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BOUNDARY VALUE PROBLEMS IN C AN $\mathbf{C}^n$

HEINRICH BEGEHR

Boundary value problems which are unconditionally solvable for one complex variable are in general not solvable for several complex variables. This phenomenon will be explained in the case of the Schwarz problem for polydiscs. Besides analytic functions, inhomogeneous Cauchy-Riemann systems are investigated. These systems in several complex variables are overdetermined.

Another overdetermined system in two complex variables is considered by introducing a proper hypercomplex variable and solved under Riemann-Hilbert boundary conditions on some submanifold of the boundary under consideration.

The theory of bianalytic functions is used to reduce the stress boundary value problem in orthotropic elasticity to boundary value problems for analytic functions in plane domains.

This paper is the improved version of [17] in which formula (16) was incorrect. As for the Poisson equation ($n = 1$) the solution should contain fundamental solutions (Green functions) in the kernels of the integrals.

1. Schwarz problem in $\mathbf{C}$ and $\mathbf{C}^n$

Let $\mathbf{D} := \{|z| < 1\}$ be the unit disc in $\mathbf{C}$, $\mathbf{D}^n := \{z = (z_1, \ldots, z_n) : |z_k| < 1\}$ the polydisc in $\mathbf{C}^n (1 \leq n)$, and $\partial\mathbf{D}^n := \{z = (z_1, \ldots, z_n) : |z_k| = 1\}$ the distinguished boundary of $\mathbf{D}^n$.

Schwarz problem. Given a real-valued function γ on $\partial\mathbf{D}^n$, find an analytic function w in $\mathbf{D}^n$ such that $\operatorname{Re} w = \gamma$ on $\partial\mathbf{D}^n$.

For $n = 1$, a solution to this problem is given by the Schwarz integral

$$S\gamma(z) := \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \, \frac{d\zeta}{\zeta}\,, \quad |z| < 1. \tag{1}$$

The general solution is $S_\gamma + ic$ with arbitrary real constant c. Its real part is the Poisson integral

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$$P\gamma(z) := \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{1-|z|^2}{|\zeta - z|^2} \, \frac{d\zeta}{\zeta} \,, \quad |z| < 1, \tag{2}$$

giving the unique solution to the Dirichlet problem for harmonic functions. For references see e.g. [12, 15, 2].

The Schwarz integral is a simple conclusion from the Cauchy formula. The latter w analytic in $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$ states for

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} w(\zeta) \frac{d\zeta}{\zeta - z} \,, \quad |z| < 1.$$

Note that

$$0 = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} w(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\,\zeta} \,, \quad |z| < 1.$$

Adding the complex conjugate of the second to the first equation gives the representation formula

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \operatorname{Re} w(\zeta) \frac{\zeta + z}{\zeta - z} \, \frac{d\zeta}{\zeta} + \frac{1}{2\pi} \int\limits_{|\zeta|=1} \operatorname{Im} w(\zeta) \frac{d\zeta}{\zeta} \,, \quad |z| < 1. \tag{3}$$

Repeating this for analytic functions in $\mathbf{D}^n$ gives

$$w(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial \mathbf{D}^n} \operatorname{Re} w(\zeta) \Big[2 \frac{\zeta}{\zeta - z} - 1 \Big] \frac{d\zeta}{\zeta} + i \operatorname{Im} w(0), \quad z \in \mathbf{D}^n, \tag{4}$$

where

$$\frac{\zeta}{\zeta - z} := \prod_{k=1}^{n} \frac{\zeta_k}{\zeta_k - z_k} \,, \quad \frac{d\zeta}{\zeta} := \prod_{k=1}^{n} \frac{d\zeta_k}{\zeta_k} \,,$$

see [14, 4, 6]. The question arises whether (4) represents an analytic function satisfying the Schwarz condition $\operatorname{Re} w = \gamma$ on $\partial \mathbf{D}^n$ when $\operatorname{Re} w(\zeta)$ is replaced by any continuous real-valued function $\gamma(\zeta)$. As can be seen, this is the case if and only if

$$
\begin{aligned}
&\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\mathrm{Re}\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}\\
&\qquad=\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1\Big]\frac{d\zeta}{\zeta}\\
&\qquad=\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\prod_{k=1}^{n}\Big[\frac{\zeta_k}{\zeta_k-z_k}+\frac{\overline{\zeta}_k}{\overline{\zeta}_k-\overline{z}_k}-1\Big]\frac{d\zeta_k}{\zeta_k}\\
&\qquad=\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\prod_{k=1}^{n}\frac{1-|z_k|^2}{|\zeta_k-z_k|^2}\ \frac{d\zeta_k}{\zeta_k}\ .
\end{aligned}
$$

A necessary and sufficient condition for this relation is

$$
\begin{aligned}
&\sum_{k=1}^{n-1}\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\Big[\Big(\prod_{\nu=1}^{k}\frac{\zeta_\nu}{\zeta_\nu-z_\nu}-1\Big)\frac{\overline{z}_{k+1}}{\overline{\zeta}_{k+1}-\overline{z}_{k+1}}\\
&\qquad+\Big(\prod_{\nu=1}^{k}\frac{\overline{\zeta}_\nu}{\overline{\zeta}_\nu-\overline{z}_\nu}-1\Big)\frac{z_{k+1}}{\zeta_{k+1}-z_{k+1}}\Big]\\
&\qquad\times\prod_{\nu=k+2}^{n}\Big(\frac{\zeta_\nu}{\zeta_\nu-z_\nu}+\frac{\overline{\zeta}_\nu}{\overline{\zeta}_\nu-\overline{z}_\nu}-1\Big)\prod_{\nu=1}^{n}\frac{d\zeta_\nu}{\zeta_\nu}=0,
\end{aligned}
\tag{5}
$$

see [4]. Obviously, there is no condition (5) for $n=1$.

