Big\{\sum\limits_{j=1}^{r}\dfrac{\alpha^j z^{j-1}}{j!}(\eta_j+\beta_0\theta_j)+\\ z(b+\beta_0a+\beta_0^2b+b\beta_0^3b)+h.o.t\Big\},\quad \text{for}\quad z\neq 0,\\ \rho+\alpha(-a(\eta_1+\beta_0\theta_1),\quad \text{for}\quad z=0,\end{cases}$$

It is clear that $h(0,0,0)=0$ and

$$\frac{\partial h}{\partial(\rho,\alpha)}(0,0,0)=\begin{pmatrix}-\beta_0,\theta_1+\beta_0\eta_1\\ 1,-\eta_1+\beta_0\theta_1\end{pmatrix}.$$

Hence

$$\det(\partial(\rho,\alpha)(0,0,0) = -\theta_1(1+\beta_0^2).$$

Therefore, if $\theta_1 \neq 0$ then there exists an open subset Ω in $\mathbf{R}^2$ not containing zero such that the topological degree $\deg(h(.,.,0),\Omega,0)$, with respect to Ω and zero is defined and $\deg(h(.,.,0),\Omega) \neq 0$. We have

Theorem 10. *If $\theta_1 \neq 0$, then $(\overline{\lambda},0)$ is a Hopf bifurcation point of periodic solutions of the equation* (8)*. More precisely, to given $\beta_0 \neq 0$, there exists a neighbourhood I_0 of zero in R such that for any $z \in I_0$ one can find $(\rho(z),\alpha(z)) \in \Omega$ for which $(z\rho(z),\lambda(z),v(z)))$ with*

$$\lambda(z) = \overline{\lambda} + z\alpha(z),$$

$$v(z) = z\phi^1 + z\beta\phi^2 + \psi(z\rho(z), z\alpha(z), z(1,\beta_0)),$$

satisfies the equation (11)*, $\lambda(z) \to \overline{\lambda}, z\rho(z) \to 0, v(z) \to 0$ as $z \to 0$, $v(z) \neq 0$ for $z \neq 0$ and the mapping $u(z)(t) = \dfrac{t}{1+z\rho(z)}$ is $(1+z\rho(z))2\pi$-periodic.*

Proof. Since $\theta_1 \neq 0$, there exists a neighborhood Ω as above such that $\deg(h(.,.,\Omega),\Omega,0) \neq 0$. By the same manner as in the proof of Theorem 5 there exist a neighborhood I_0 of zero in R, $I_0 \subset I_2$ such that $h(\rho,\alpha,tz) \neq 0$ for all $(\rho,\alpha) \in \partial\Omega$, $t \in [0,1]$, $z \in I_0$. It follows that the topological degree $h(.,.,tz)$ is defined and

$$\deg(h(.,.,z),\Omega,0) = \deg(h(.,.,0),\Omega,0) \neq 0.$$

Therefore for any $z \in I_0$ there is $(\rho(z),\alpha(z)) \in \Omega$ with $h(\rho(z),\alpha(z),z) = 0$. Multiplying both sides of this equality with z^2 we obtain

$$\Big\langle -z\rho(z)\frac{d(v(z))}{d\tau} + (1+z\rho(z))\Big\{\sum_{j=1}^{r}\frac{(z\alpha(z))^j}{j!}L_{\lambda\ldots\lambda}(\overline{\lambda},0)(v(z))$$

$$+H(\lambda(z),v(z)) + K(\lambda(z),v(z))\Big\},\psi^k\Big\rangle = 0, \quad k=1,2,$$

with $v(z) = z\phi^1 + z\beta_0\phi^2 + \psi(z\rho(z), z\alpha(z), (z,\beta_0 z))$, $\lambda(z) = \overline{\lambda} + z\alpha(z)$. Together with the fact that

$$(1+z\rho(z))\Big\langle \frac{d(v(z))}{d\tau} + T(v(z)) + L(\overline{\lambda},v(z)),\psi^k\Big\rangle = 0, \quad k=1,2,$$

$$\begin{aligned}(13)\qquad &\Big\langle \frac{d(v(z))}{d\tau} + (1+z\rho(z))\{T(v(z)) + L(\lambda, v(z)) + \\ &H(\lambda(z), v(z)) + K(\lambda(z), v(z))\}, \psi^k \Big\rangle = 0.\end{aligned}$$

A combination of (12) and (13) yields

$$\frac{d(v(z))}{d\tau} + (1+z\rho(z))\{T(v(z)) + L(\lambda, v(z)) +$$

$$H(\lambda(z), v(z)) + K(\lambda(z), v(z))\} = 0.$$

Obviously, we have $\lambda(z) \to \overline{\lambda}$, $z\rho(z) \to 0$, $v(z) \to 0$ and $v(z) \neq 0$ for $z \neq 0$ and the mapping $u(z)$ defined as above is $(1 + z\rho(z))2\pi$-periodic. This completes the proof of the theorem.

