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THREE SPECIES COMPETITION IN PERIODIC ENVIRONMENT

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

INTRODUCTION

Consider the Lotka-Volterra system for 3-competing species

$$(0.1)\qquad \begin{aligned} x_1'(t) &= x_1(t)(1 - x_1(t) - \alpha(t)x_2(t) - \beta(t)x_3(t)),\\ x_2'(t) &= x_2(t)(1 - \beta(t)x_1(t) - x_2(t) - \alpha(t)x_3(t)),\\ x_3'(t) &= x_3(t)(1 - \alpha(t)x_1(t) - \beta(t)x_2(t) - x_3(t)),\end{aligned}$$

where $\alpha,\ \beta : \mathbf{R}^1 \to \mathbf{R}^1$ are continuous, nonnegative and T-periodic for some common period $T > 0$. Set

$$\begin{aligned} \mathbf{R}^3_+ &= \{x = (x_1, x_2, x_3) \in \mathbf{R}^3\ :\ x_i \geq 0;\ i = 1, 2, 3\},\\ L &= \{x \in \mathbf{R}^3_+\ :\ x_1 = x_2 = x_3\}.\end{aligned}$$

In this paper we shall show that system (0.1) has at least one T-periodic solution with strictly positive components. If $\max\limits_{0\leq t\leq T}(\alpha(t) + \beta(t)) < 2$, then such a solution is unique and globally asymptotically stable (or attractive) in int $(\mathbf{R}^3_+) := \{x \in \mathbf{R}^3_+ : x_i > 0, i = 1, 2, 3\}$. Furthermore, if $\min\limits_{0\leq t\leq T}\{\alpha(t)+\beta(t)\} > 2$ then a positive (componentwise) T-periodic solution of (0.1) is also unique but $\operatorname{dist}(x(t), \partial\mathbf{R}^3_+) \to 0$ as $t \to +\infty$ for every solution $x(t)$ of (0.1) with $x(t_0) \in \operatorname{int}(\mathbf{R}^n_+) \setminus L$ for some $t_0 \in \mathbf{R}$, where $\operatorname{dist}(x(t), \partial\mathbf{R}^3_+)$ is the distance from $x(t)$ to $\partial\mathbf{R}^3_+$.

The case that α, β are positive constants was already studied by R. M. May and W. J. Leonard [6], P. Schuster, K. Sigmund and R. Wolff [8].

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The study of the nonautonomous Lotka-Volterra competition equations have been the subject of several recent papers (see e.g. [1, 2, 4, 9]). K. Gopalsamy [4] and A. Tineo and C. Alvarez [9] considered the system

$$x_i'(t) = x_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right] \quad (1 \le i \le n), \tag{0.2}$$

when $n \ge 2$ and $a_{ij}, b_i : \mathbf{R}^1 \to \mathbf{R}^1$ are continuous, positive and T-periodic for some common periodic $T > 0$.

In [9], it was shown that the two sets of conditions

(T_1) $b_i(t) > \sum_{j \in J_i} a_{ij}(t)U_j^0(t)$ $(1 \le i \le n, \quad t \in \mathbf{R})$, where $U_i^0(t)$ is the unique T-periodic and positive solution of the logistic equation

$$U'(t) = U(t)(b_i(t) - a_{ii}(t)U(t)),$$

and $J_i = \left\{1, 2, \dots, i-1, i+1, \dots, n\right\}$,

(T_2) there are positive constants $\alpha_1, \dots, \alpha_n$ such that

$$\alpha_i a_{ii}(t) > \sum_{j \in J_i} \alpha_j a_{ji}(t) \quad (1 \le i \le n, \quad t \in \mathbf{R}),$$

imply that the system (0.2) has a T-periodic solution $x^0(t)$ whose components are positive and $x(t) - x^0(t) \to 0$ as $t \to +\infty$, for any positive solution $x(t)$ of (0.2).

In the case of the system (0.1), conditions (T_1) and (T_2) become $\alpha(t) + \beta(t) < 1$, for $t \in \mathbf{R}$, which is more restrictive than ours. The ecological significance of the system (0.1) is discussed in [4, 5, 6, 8].

We also consider the case that $\alpha(t)$ and $\beta(t)$ are continuous, nonnegative and almost periodic in Section 3 of this paper.

1. Existence

The Cauchy problem for (0.1) has a unique solution whenever the initial data $x_i(t_0) = x_{i0}$ belong to $\mathbf{R}_+$ $(i = 1, 2, 3)$. Any solution satisfies the condition

$$0 \le x_i(t) \le \max\{1, x_{i0}\} \quad (i = 1, 2, 3, \quad t \ge t_0).$$

Let

$$M_1 = \Big\{x \in R^3_+ \ : \ x_i \le 1\Big\}, \quad M_2 = \Big\{x \in M_1 \ : \ x_1 + x_2 + x_3 \ge \eta\Big\},$$

where $\eta = \frac{1}{4} \max_{0 \le t \le T} (1 + \alpha^2(t) + \beta^2(t))^{-1}$.