Theorem 1. *Let γ be real-valued continuous on $\partial\mathbf{D}^n$ and satisfy* (5) *in $\mathbf{D}^n$. Then for any real c,*

$$
\varphi(z):=\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\Big[2\frac{\zeta}{\zeta-z}-1\Big]\frac{d\zeta}{\zeta}+ic
\tag{6}
$$

is analytic in $\mathbf{D}^n$ with

$$
\mathrm{Re}\,\varphi=\gamma\quad on\quad\partial\mathbf{D}^n.
$$

A particular solution to the inhomogeneous Cauchy-Riemann equation

$w_{\overline{z}} = f$ in plane domains D with $f \in L_p(\overline{D})$, $1 \le p$, is given by the Pompeiu operator

$$(7) \qquad Tf(z) := -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}\,, \quad \zeta = \xi + i\eta,\ z \in \mathbf{C}.$$

Boundary value problems which can be solved for analytic functions can also be treated for inhomogeneous Cauchy-Riemann equations, see e.g. [15, 10, 11, 5, 16, 8]. In the case of several complex variables, the overdetermined inhomogeneous Cauchy-Riemann system

$$w_{\overline{z}_k} = f_k, \quad f_{k\,\overline{z}_\ell} = f_{\ell\,\overline{z}_k}, \quad 1 \le k, \ell \le n,$$

in polydomains $D^n := \mathop{\times}\limits_{k=1}^{n} D_k$ can be solved similarly by iterating the Pompeiu operators T_k of the involved domains D_k. A particular solution is

$$\sum_{\nu=1}^{n}(-1)^{\nu+1}\sum_{1\le k_1<k_2<\cdots<k_\nu\le n} T_{k_\nu}T_{k_{\nu-1}}\ldots T_{k_1}f_{k_1\,\overline{\zeta}_{k_2}\,\overline{\zeta}_{k_3}\ldots\overline{\zeta}_{k_\nu}}\ ,$$

where proper differentiability for f_k is assumed, see [4]. Here the second sum is taken over all multi-indices $(k_1, \ldots, k_\nu)$ satisfying $1 \le k_1 < k_2 < \cdots < k_\nu \le n$.

Theorem 2. *Let f_k, $1 \le k \le n$, have mixed derivatives of the first order with respect to the variables z_ℓ, $1 \le \ell \le n$ and their complex conjugate counterparts $\overline{z}_\ell$, $\ell \ne k$, in $L_1(\overline{\mathbf{D}}^n)$ and satisfy the compatibility conditions*

$$f_{k\,\overline{z}_\ell} = f_{\ell\,\overline{z}_k} \quad 1 \le k, \ell \le n,\ \ell \ne k, \quad \textit{in} \quad \mathbf{D}^n.$$

Let γ be real-valued and continuous on $\partial\mathbf{D}^n$ and

$$Re\Big\{\sum_{k=1}^{n-1}\frac{1}{(2\pi i)^n}\int_{\partial\mathbf{D}^n}\gamma(\zeta)\Big[\prod_{\nu=1}^{k}\frac{\zeta_\nu}{\zeta_\nu - z_\nu} - 1\Big]$$
$$\times\frac{\overline{z}_{k+1}}{\overline{\zeta}_{k+1} - \overline{z}_{k+1}}\prod_{\nu=k+2}^{n}\Big(\frac{\zeta_\nu}{\zeta_\nu - z_\nu} + \frac{\overline{\zeta}_\nu}{\overline{\zeta}_\nu - \overline{z}_\nu} - 1\Big)\prod_{\nu=1}^{n}\frac{d\zeta_\nu}{\zeta_\nu}$$

$$-\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n\\ cd\{k_1,\dots,k_\nu\}=\nu}}\frac{(-1)^\nu}{\nu}\int\limits_{\mathbf{D}_{k_1}}\cdots\int\limits_{\mathbf{D}_{k_\nu}} f_{k_1\,\overline{\zeta}_{k_2}\dots\overline{\zeta}_{k_\lambda}\zeta_{k_{\lambda+1}}\dots\zeta_{k_\nu}}$$

$$\times\frac{\overline{z}_{k_1}}{1-\overline{z}_{k_1}\,\zeta_{k_1}}\cdots\frac{\overline{z}_{k_\lambda}}{1-\overline{z}_{k_\lambda}\,\zeta_{k_\lambda}}\;\frac{z_{k_{\lambda+1}}}{1-z_{k_{\lambda+1}}\,\overline{\zeta}_{k_{\lambda+1}}}\cdots$$

$$\cdots\frac{z_{k_\nu}}{1-z_{k_\nu}\,\overline{\zeta}_{k_\nu}}\,d\xi_{k_1}d\eta_{k_1}\dots d\xi_{k_\nu}d\eta_{k_\nu}\Big\}=0. \tag{8}$$

Then the Schwarz problem

$$Re\,w=\gamma\quad on\quad \partial\mathbf{D}^n$$

for the inhomogeneous Cauchy-Riemann system

$$w_{\overline{z_k}}=f_k,\quad 1\le k\le n,\quad in\quad \mathbf{D}^n$$

is solvable. The solution is

$$w(z)=\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n}\gamma(\zeta)\Big[2\prod_{\nu=1}^{n}\frac{\zeta_\nu}{\zeta_\nu-z_\nu}-1\Big]\prod_{\nu=1}^{n}\frac{d\zeta_\nu}{\zeta_\nu}$$

$$+\sum_{\nu=1}^{n}\sum_{1\le k_1<\cdots<k_\nu\le n}\frac{(-1)^\nu}{\pi^\nu}$$

$$\times\int\limits_{\mathbf{D}_{k_1}}\cdots\int\limits_{\mathbf{D}_{k_\nu}}\Big\{\overline{f_{k_1\,\overline{\zeta}_{k_2}\dots\overline{\zeta}_{k_\nu}}}\frac{z_{k_1}}{1-z_{k_2}\overline{\zeta}_{k_1}}\cdots\frac{z_{k_\nu}}{1-z_{k_\nu}\,\overline{\zeta}_{k_\nu}}$$

$$-(-1)^\nu f_{k_1\,\overline{\zeta}_{k_2}\dots\overline{\zeta}_{k_\nu}}\;\frac{1}{\zeta_{k_1}-z_{k_1}}\cdots\frac{1}{\zeta_{k_\nu}-z_{k_\nu}}\Big\}d\xi_{k_1}d\eta_{k_1}\dots d\xi_{k_\nu}d\eta_{k_\nu}$$

$$+\sum_{\nu=2}^{n}\sum_{\lambda=1}^{\nu-1}\sum_{\substack{1\le k_1<\cdots<k_\lambda\le n\\ 1\le k_{\lambda+1}<\cdots<k_\nu\le n}}\frac{(-1)^\lambda}{\pi^\nu}$$

$$\times\int\limits_{\mathbf{D}_{k_1}}\cdots\int\limits_{\mathbf{D}_{k_\nu}}\overline{f_{k_1\,\overline{\zeta}_{k_2}\dots\overline{\zeta}_{k_\lambda}\,\zeta_{k_{\lambda+1}}\dots\zeta_{k_\nu}}}\frac{z_{k_1}}{1-z_{k_1}\,\overline{\zeta}_{k_1}}\cdots$$

$$\cdots\frac{z_{k_\lambda}}{1-z_{k_\lambda}\,\overline{\zeta}_{k_\lambda}}\frac{1}{\zeta_{k_{\lambda+1}}-z_{k_{\lambda+1}}}\cdots\frac{1}{\zeta_{k_\nu}-z_{k_\nu}}d\xi_{k_1}d\eta_{k_1}\dots d\xi_{k_\nu}d\eta_{k_\nu}+ic \tag{9}$$

with arbitrary real constant c.