Next, we consider the case $\theta_1 = 0$, $\eta_1 \neq 0$. In this case the matrix $\dfrac{\partial h}{\partial(\rho, \alpha)}(0, 0, 0)$ has the form

$$B = \frac{\partial h}{\partial(\rho, \alpha)}(0, 0, 0) = \begin{pmatrix} -\beta_0, \beta_0\eta_1 \\ 1, -\eta_1 \end{pmatrix},$$

which is singular and we can not use the above proof.

We set

$$\begin{aligned}
&R_0 = \{y \in R^2 \ / \ By = 0\} = \left[\begin{pmatrix} \eta_1 \\ 1 \end{pmatrix}\right] = [\varphi], \\
&R_1 = \{y \in R^2 \ / \ (y, \varphi) = 0\} = \left[\begin{pmatrix} 1 \\ -\eta_1 \end{pmatrix}\right] = [\xi], \\
(14)\qquad &R_0^* = \{y \in R^2 \ / \ By^* = 0\} = \left[\begin{pmatrix} 1 \\ \beta_0 \end{pmatrix}\right] = [\varphi^*], \\
&R_1^* = \{y \in R^2 \ / \ (y, \varphi^*) = 0\} = \left[\begin{pmatrix} -\beta_0 \\ 1 \end{pmatrix}\right] = [\xi^*].
\end{aligned}$$

One can easily verify that $R^2 = R_0 \oplus R_1 = R_0^* \oplus R_1^*$, the linear mapping B maps R_0 into R_0^* and R_1 onto R_1^*. Let P_0, Q_0, P_0^*, Q_0^* be the projectors of R^2 onto R_0, R_1, R_0^*, R_1^*, respectively. Then the totality of solutions of the equation

$$h(\rho, \alpha, z) = 0$$

can be reduced to solving two equations

$$\begin{aligned} Q_0 h(\rho, \alpha, z) &= 0, \\ P_0 h(\rho, \alpha, z) &= 0. \end{aligned}$$

A simple calculation yields

$$Q_0 h(\rho, \alpha, z)$$
$$= \begin{cases} -(\beta_0 + \eta_1)\rho + (1 + z\rho)\Big\{ \sum\limits_{j=1}^{r} \dfrac{\alpha^j z^{j-1}}{j!}((\beta_0 + \eta_1)\eta_j + \theta_j(1 - \beta_0\eta_1)) \\ +z(1 + \beta_0^2)(a(1 - \beta_0\eta_1) - b(\beta_0 + \eta_1)) + h.o.t\Big\} \text{ for } z \neq 0, \\ -(\beta_0 + \eta_1)\rho + \alpha(\beta_0 + \eta_1)\eta_1, \text{ for } z = 0. \end{cases}$$

$$P_0 h(\rho, \alpha, z)$$
$$= \begin{cases} (1 - \beta_0\eta_1)\rho + (1 + z\rho)\Big\{ \sum\limits_{j=1}^{r} \dfrac{\alpha^j z^{j-1}}{j!}((\beta_0 + \eta_1) + \theta_j - (1 - \beta_0\eta_1)\eta_j) \\ +z(1 + \beta_0^2)(a(\eta_1 + \beta_0) + b(1 - \beta_0\eta_1)) + h.o.t\Big\} \text{ for } z \neq 0, \\ (1 - \beta_0\eta_1)\rho - \alpha(1 - \beta_0\eta_1)\eta_1, \text{ for } z = 0. \end{cases}$$

By the same arguments as in the proof of Theorem 1, there exists a point $\bar{t} \in (0, 1]$ such that for any $\alpha, z \in \bar{t}I_2$ one can find $\rho(\alpha, z) \in \bar{t}I_2$, $\rho(0, 0) = 0$ satisfying

$$Q_0 h(\rho(\alpha, z), \alpha, z) = 0 \text{ and } \lim_{z \to 0} |\frac{\rho(\alpha z, z)}{z}| = 0. \tag{15}$$