In the following lemma we do not assume periodic conditions on the coefficients $\alpha(t)$ and $\beta(t)$.

Lemma 1.1. *The sets M_1 and M_2 are positively invariant and attractive with respective to $\mathbf{R}^3_+$ and $\mathbf{R}^3_+ \setminus \{0\}$, respectively.*

Proof. It is easy to see that M_1 is positively invariant and attractive. We now prove the attractivity and invariance of M_2.

Let $x \in M_1 \setminus (M_2 \cup \{0\})$. Then $0 \le x_i \le 1$ $(i = 1,2,3)$ and there exists at least one index $j \in \{1,2,3\}$ such that $x_j > 0$. From (0.1) we have

$$\begin{aligned} x_j' &\ge x_j\big(1 - \sqrt{(1 + \alpha^2 + \beta^2)(x_1^2 + x_2^2 + x_3^2)}\big) \\ &\ge x_j\big(1 - \sqrt{(1 + \alpha^2 + \beta^2)(x_1 + x_2 + x_3)}\big) \\ &\ge x_j\big(1 - \sqrt{(1 + \alpha^2 + \beta^2)\eta}\big) \ge \frac{1}{2} x_j > 0. \end{aligned}$$

This fact and the invariance and attractivity of M_1 imply the invariant and attractivity of M_2. The lemma is proved.

By Lemma 1.1, as far as the asymptotic behavior is concerned, we may confine our attention to M_2.

Lemma 1.2. *The set L is invariant and the system* (0.1) *has a unique T-periodic solution $x^*(t)$ in L with positive components.*

Proof. Suppose that $x_0 = (x_{01}, x_{02}, x_{03}) \in L$ with $x_{0i} > 0$ $(i = 1,2,3)$. For each $i = 1,2,3$ let us denote $x_{0i}(t)$ the solution of the equation

(1.1) $$x_i'(t) = x_i(t)\big(1 - (1 + \alpha(t) + \beta(t))x_i(t)\big),$$

(1.2) $$x_i(t_0) = x_{0i}.$$

Clearly, $x_{01}(t) = x_{02}(t) = x_{03}(t_0)$.

It is easy to see that $x_0(t) = \big(x_{01}(t), x_{02}(t), x_{03}(t)\big)$ is the solution of (0.1) with initial condition $x_0(t_0) = x_0$. The invariant property of L

follows from the uniqueness of solution of the Cauchy problem. Since the T-periodic logistic equation (1.1) has a unique positive and T-periodic solution $x_i^*(t)$ (see, for example, [2]), it follows that the system (0.1) has a unique positive T-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))$ in L. The lemma is proved.

2. Uniqueness and asymptoticity

In this section we prove the assertion mentioned before. To do this, we need the following result in [7, p. 289]. Consider

$$x'(t) = f(t, x(t)), \tag{2.1}$$

$$x(t_0) = x_0, \tag{2.2}$$

where $f : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n$ is such that the Cauchy problem (2.1) - (2.2) has a unique solution for $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^n$. If $g : \mathbf{R} \to \mathbf{R}$ is a function, we denote by $D^+g(t_0) = \limsup_{t\to t_0^+} \dfrac{g(t) - g(t_0)}{t - t_0}$, the (upper right) Dini derivative of g at $t = t_0$.

Furthermore, if $V : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a function then D^+V denotes the Dini derivative of V along solutions of (2.1).

Theorem 2.1 (see [7, p. 289]). *Suppose that $\Omega \subset \mathbf{R}^n$ is open and connected. Let S be a subset of Ω, closed with respect to Ω. Let $V : [t_0, +\infty)\times\Omega \to \mathbf{R}$ be a function which is locally Lipschitzian in x and continuous, and let $\psi : \Omega \to \mathbf{R}$ be a locally Lipschitzian function. If there exists a number $B > 0$ such that for every $(t, x) \in [t_0, +\infty) \times S$:*
(i) $D^+V(t,x) \le -\psi(x)$,
(ii) $\psi(x) \ge 0$,
(iii) $D^+\psi(x) \ge -B$, *(or $D^+\psi(x) \le B$), and*
(iv) *for every compact set $C \subset \mathbf{R}^n$, there exists a number $A > 0$ such that*

$$V(t, x) \ge -A \quad on \quad [t_0, +\infty) \times (C \cap \Omega),$$

then any solution $x(t)$ of (2.1) for which the right maximal interval of existence is $[t_0, +\infty)$ and $x(t) \in S$ for $t \in [t_0, +\infty)$ has the ω-limit set Λ^+ satisfying

$$\Lambda^+ \cap \Omega \subset E = \Big\{x \in S : \psi(x) = 0\Big\}.$$

Theorem 2.2. *Let $x^*(t)$ be the T-periodic solution of (0.1) as in Lemma 1.2. If $\max_{0\le t\le T}\{\alpha(t) + \beta(t)\} < 2$, then $x(t) - x^*(t) \to 0$ as $t \to +\infty$ for any solution $x(t)$ of (0.1) with $x(t_0) \in int\,(\mathbf{R}_+^3)$, $t_0 \in \mathbf{R}$.*

Proof. Suppose that $x(t)$ is a solution of (0.1) with $x_i(t_0) > 0$ $(i = 1, 2, 3)$. Denote by Λ^+ the ω-limit set of $x(t)$. Without loss of generality, we may assume that $x(t_0) \in M_2$, since M_2 is attractive relative to $\mathbf{R}^3_+ \setminus \{0\}$.