This result even holds for $n = 1$. For a proof see [4].

2. Dirichlet problem for second order systems

Any complex harmonic function, i.e. any solution u to $u_{z\overline{z}} = 0$ can be represented by two analytic functions φ and ψ as $u = \varphi + \overline{\psi}$. In a bounded smooth domain $D \subset \mathbf{C}$ the Dirichlet problem $u = \gamma$ on ∂D for given complex continuous γ on ∂D is solvable. For $D = \mathbf{D}$ we have the Schwarz problems

$$\mathrm{Re}(\varphi + \psi) = \mathrm{Re}\gamma, \quad \mathrm{Re}\, i(\varphi - \psi) = -\mathrm{Im}\gamma \quad \text{on} \quad \partial\mathbf{D},$$

and hence, by Theorem 1, with arbitrary real constants c_1, c_2

$$\varphi + \psi = S\mathrm{Re}\gamma + ic_1, \quad \varphi - \psi = iS\mathrm{Im}\gamma + c_2.$$

That means

$$u(z) = \varphi(z) + \overline{\psi}(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1\Big] \frac{d\zeta}{\zeta}$$

$$(10) \qquad = \frac{1}{4\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \Big[\frac{\zeta + z}{\zeta - z} + \frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}}\Big] \frac{d\zeta}{\zeta}\ , \ |z| < 1.$$

Up to the last equality these considerations also hold for pluriharmonic functions. They are solutions to systems

$$u_{z_k \overline{z_\ell}} = 0, \quad 1 \le k, \ell \le n,$$

see [1]. As in Section 1, inhomogeneous systems

$$(11) \qquad u_{z_k \overline{z_\ell}} = f_{k\ell}, \quad 1 \le k, \ell \le n,$$

can be handled if the compatibility conditions

$$f_{k\ell z_i} = f_{i\ell z_k}, \quad f_{k\ell \overline{z}_j} = f_{kj\, \overline{z}_\ell}, \quad 1 \le i, j, k, \ell \le n,$$

are satisfied. Treating these systems for fixed ℓ, $1 \le \ell \le n$, analogously to Section 1, one gets a particular solution

$$u_{0\, \overline{z}_\ell} = \sum_{\mu=1}^{n} (-1)^{\mu+1} \sum_{1 \le k_1 < \cdots < k_\mu \le n} \overline{T}_{k_\mu} \ldots \overline{T}_{k_1} f_{k_1 \ell \zeta_{k_2} \ldots \zeta_{k_\mu}} + \overline{\psi}_\ell =: F_\ell,$$

$$\psi_{\ell\, \overline{z}_k} = 0, \quad 1 \le k \le n.$$

Choosing the ψ_ℓ such that F_ℓ satisfies the compatibility conditions $F_{\ell\,\overline{z}_j} = F_{j\,\overline{z}_\ell}$ and using the results from Section 1 one can show that

$$
\begin{aligned}
u_0 = &\sum_{\mu,\nu=1}^{n} (-1)^{\mu+\nu} \sum_{\substack{1\le k_1<\dots<k_\mu\le n\\ 1\le \ell_1<\dots<\ell_\nu\le n}} T_{\ell_\nu}\dots T_{\ell_1}\,\overline{T}_{k_\mu}\dots\overline{T}_{k_1}\, f_{k_1\ell_1\zeta_{k_2}\dots\zeta_{k_\nu}\,\overline{\zeta}_{\ell_2}\dots\overline{\zeta}_{\ell_\nu}} \\
&+\sum_{\nu=1}^{n}(-1)^{\nu+1} \sum_{1\le\ell_1<\dots<\ell_\nu\le n} T_{\ell_\nu}\dots T_{\ell_1}\,\overline{\psi_{\ell_1\zeta_{\ell_2}\dots\zeta_{\ell_\nu}}}
\end{aligned}
\tag{12}
$$

is a particular solution to (11). The general solution is of the form

$$u = \varphi + \overline{\psi} + u_0, \tag{13}$$

with two arbitrary analytic functions φ, ψ. Prescribing Dirichlet boundary conditions for u on $\partial\mathbf{D}^n$,

$$u = \gamma \quad \text{on} \quad \partial\mathbf{D}^n,$$

and using Theorem 1 as before in the case $n = 1$, the polyanalytic function $\varphi + \overline{\psi}$ is fixed because

$$\varphi(z) + \overline{\psi}(z) = \frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n} (\gamma - u_0)(\zeta)\Big[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1\Big]\frac{d\zeta}{\zeta} \tag{14}$$

if and only if the solvability condition

$$\frac{1}{(2\pi i)^n}\int\limits_{\partial\mathbf{D}^n} (\gamma - u_0)(\zeta)K(\zeta, z)\frac{d\zeta}{\zeta} = 0 \tag{15}$$

is satisfied with

$$
\begin{aligned}
K(\zeta, z) := &\sum_{k=1}^{n-1}\Big[\Big(\prod_{\nu=1}^{k}\frac{\zeta_\nu}{\zeta_\nu - z_\nu} - 1\Big)\frac{\overline{z}_{k+1}}{\overline{\zeta}_{k+1} - \overline{z}_{k+1}} \\
&+\Big(\prod_{\nu=1}^{k}\frac{\overline{\zeta}_\nu}{\overline{\zeta}_\nu - \overline{z}_\nu} - 1\Big)\frac{z_{k+1}}{\zeta_{k+1} - z_{k+1}}\Big]\prod_{\nu=k+2}^{n}\Big(\frac{\zeta_\nu}{\zeta_\nu - z_\nu} + \frac{\overline{z}_\nu}{\overline{\zeta}_\nu - \overline{z}_\nu}\Big).
\end{aligned}
$$

This basically leads to the next result.