Further, we define the mapping $\ell : \bar{t}I_2 \times \bar{t}I_2 \to \mathbf{R}$ by

$$\ell(\alpha, z) = \begin{cases} (1 - \beta_0\eta_1)\frac{\rho(\alpha z, z)}{z} + \\ (1 + z\rho(\alpha z, z))\Big\{ \sum\limits_{j=1}^{r} \alpha^j z^{2(j-1)}((\beta_0 + \eta_1)\theta_j - (1 - \beta_0\eta_1)\eta_j) \\ +(1 + \beta_0^2)(a(\eta_1 + \beta_0) + b(1 - \beta_0\eta_1)) + h.o.t\Big\} \text{ for } z \neq 0, \\ -\alpha(1 - \beta_0\eta_1)\eta_1, \text{ for } z = 0. \end{cases}$$

We have $\ell(\alpha, 0) = -\alpha(1 - \beta_0\eta_1)\eta_1$. Take $\beta_0 \neq \dfrac{1}{\eta_1}$ we have $\dfrac{\partial \ell}{\partial \alpha}(0, 0) = -(1 - \beta_0\eta_1) \neq 0$. It then follows that there exists a neighborhood $\overline{I}$ of zero in R such that the topological degree, $\deg(\ell(.,0),\overline{I},0)$, of $\ell(.,0)$ with respect to $\overline{I}$ and zero is defined. By the definition of the topological degree we have

$$\deg(\ell(.,0),\overline{I},0)\neq 0.$$

By the same proof as in Theorem 5, we conclude that there exists a neighborhood $\overline{I}_0$ of zero such that for any $z\in\overline{I}_0$ one can find $\alpha(z)\in I$ with $\ell(\alpha(z),z)=0$. Multiplying both sides of this equality with z we obtain

$$(1-\beta_0\eta_1)(\rho(\alpha(z)z,z)+(1+z\rho(z\alpha(z),z)\Big\{\sum_{j=1}^{r}\frac{(z\alpha(z))^j z^{j-1}}{j!}((\beta_0+\eta_1)\theta_j$$

$$-(1-\beta_0\eta_1)\eta_j)+z(1+\beta_0^2)(a(\eta_1+\beta_0)+b(1-\beta_0\eta_1))+h.o.t\Big\}=0,$$

or

$$P_0h(\rho(z\alpha(z),z),\alpha(z),z))=0. \tag{16}$$

A combination of (15) and (16) yields

$$h(\rho(z\alpha(z),z),\alpha(z),z)=0.$$

Therefore, we have

Theorem 11. *If $\eta_1\neq 0$, then $(\overline{\lambda},0)$ is a Hopf bifurcation point of periodic solutions of the equation* (8). *More precisely, to given $\beta_0\neq 1/\eta_1$ there exists a neighborhood I_0 of zero in $\mathbf{R}$ such that for any $z\in I_0$ one can find $(\alpha(z),\rho(z\alpha(z),z)$ for which $z\rho(z\alpha(z),\alpha(z),z)$, $\lambda(z)$, $v(z)$ with*

$$\lambda(z)=\overline{\lambda}+z\alpha(z),$$

$$v(z)=z\phi^1+z\beta_0\phi^2+\psi(z\rho(z\alpha(z)),\alpha(z),z\alpha(z),(z,z\beta_0)),$$

satisfies the equation (11), *$\lambda(z)\to\overline{\lambda}$, $z\rho(z\alpha(z))\to 0$, $v(z)\to 0$ as $z\to 0$, $v(z)\neq 0$ for $z\neq 0$, and the mapping $u(z)=v(z)\big(\frac{t}{1+z\rho(z\alpha(z),z)}\big)$ is $(1+z\rho(z\alpha(z),z))2\pi$-periodic.*

Proof. The proof of this theorem follows from (15) and the proof of Theorem 10.