Theorem 3. *Let $f_{k\ell}$, $1 \le k, \ell \le n$, have mixed first order derivatives with respect to all variables $(z_1, \dots, z_n) \in \mathbf{D}^n$ and $(\overline{z}_1, \dots, \overline{z}_n) \in \mathbf{D}^n$ in $L_1(\overline{\mathbf{D}}^n)$. Assume that $f_{k\ell}$ satisfy the compatibility conditions and*

$$\frac{1}{(2\pi i)^n} \int\limits_{\partial \mathbf{D}^n} \overline{f}_{j\ell} \prod_{\mu \neq j,k} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \frac{d\zeta_k}{(\zeta_k - z_k)^2} \frac{d\overline{\zeta}_j}{\zeta_j - z_j} =$$
$$= \frac{1}{(2\pi i)^n} \int\limits_{\partial \mathbf{D}^n} \overline{f}_{k\ell} \prod_{\mu \neq j,k} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \frac{d\overline{\zeta}_k}{\zeta_k - z_k} \frac{d\zeta_j}{(\zeta_j - z_j)^2}, \; j \neq k, \; 1 \le j, k \le n.$$

Let $\gamma \in C(\partial \mathbf{D})$ be complex-valued and satisfy

$$\frac{1}{(2\pi i)^n} \int\limits_{\partial \mathbf{D}^n} \gamma(\zeta) K(\zeta, z) \frac{d\zeta}{\zeta} = \sum_{\mu=2}^{n} \sum_{\nu=1}^{\mu-1} \sum_{\substack{1 \le k_1 < \cdots < k_{\mu-\nu} \le n \\ 1 \le \ell_1 < \cdots < \ell_\nu \le n \\ cd\{k_1, \dots, k_{\mu-\nu}, \ell_1, \dots, \ell_\nu\} = \mu}} \frac{(-1)^\mu}{\pi^\mu}$$
$$\times \int\limits_{\mathbf{D}_{k_1}} \cdots \int\limits_{\mathbf{D}_{k_{\mu-\nu}}} \int\limits_{\mathbf{D}_{\ell_1}} \cdots \int\limits_{\mathbf{D}_{\ell_\nu}} f_{k_1 \ell_1 \zeta_{k_2} \dots \zeta_{k_{\mu-\nu}} \overline{\zeta}_{\ell_2} \dots \overline{\zeta}_{\ell_\nu}} \frac{z_{k_1}}{1 - z_{k_1} \overline{\zeta}_{k_1}}$$
$$\cdots \frac{z_{k_{\mu-\nu}}}{1 - z_{k_{\mu-\nu}} \overline{\zeta}_{k_{\mu-\nu}}} \frac{\overline{z}_{\ell_1}}{1 - \overline{z}_{\ell_1} \zeta_{\ell_1}} \cdots \frac{\overline{z}_{\ell_\nu}}{1 - \overline{z}_{\ell_\nu} \zeta_{\ell_\nu}}$$
$$\times d\xi_{k_1} d\eta_{k_1} \dots d\xi_{k_{\mu-\nu}} d\eta_{k_{\mu-\nu}} d\xi_{\ell_1} d\eta_{\ell_1} \dots d\xi_{\ell_\nu} d\eta_{\ell_\nu}.$$

Then the Dirichlet problem for the inhomogeneous pluriharmonic system (11) *is uniquely solvable in $\mathbf{D}^n$. The solution is given by*

$$u(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial \mathbf{D}^n} \gamma(\zeta) \prod_{k=1}^{n} \frac{1 - |z_k|^2}{|\zeta_k - z_k|^2} \frac{d\zeta_k}{\zeta_k}$$
$$+ \sum_{\mu,\nu=1}^{n} \Big[\sum_{\substack{\rho=0 \\ \mu+\nu \le n}}^{\min\{\mu,\nu\}} + \sum_{\substack{\rho=\mu+\nu-n \\ n < \mu+\nu}}^{\min\{\mu,\nu\}} \Big] \sum_{\substack{1 \le \sigma_1 < \cdots < \sigma_\rho \le n \\ 1 \le k_1 < \cdots < k_{\mu-\rho} \le n \\ 1 \le \ell_1 < \cdots < \ell_{\nu-\rho} \le n}} \frac{(-1)^{\mu+\nu}}{\pi^{\mu+\nu-\rho}}$$
$$\times \int\limits_{\mathbf{D}_{\sigma_1}} \cdots \int\limits_{\mathbf{D}_{\sigma_\rho}} \int\limits_{\mathbf{D}_{k_1}} \cdots \int\limits_{\mathbf{D}_{k_{\mu-\rho}}} \int\limits_{\mathbf{D}_{\ell_1}} \cdots$$
$$(16) \qquad \cdots \int\limits_{\mathbf{D}_{\ell_{\nu-\rho}}} f_{\sigma_1 \sigma_1 \zeta_{\sigma_2} \dots \zeta_{\sigma_\rho} \zeta_{k_1} \dots \zeta_{k_{\mu-\rho}} \overline{\zeta}_{\sigma_2} \dots \overline{\zeta}_{\sigma_\rho} \overline{\zeta}_{\ell_1} \dots \overline{\zeta}_{\ell_{\nu-\rho}}}$$

$$\times \prod_{\lambda=1}^{\rho} \log\left|\frac{\zeta_{\sigma_\lambda} - z_{\sigma_\lambda}}{1 - \overline{z}_{\sigma_\lambda}\,\zeta_{\sigma_\lambda}}\right|^2 d\xi_{\sigma_\lambda} d\eta_{\sigma_\lambda} \prod_{\lambda=1}^{\mu-\rho} \frac{d\xi_{k_\lambda} d\eta_{k_\lambda}}{\overline{\zeta}_{k_\lambda} - \overline{z}_{k_\lambda}}$$
$$\times \prod_{\lambda=1}^{\nu-\rho} \Big(\frac{1}{\zeta_{\ell_\lambda} - z_{\ell_\lambda}} + \frac{\overline{z}_{\ell_\lambda}}{1 - \overline{z}_{\ell_\lambda}\,\zeta_{\ell_\lambda}}\Big) d\xi_{\ell_\lambda} d\eta_{\ell_\lambda},$$

where the last sum is taken over mutually disjoint ordered sets $\{\sigma_1, \dots, \sigma_\rho\}$, $\{k_1, \dots, k_{\mu-\rho}\}$, $\{\ell_1, \dots, \ell_{\nu-\rho}\}$.