Next, we consider the case $\theta_1 = \eta_1 = 0$. We assume that $c > 1$ is the smallest integer such that $\theta_c \neq 0$. We define the mapping $f : I_2 \times I_2 \times I_2 \to R^2$, $f = (f_1, f_2)$, by

$$\begin{cases} \Big\langle -\rho\frac{d(\phi^1+\beta_0\phi^2)}{d\tau} + (1+z\rho)\Big\{ \sum\limits_{j=1}^{r} \dfrac{\alpha^j z^{j-1}}{j!} L_{\lambda\ldots\lambda}(\overline{\lambda},0)(\phi^1+\beta_0\phi^2) \\ +z^{c-1} H(\overline{\lambda}+z\alpha, \phi^1+\beta_0\phi^2) + h.o.t\Big\}, \psi\Big\rangle, \text{ for } z \neq 0, \\ \Big\langle -\rho\dfrac{d(\phi^1+\beta_0\phi^2)}{d\tau} + \alpha L_1(\overline{\lambda},0)(\phi^1+\beta\phi^2), \psi^k\Big\rangle >, \text{ for } z = 0,\ k = 1,2. \end{cases}$$

A simple calculation yields

$$f_1(\rho,\alpha,z) = \begin{cases} -\rho\beta_0 + (1+z\rho)\Big\{ \sum\limits_{j=c}^{r} \dfrac{\alpha^j z^{j-1}}{j!}(\theta_j + \beta_0\eta_j) + \\ +z^{c-1}(a - \beta_0 b + \beta_0^2 a - \beta_0^3 b) + h.o.t\Big\}, \quad \text{for } z \neq 0, \\ -\rho\beta_0, \text{ for } z = 0. \end{cases}$$

$$f_2(\rho,\alpha,z) = \begin{cases} \rho + (1+z\rho)\Big\{ \sum\limits_{j=c}^{r} \dfrac{\alpha^j z^{j-1}}{j!}(-\eta_j + \beta_0\theta_j) + \\ +z^{c-1}(b + \beta_0 a + \beta_0^2 b + \beta_0^3 b) + h.o.t\Big\}, \quad \text{for } z \neq 0, \\ \rho, \text{ for } z = 0. \end{cases}$$

We have

$$f(\rho,\alpha,0) = (-\rho\beta_0, \rho),$$

and

$$C = \Big(\frac{\partial f_k}{\partial(\rho,\alpha)}(0,0,0)\Big)_{k=1,2} = \begin{pmatrix} -\beta_0, 0 \\ 1, 0 \end{pmatrix}.$$

Let R_0, R_1, R_0^*, R_1^* be defined as in (14) with $\eta_1 = 0$.

By the same arguments as in the proof of Theorem 1, there is a point $\overline{t} \in (0,1]$ such that for any $\alpha, z \in \overline{t}I_2$ one can find $\rho(\alpha,z) \in \overline{t}I_2$, $\rho(0,0) = 0$ satisfying

$$Q_0^* f(\rho(\alpha,z),\alpha,z) = 0. \tag{17}$$

Further, we have

$$P_0^* f(\rho,\alpha,z) = \begin{cases} (1+z\rho)(1+\beta_0^2)\Big\{ \sum\limits_{j=c}^{r} \alpha^j z^{j-1} j!\theta_j + \\ +a(1+\beta_0^2)z^{c-1} + h.o.t\Big\}, \quad \text{for} \quad z \neq 0, \\ (1+\beta_0^2)\theta_c\alpha^c, \quad \text{for} \quad z = 0. \end{cases}$$

We define the mapping $q : \bar{t}I_2 \times \bar{t}I_2 \to R$ by

$$q(\alpha, z) = \begin{cases} (1 + z\rho(\alpha, z))(1 + \beta_0^2)\Big\{ \sum\limits_{j=1}^{r} \dfrac{\alpha^j z^{j-1}}{j!}\theta_j + \\ +a(1 + \beta_0^2) + h.o.t\Big\}, \quad \text{for} \quad z \neq 0, \\ (1 + \beta_0^2)\{\theta_c \alpha^c + a(1 + \beta_0^2)\}, \quad \text{for} \quad z = 0. \end{cases}$$

It then follows that if c is an odd integer, there exists a neighborhood Ω of the point $\overline{\alpha} = \sqrt[c]{-a(1 + \beta_0^2)/\theta_c}$ such that $\deg(q(.,0), \Omega, 0)$ is defined and different from zero. Therefore, we have