3. A boundary value problem for first order systems in $\mathbf{C}^2$

Any first order system

$$\boldsymbol{w}_{z_2} + \boldsymbol{\Lambda}\,\boldsymbol{w}_{z_1} = \boldsymbol{f} \quad \text{in} \quad D_0 \subset \mathbf{C}^2 \tag{17}$$

of $N \geq 1$ equations with given analytic $N \times N$ matrix function $\boldsymbol{\Lambda}$ and N-dimensional analytic vector $\boldsymbol{f}$ for the unknown analytic vector function $\boldsymbol{w}$ can be transformed into a system with the matrix in Jordan normal form if the eigenvalues λ_κ, $1 \leq \kappa \leq k$, have constant multiplicities n_κ throughout D_0. Therefore we may assume that $\boldsymbol{\Lambda}$ has the Jordan normal form.

Let us consider just one $n \times n$ block

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \lambda & 0 \\ 0 & \cdots & \cdots & 1 & \lambda \end{bmatrix},$$

where λ is an analytic function in D_0. Writing this system componentwise we get

$$w_{0z_2} + \lambda w_{0z_1} = f_0, \quad w_{\nu z_2} + \lambda w_{\nu z_1} + w_{\nu-1 z_1} = f_\nu, \quad 1 \leq \nu \leq n-1. \tag{18}$$

Let there be an integrating factor μ different from zero and analytic in D_0 for the differential form $dz_1 - \lambda dz_2$ such that

$$d\zeta_1 := \mu dz_1 - \mu\lambda dz_2$$

is a total differential. Then the differentiable function ζ_1 satisfies $\zeta_{1z_1} = \mu$, $\zeta_{1z_2} = -\mu\lambda$. Setting $\zeta_2(z_1, z_2) \equiv z_2$, the mapping $\boldsymbol{\zeta} := (\zeta_1, \zeta_2)$ maps D_0 one-to-one onto a domain $D \subset \mathbf{C}^2$. The Jacobian of this map is

$$\begin{vmatrix} \mu & -\mu\lambda \\ 0 & 1 \end{vmatrix} = \mu \neq 0$$

in D_0, and the Jacobian of its inverse $z = z(\zeta_1, z_2)$ is

$$\begin{vmatrix} \frac{1}{\mu} & \lambda \\ 0 & 1 \end{vmatrix} = \frac{1}{\mu} \, .$$

Introducing $\boldsymbol{\omega}(\zeta_1, \zeta_2) := \boldsymbol{w}(z_1(\zeta_1, \zeta_2), z_2(\zeta_1, \zeta_2))$, which is analytic in D, gives the system

$$\boldsymbol{\omega}_{\zeta_1} = \frac{1}{\mu}\boldsymbol{w}_{z_1}, \quad \boldsymbol{\omega}_{\zeta_2} = \lambda\boldsymbol{w}_{z_1} + \boldsymbol{w}_{z_2},$$

or in component form

$$\omega_{0\zeta_2} = \tilde{f}_0, \quad \omega_{\nu\zeta_2} + \mu\omega_{\nu-1\,\zeta_1} = \tilde{f}_\nu,$$
$$\tilde{f}_\nu(\zeta_1, \zeta_2) := f_\nu(z_1(\zeta_1, \zeta_2), z_2(\zeta_1, \zeta_2)), \quad 0 \leq \nu \leq n-1.$$

Instead of (18) one considers the system

$$w_{0z_2} = f_0, \quad w_{\nu z_2} + \mu w_{\nu-1\,z} = f_\nu, \quad 1 \leq \nu \leq n-1, \tag{19}$$

in $D_0 \subset \mathbf{C}^2$. Taking integration leads to

$$w_0(z_1, z_2) = \varphi_0(z_1) + \int_0^{z_2} f_0(z_1, t)dt,$$
$$w_\nu(z_1, z_2) = \varphi_\nu(z_1) + \int_0^{z_2} \{f_\nu(z_1, t) - \mu(z, t)w_{\nu-1z_1}(z_1, t)\}dt,$$
$$1 \leq \nu \leq n-1.$$

In vector form (19) is

$$\boldsymbol{w}_{z_2} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mu & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu & 0 \end{bmatrix} \boldsymbol{w}_{z_1} = \boldsymbol{f}. \tag{20}$$

This system can formally be simplified by introducing the nilpotent element

$$\mathbf{e} := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \mathbf{e}^n = \mathbf{0},$$

together with the hypercomplex quantities

$$\boldsymbol{f} = \sum_{\nu=0}^{n-1} f_\nu \mathbf{e}^\nu, \quad \boldsymbol{w} = \sum_{\nu=0}^{n-1} w_\nu \mathbf{e}^\nu,$$

and the hypercomplex differential operator

$$\boldsymbol{D} := \frac{\partial}{\partial z_2} + \mu \mathbf{e} \frac{\partial}{\partial z_1}$$

giving

$$\boldsymbol{D}\,\boldsymbol{w} = \boldsymbol{f} \quad \text{in} \quad D. \tag{21}$$

Let $\boldsymbol{t} = \sum\limits_{\nu=0}^{n-1} t_\nu \mathbf{e}^\nu$ be a solution to the homogeneous equation $\boldsymbol{D}\,\boldsymbol{t} = 0$ with $t_0(z_1, z_2) \equiv z_1$. One can choose

$$t_\nu = t_\nu(z_1, z_2) = (-1)^\nu \mu_\nu(z_1, z_2)$$

with

$$\mu_0(z_1, z_2) \equiv z_1, \quad \mu_\nu(z_1, z_2) = \int_0^{z_2} \mu(z_1, t) \mu_{\nu-1 z_1}(z_1, t) dt, \ 1 \leq \nu \leq n-1,$$

see [3]. Denoting

$$\boldsymbol{t}_1(z_1, z_2) := \boldsymbol{t}(z_1, z_2), \quad \boldsymbol{t}_2(z_1, z_2) := z_2$$

and passing from the complex variables (z_1, z_2) to the independent hypercomplex variables $(\boldsymbol{t}_1, \boldsymbol{t}_2)$, the operator $\boldsymbol{D}$ becomes $\partial/\partial \boldsymbol{t}_2$. Thus, system (17) is reduced to

$$\frac{\partial}{\partial \boldsymbol{t}_2} \boldsymbol{w}(z_1(\boldsymbol{t}_1, \boldsymbol{t}_2), \boldsymbol{t}_2) = \boldsymbol{f}(z_1(\boldsymbol{t}_1, \boldsymbol{t}_2), \boldsymbol{t}_2). \tag{22}$$

This change of variables has the Jacobian

$$J := \begin{vmatrix} \boldsymbol{t}_{1z_1} & \boldsymbol{t}_{1z_2} \\ \boldsymbol{t}_{2z_1} & \boldsymbol{t}_{2z_2} \end{vmatrix} = \boldsymbol{t}_{z_1} = 1 + \sum_{\nu=1}^{n-1} t_{\nu z_1} \boldsymbol{e}^\nu \neq 0.$$

Moreover,
$$\frac{\partial}{\partial z_1} = J\frac{\partial}{\partial \boldsymbol{t}_1}\,, \quad \frac{\partial}{\partial z_2} = -\mu\,\boldsymbol{e}\,J\frac{\partial}{\partial \boldsymbol{t}_1} + \frac{\partial}{\partial \boldsymbol{t}_2}\,,$$
and
$$\boldsymbol{D}\,\boldsymbol{t}_2 = 1, \quad \boldsymbol{D}\,\boldsymbol{t}_1 = 0, \quad \frac{\partial \boldsymbol{t}_2}{\partial \boldsymbol{t}_1} = \frac{1}{J}\frac{\partial z_2}{\partial z_1} = 0.$$

A particular solution to (21) is

$$\boldsymbol{w}_0(\boldsymbol{t}_1, \boldsymbol{t}_2) := \int_0^{\boldsymbol{t}_2} \boldsymbol{f}(z_1(\boldsymbol{t}_1, \boldsymbol{t}), \boldsymbol{t})d\boldsymbol{t}.$$

The general solution to the homogeneous equation (21), with $\boldsymbol{f} = 0$, is given by an arbitrary analytic hypercomplex function $\boldsymbol{\varphi}$ as $\boldsymbol{\varphi}(\boldsymbol{t}_1(z_1, z_2))$. Hence, the general solution to (21) is

$$\boldsymbol{w}(z_1, z_2) = \boldsymbol{w}_0(\boldsymbol{t}_1(z_1, z_2), z_2) + \boldsymbol{\varphi}(\boldsymbol{t}_1(z_1, z_2)). \tag{23}$$

In order to determine $\boldsymbol{\varphi}$, the Riemann-Hibert boundary condition

$$\mathrm{Re}\{\overline{G}(z_1)\boldsymbol{w}(z_1, 0)\} = \boldsymbol{g}(z_1) \quad \text{on} \quad \partial D_1$$

can be imposed on the boundary of the domain

$$D_1 := D_0 \cap \{(z_1, z_2) \in \mathbf{C}^2 : z_2 = 0\} \subset \mathbf{C}.$$

For $z_1 \in D_1,\ z_2 = 0$ we have

$$\boldsymbol{w}(z_1, 0) = \boldsymbol{w}_0(\boldsymbol{t}_1(z_1, 0), 0) + \varphi(\boldsymbol{t}_1(z_1, 0)) = \boldsymbol{\varphi}(z_1),$$

so that
$$\mathrm{Re}\ \{\overline{G}(z_1)\,\boldsymbol{\varphi}(z_1)\} = \boldsymbol{g}(z_1) \quad \text{on} \quad \partial D_1$$

determines the analytic vector $\boldsymbol{\varphi}$ according to the well-known Riemann-Hilbert boundary value problem, see e.g. [12].

Theorem 4. *The general solution to system* (17) *with* $\boldsymbol{\Lambda}$ *having just one eigenvalue, where* $\boldsymbol{\Lambda}$ *and* $\boldsymbol{f}$ *are an analytic matrix- and vector-function, respectively, in the domain* $D_0 \subset \mathbf{C}^2$, $(0,0) \in D_0$, *is given by* (23). *Here the domains*

$$D_{z_1^0} = \{(z_1, z_2) \in D_0,\ z_1 = z_1^0\} \subset \mathbf{C}, \quad z_1^0 \in \mathrm{proj}_{z_2} D_0$$

are assumed to be simply connected. Moreover, $\boldsymbol{\varphi}(z_1)$ is an arbitrary analytic vector-function in

$$D_1 = \{(z_1, z_2) \in D_0,\ z_2 = 0\}.$$

Provided D is simply connected and regular $\boldsymbol{\varphi}$ is determined by Riemann-Hibert boundary conditions depending on the index of the coefficient.

In [2] the m-th order complex equation

$$\frac{\partial^m u}{\partial z_2^m} - \frac{\partial u}{\partial z_1} = 0 \quad \text{in} \quad |z_1|^2 + |z_2|^2 < 1,$$

for $2 \le m$ is treated in the same manner.

4. Stress boundary value problem in orthotropic elasticity

The equilibrium equations

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_1 = 0, \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_2 = 0, \end{aligned} \tag{24}$$

for the stress components σ_x, σ_y, τ_{xy} and the body force vector $\boldsymbol{F} = (F_1, F_2)$ of an orthotropic elastic body together with the Hooke law

$$\begin{aligned} \sigma_x &= \frac{1}{1-\nu_{12}\nu_{21}}\Big(E_{11}\frac{\partial u}{\partial x} + \nu_{12}E_{22}\frac{\partial v}{\partial y}\Big), \\ \sigma_y &= \frac{1}{1-\nu_{12}\nu_{21}}\Big(\nu_{21}E_{11}\frac{\partial u}{\partial x} + E_{22}\frac{\partial v}{\partial y}\Big), \\ \tau_{xy} &= G_{12}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big), \end{aligned} \tag{25}$$

in the case of small deformations (u, v) lead to a system of second order equations for (u, v) if (u, v) are twice continuously differentiable. The coefficients in (25) are the modulus of motion G_{12}, the Poisson ratios ν_{12}, ν_{21} and the Young moduli E_{11}, E_{22} related by

$$\nu_{12}E_{22} = \nu_{21}E_{11},$$

see [9]. Substituting (25) into (24) gives

$$\Big[A_{11}\frac{\partial^2}{\partial x^2} + A_{12}\frac{\partial^2}{\partial x \partial y} + A_{21}\frac{\partial^2}{\partial y \partial x} + A_{22}\frac{\partial^2}{\partial y^2}\Big]\binom{u}{v} + \binom{F_1}{F_2} = \binom{0}{0} \tag{26}$$

with

$$A_{11} = \begin{bmatrix} E_{11}/(1-\nu_{12}\nu_{21}) & 0 \\ 0 & G_{12} \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 & \nu_{12}E_{22}/(1-\nu_{12}\nu_{21}) \\ G_{12} & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & G_{12} \\ \nu_{12}E_{11}/(1-\nu_{12}\nu_{21}) & 0 \end{bmatrix}, \; A_{22} = \begin{bmatrix} G_{12} & 0 \\ 0 & E_{22}/(1-\nu_{12}\nu_{21}) \end{bmatrix}.$$