Theorem 12. *If $\theta_c \neq 0$ with c being an odd number, then $(\overline{\lambda}, 0)$ is a Hopf bifurcation point of periodic solutions of the equation* (8)*. More precisely, to given β_0 there exists a neighborhood I_0 of zero in R such that for $z \in I_0$ one can find $\alpha(z) \in \Omega$ for which $(z\rho(\alpha(z), z)), \lambda(z), v(z))$ with*

$$\lambda(z) = \overline{\lambda} + z\alpha(z),$$

$$v(z) = z^{c/2}\phi^1 + z^{c/2}\beta_0\phi^2 + \psi(z\rho(\alpha(z), z), z\alpha(z), z^{c/2}(1, \beta_0)),$$

satisfies the equation (11)*, $\lambda(z) \to \overline{\lambda}$, $z\rho(\alpha(z), z) \to 0$, $v(z) \to 0$ as $z \to 0$, $v(z) \neq 0$ for $z \neq 0$ and the mapping $u(z)(t) = v(z)\big(\dfrac{t}{1 + z\rho(\alpha(z), z)}\big)$ is $(1 + z\rho(\alpha(z), z))2\pi$-periodic.*

Proof. By the same arguments as in the proof of Theorem 5 we conclude that there exists a neighborhood I_0 of zero in R, $I_0 \subset \bar{t}I_2$ such that for $z \in I_0$ one can find $\alpha(z) \in \Omega$ with $q(\alpha(z), z) = 0$. Multiplying both sides of this equality with z^{c-1} we obtain $P_0^* f(\rho(\alpha(z), z), \alpha(z), z) = 0$. Together with (17) we obtain $f(\rho(\alpha(z), z), \alpha(z), z) = 0$. Again, multiplying both sides of this equality with $z^{c/2+1}$ we conclude

$$\Big\langle -\rho(\alpha(z), z)\frac{d(z^{c/2}\phi^1 + z^{c/2}\phi^2)}{d\tau} +$$

$$(1 + z\rho(\alpha(z), z)\Big\{ \sum_{j=1}^{r} \frac{(z\alpha(z))^j}{j!} L_{\lambda\ldots\lambda}(\overline{\lambda}, 0)(z^{c/2}\phi^1 + z^{c/2}\beta_0 t^2)$$

$$+H(\overline{\lambda}z\alpha(z), z^{c/2}\phi^1 + z^{c/2}\beta_0\phi^2) + h.o.t\Big\}, \psi^k\Big\rangle \ = 0,$$

for $k = 1, 2$. Together with (16) we obtain

$$\dot{v}(z) + (1 + z\rho(\alpha(z), z))\{T(u(z)) + L(\overline{\lambda}, v(z)) +$$

$$\sum_{j=1}^{r} \frac{(z\alpha(z))^j}{j!} L_{\lambda\ldots\lambda}(\overline{\lambda},0)(v(z)) + \mathcal{O}(z|\alpha(z)|^{r+1})$$

$$+H(\lambda(z),v(z)) + K(\lambda(z),v(z))\} = 0$$

with

$$\lambda(z) = \overline{\lambda} + z\alpha(z),$$

$$v(z) = z^{c/2}\phi^1 + z^{c/2}\beta_0\phi^2 + \psi(z\rho(\alpha(z),z), z\alpha(z), z^{c/2}(1,\beta_0)).$$

This completes the proof of the theorem.

Further, if $a \neq 0$ and c is an even number and $a\theta_0 < 0$, we conclude that there exists a neighborhood $\Omega_\pm$ of the point $\overline{\alpha}^{\pm} = \pm\sqrt[c]{-(1+\beta_0^2)a/\theta_c}$ such that the topological degree, $\deg(q(.,0),\Omega_\pm,0)$, of $q(.,0)$ with respect to $\Omega_\pm$ and zero is defined and different from zero is defined and different from zero. Therefore, we have

Theorem 13. *If c is an even number and $a\theta_c < 0$, then $(\overline{\lambda},0)$ is a Hopf bifurcation point of periodic solutions of the equation* (8). *More precisely, to given β_0 there exists a neighborhood I_0 of zero in R such that for $z \in I_0$ one can find $\alpha_\pm(z) \in \Omega_\pm$, $\alpha_\pm(0) = \pm\sqrt[c]{-(1+\beta_0^2)a/\theta_c}$, for which the same conclusions of Theorem 12 continue to hold with $(z\rho(\alpha(z),z),\lambda(z),v(z))$ replaced by $(z\rho(\alpha_\pm(z),z),\lambda_\pm(z),v_\pm(z))$ with*

$$\lambda_\pm(z) = \overline{\lambda} + z\alpha_\pm(z),$$

$$v_\pm(z) = z^{c/2}\phi^1 + z^{c/2}\beta_0\phi^2 + \psi(z\rho(\alpha_\pm(z),z), z\alpha_\pm(z), z^{c/2}(1,\beta_0)).$$

Proof. The proof of this theorem proceeds exactly as the one of Theorem 12.

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