Let $D \subset \mathbf{C}$ be a simply connected domain occupied by the elastic body and as well the displacement (u, v) as the stresses (X_n, Y_n) be specified at all points of ∂D. Here $\boldsymbol{n}$ is the exterior normal vector to ∂D and X_n, Y_n are the components of the external stress along the axes. Then the stress boundary value problem is

$$\begin{aligned} \sigma_x \cos(n,x) + \tau_{xy}\cos(n,y) &= X_n, \\ \tau_{xy}\cos(n,x) + \sigma_y \cos(n,y) &= Y_n, \end{aligned} \quad \text{on} \quad \partial D.$$

Using (25) these conditions become

$$\begin{aligned} &\Big[\cos(n,x)\Big(A_{11}\frac{\partial}{\partial x} + A_{12}\frac{\partial}{\partial y}\Big) \\ &+ \cos(n,y)\Big(A_{21}\frac{\partial}{\partial x} + A_{22}\frac{\partial}{\partial y}\Big)\Big]\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \quad \text{on} \quad \partial D. \end{aligned} \tag{27}$$

Introducing new parameters

$$\nu^2 = \nu_{12}\nu_{21}, \quad E^2 = E_{11}E_{22},$$
$$\delta^2 = \sqrt{\frac{E_{11}}{E_{22}}}\,, \quad k_1 = \frac{E}{2G_{12}} - \nu,$$
$$\lambda = \frac{1-\nu^2}{2(k_1+\nu)^k}\,, \quad k = k_1 - \sqrt{k_1^2 - 1}\,,$$

and new dependent and independent variables

$$x = \delta\xi, \quad y = \frac{1}{\sqrt{k}}\eta, \quad u = \frac{1-k\nu}{(k+\nu)\sqrt{k}}\,\tilde{u}, \quad v = -\delta\,\tilde{v},$$

(26) is transformed into

$$\begin{aligned} &\left[\begin{bmatrix} 1 & 0 \\ 0 & -\lambda/k \end{bmatrix}\frac{\partial^2}{\partial\xi^2} + \begin{bmatrix} 0 & \lambda/k - k \\ 1-\lambda & 0 \end{bmatrix}\frac{\partial^2}{\partial\xi\partial\eta} + \begin{bmatrix} \lambda & 0 \\ 0 & -k \end{bmatrix}\frac{\partial^2}{\partial\eta^2}\right]\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \\ &= \frac{\nu^2-1}{E}\begin{pmatrix} \dfrac{(k+\nu)\sqrt{k}}{1+k\nu}F_1 \\ F_2 \end{pmatrix}. \end{aligned} \tag{28}$$

The left-hand side can be factorized. Thus (28) can be written as

$$\Big[\begin{bmatrix} k & 0 \\ 0 & \lambda \end{bmatrix}\frac{\partial}{\partial x}+\begin{bmatrix} 0 & \lambda \\ k & 0 \end{bmatrix}\frac{\partial}{\partial y}\Big]\Big[\begin{bmatrix} 1/k & 0 \\ 0 & 1/k \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial y}\Big]\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

where (ξ,η) and $(\tilde u,\tilde v)$ were replaced by (x,y) and (u,v), respectively and the coefficients on the right-hand side were neglected. Using the notations

$$\frac{1}{k}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\theta,\qquad \frac{\partial u}{\partial y}+\frac{1}{k}\frac{\partial v}{\partial x}=\omega,$$

we get the following relations in some plane domain $D\subset \mathbf{C}$

$$k\frac{\partial\theta}{\partial x}+\lambda\frac{\partial\omega}{\partial y}=F_1,\qquad k\frac{\partial\theta}{\partial y}-\lambda\frac{\partial\omega}{\partial x}=F_2.$$

By introducing complex variables we have

$$\begin{aligned} &\Big(\frac{\partial}{\partial x}+ik\frac{\partial}{\partial y}\Big)(u+iv)=k\theta+ik\omega,\\ &\frac{1}{2}\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big)(k\theta-i\lambda\omega)=F_1+iF_2. \end{aligned}\tag{29}$$

This system in the homogeneous case $F:=F_1+iF_2=0$ reduces to the system of bianalytic functions, see [13], With

$$\begin{aligned} &f=u+iv,\quad \varphi:=k\theta-i\lambda\omega,\quad z=x+iy,\\ &\zeta=\zeta(z):=\frac{k+1}{2k}z+\frac{k-1}{2k}\,\overline{z}=x+iy/k \end{aligned}$$

(29) is equivalent to

$$\frac{\partial f}{\partial\overline{\zeta}}=\frac{\lambda-k}{4\lambda}\varphi+\frac{\lambda+k}{4\lambda}\,\overline{\varphi},\qquad \frac{\partial\varphi}{\partial\,\overline{z}}=F.\tag{30}$$

A particular solution to the last equation of (30) is $\varphi_0=T_DF$. Similarly, a particular solution to the first equation of system (30) is

$$\begin{aligned} f_0(z)=-\frac{1}{\pi}\int_G\Big\{&\frac{\lambda-k}{4\lambda}\varphi_0\Big(\frac{k+1}{2}\,\overline{\zeta}+\frac{1-k}{2}\,\overline{\overline{\zeta}}\Big)\\ &+\frac{\lambda+k}{4\lambda}\,\overline{\varphi}_0\Big(\frac{k+1}{2}\,\tilde\zeta+\frac{1-k}{2}\,\overline{\tilde\zeta}\Big)\Big\}\frac{d\tilde\xi\,d\tilde\eta}{\tilde\zeta-\zeta(z)}\ , \end{aligned}$$

when G is the image of D under the transformation $\zeta = \zeta(z)$. Consequently (26) has the particular solution

$$\begin{aligned} u_0(x,y) &= \frac{1+k\nu}{(k+\nu)\sqrt{k}}\ \mathrm{Re}\ f_0\Big(\frac{x}{\delta} + i\sqrt{k}\,y\Big), \\ v_0(x,y) &= -\delta\,\mathrm{Im}\, f_0\Big(\frac{x}{\delta} + i\sqrt{k}\,y\Big). \end{aligned} \tag{31}$$

The theory of bianalytic functions, see [13], shows that the general solution of the homogeneous system (30), with $F \neq 0$, is given in the form

$$f(z) = \frac{\lambda - k}{2(1-k)\lambda}\Phi(\zeta(z)) + \frac{\lambda + k}{2(1+k)\lambda}\,\overline{\Phi}(\zeta(z)) + \Psi\Big(\frac{k+1}{2k}\zeta(z) - \frac{k-1}{2k}\,\overline{\zeta}(z)\Big),$$

where for arbitrary analytic function $\varphi(z)$

$$\Phi(\zeta) := \int_{\zeta_0}^{\zeta} \varphi(z)dz$$

with some arbitrary fixed $\zeta_0 \in G$ and Ψ is an arbitrary analytic function. Thus the general solution to (28) has the form

$$\begin{aligned} u(x,y) &= -\frac{k+\nu}{E\sqrt{k}}\,\mathrm{Re}\,\Big[\alpha\Phi\Big(\frac{x}{\delta} + i\sqrt{k}\,y\Big)\Big] - \frac{1+k\nu}{kE\sqrt{k}}\,\mathrm{Re}\,\Big[k\beta\Psi\Big(\frac{x}{\delta} + i\frac{y}{\sqrt{k}}\Big)\Big] \\ &\quad + \frac{1+k\nu}{(k+\nu)\sqrt{k}}\ \mathrm{Re}\, f_0\Big(\frac{x}{\delta} + i\sqrt{k}\,y\Big), \\ v(x,y) &= \frac{1+\nu k}{E_{22}\delta k}\,\mathrm{Im}\,\Big[\alpha\Phi\,\Big(\frac{x}{\delta} + i\sqrt{k}\,y\Big)\Big] + \frac{\nu+k}{E_{22}\delta k}\,\mathrm{Im}\,\Big[k\beta\Psi\,\Big(\frac{x}{\delta} + i\frac{y}{\sqrt{k}}\Big)\Big] \\ &\quad - \delta\,\mathrm{Im}\, f_0\,\Big(\frac{x}{\delta} + i\sqrt{k}\,y\Big) \end{aligned}$$

with

$$\alpha = \frac{1+k\nu}{1-k^2}\ \frac{E}{1-\nu^2}\ , \quad \beta := -\frac{E}{k+\nu}\ .$$

The stress boundary conditions lead to boundary value problems for the unknown analytic functions Φ, Ψ. At first (25) is expressed by the new parameters,

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}\Big(\delta^2\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\Big), \\ \sigma_y &= \frac{E}{1-\nu^2}\Big(\nu\frac{\partial u}{\partial x} + \delta^{-2}\frac{\partial v}{\partial y}\Big), \\ \tau_{xy} &= \frac{E}{2(k_1+\nu)}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big). \end{aligned}$$

Then changing the variables to $z_1 = \dfrac{x}{\delta} + i\sqrt{k}\,y$ and $z_2 = \dfrac{x}{\delta} + i\dfrac{y}{\sqrt{k}}$, respectively, and introducing the functions $\psi(x_2) = \Psi'(z_2)$, $\varphi(z_1) = \Phi'(z_1)$ one gets

$$\begin{aligned}
\sigma_x &= -\delta\sqrt{k}\,\alpha \mathrm{Re}\,\varphi(z_1) - \frac{\delta}{\sqrt{k}}\beta \mathrm{Re}\,\psi(z_2)\\
&\quad + \frac{E\delta\sqrt{k}}{1-\nu^2}\Big(\frac{1-k\nu}{(k+\nu)k} - \nu\Big)\,\mathrm{Re}\, f_0'(z_1),\\
\sigma_y &= \frac{1}{\delta\sqrt{k}}\alpha\mathrm{Re}\varphi(z_1) + \frac{\sqrt{k}}{\delta}\,\beta\mathrm{Re}\psi(z_2)\\
&\quad + \frac{E\sqrt{k}}{(1-\nu^2)\delta}\Big(\frac{\nu(1+k\nu)}{(k+\nu)k} - 1\Big)\,\mathrm{Re}\, f_0'(z_1),\\
\tau_{xy} &= \alpha\mathrm{Im}\,\varphi(z_1) + \beta\,\mathrm{Im}\,\psi(z_2) - \frac{E}{2(k_1+\nu)}\Big(\frac{1+k\nu}{k+\nu} + 1\Big)\mathrm{Im}\, f_0'(z_1).
\end{aligned}$$

Observe on ∂D with $x = \delta x_1$, $\sqrt{k}\,y = y_1$, that

$$\cos(n,x) = \frac{dy}{ds} = \frac{1}{\sqrt{k}}\,\frac{dy_1}{ds}\ , \quad \cos(n,y) = -\frac{dx}{ds} = -\delta\frac{dx_1}{ds}\ ,$$

then on ∂D

$$\begin{aligned}
X_n ds &= -d\big[\mathrm{Im}\,\{\delta\alpha\Phi(z_1) + \delta\beta\Psi(z_2)\} + \delta g_2(z_1)\big],\\
Y_n ds &= -d\Big[\mathrm{Re}\Big\{\frac{\alpha}{\sqrt{k}}\Phi(z_1) + \sqrt{k}\,\beta\Psi(z_2)\Big\} + \frac{1}{\sqrt{k}}\,g_1(z_1)\Big],
\end{aligned}$$

where g_1, g_2 are functions defined by $f_0(z_1)$, i.e. by the inhomogeneous right-hand side in (28). Integrating along ∂D from some initial point with $s = 0$ to $z \in \partial D$ with arc-length parameter $s > 0$ one gets for $z = x + iy \in \partial D$

$$\begin{aligned}
&\mathrm{Im}\,\{\alpha\Phi(z_1) + \beta\Psi(z_2)\} = g_2(z_1) - \frac{1}{\delta}\int_0^s X_n d\,\tilde{s} + c_2,\\
(32)\qquad &\\
&\mathrm{Re}\,\{\alpha\Phi(z_1) + k\beta\Psi(z_2)\} = g_1(z_1) - \sqrt{k}\int_0^s Y_n d\,\tilde{s} + c_1
\end{aligned}$$

with some integration constants c_2 and c_1. These are boundary value problems for the analytic functions Φ and Ψ. In the case of a multi-connected domain D, these functions are multi-valued in general (see [7], where D is doubly connected).

Denote the right-hand side of (32) by $-\beta f_2$, βf_1, respectively, and assume that $D_1 := \{z_1 : z_1 = \frac{x}{\delta} + i\sqrt{k}\, y,\ x + iy \in D\}$ has a Schwarz operator S_1, see [2]. By eliminating $\Phi(z_1)$, the system (32) can be reduced to the integral equation

$$S_1\Big(\frac{k-1}{2}\Psi(z_2) + \frac{k+1}{2}\overline{\Psi(z_2)}\Big) = S_1(f_1 - if_2) + d \tag{33}$$

with arbitrary complex constant d.

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I. Mathematisches Institut, FU Berlin
Arnimallee 3, D-14195 Berlin, Germany