Define a Liapunov function

$$V(x) = -x_1 x_2 x_3 (x_1 + x_2 + x_3)^{-3} \quad \text{on} \quad M_2.$$

Then we have

$$\begin{aligned} D^+V &= -\Big(1 - \frac{\alpha + \beta}{2}\Big) x_1 x_2 x_3 (x_1 + x_2 + x_3)^{-4} \times \\ &\quad \big[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\big] \\ &\le -\Big(1 - \frac{1}{2} \max_{0 \le t \le T} (\alpha(t) + \beta(t))\Big) x_1 x_2 x_3 (x_1 + x_2 + x_3)^{-4} \times \\ &\quad \big[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\big] =: -\psi(x). \end{aligned}$$

It is easy to verify that V and ψ satisfy all conditions in Theorem 2.1 with $\Omega = S = \operatorname{int}(M_2)$. Therefore

$$\Lambda^+ \cap \Omega \subset E = \Big\{x : \psi(x) = 0\Big\} = \Big\{x : \eta < x_1 = x_2 = x_3 < 1\Big\}.$$

Since M_2 is positive invariant, we can conclude that

$$\Lambda^+ \subset \Big\{x : \eta < x_1 = x_2 = x_3 < 1\Big\} \cup \partial M_2.$$

Clearly,

$$\begin{aligned} D^+V(x) < 0 \quad &\text{for} \quad x \in M_2 \setminus \Big\{\partial \mathbf{R}^3_+ \cup \{x : \eta \le x_1 = x_2 = x_3 \le 1\}\Big\}, \\ V(x) = -\frac{1}{27} \quad &\text{for} \quad x \in \Big\{x : \eta \le x_1 = x_2 = x_3 \le 1\Big\}, \end{aligned}$$

and $V(x) = 0$ for $x \in \partial \mathbf{R}^3_+ \cap \partial M_2$. Consequently,

$$\Lambda^+ \subset \Big\{x : \eta \le x_1 = x_2 = x_3 \le 1\Big\}.$$

Since Λ^+ is bounded, we have $x(t) \to \Lambda^+$ as $t \to +\infty$.

We now prove that $x_i(t) - x_i^*(t) \to 0$ as $t \to +\infty$ $(i = 1, 2, 3)$. It is enough to prove $x_1(t) - x_1^*(t) \to 0$ as $t \to +\infty$.

Suppose that $W(t) = \dfrac{1}{x_1(t)}$, $W^*(t) = \dfrac{1}{x_1^*(t)}$. Then

$$W'(t) = 1 - W(t) + \big(\alpha(t)x_2(t) + \beta(t)x_3(t)\big)\frac{1}{x_1(t)},$$
$$W^{*\prime}(t) = 1 - W^*(t) + \alpha(t) + \beta(t).$$

Hence,

$$(W-W^*)'(t) = -(W-W^*)(t) + \alpha(t)\Big(\frac{x_2(t)}{x_1(t)} - 1\Big) + \beta(t)\Big(\frac{x_3(t)}{x_1(t)} - 1\Big). \tag{2.3}$$

There are two possibilities:

(1) There exists $t_1 \ge t_0$ such that $(W - W^*)'(t) \ne 0$ for $t \ge t_1$.

(2) There exists a sequence of numbers $\{S_n\}_1^\infty$ in $[t_0, +\infty)$ such that for $n \ge 1$, $S_n < S_{n+1}$, $(W - W^*)'(S_n) = 0$, and $S_n \to +\infty$ as $n \to \infty$.

If (1) holds, then $W(t) - W^*(t)$ is monotonic on $[t_1, +\infty)$. Therefore $\lim\limits_{t\to+\infty}(W(t) - W^*(t))$ exists. If $\lim\limits_{t\to+\infty}(W(t) - W^*(t)) = 0$ then, since $x_1(t)$ and $x_1^*(t)$ are bounded and $x_1^*(t) - x_1(t) = x_1^*(t)x_1(t)(W(t) - W^*(t))$, it follows that $x_1^*(t) - x_1(t) \to 0$ as $t \to +\infty$.

Suppose now that (1) holds and $\lim\limits_{t\to+\infty}(W(t) - W^*(t)) = \gamma \ne 0$. Since $x(t) \to \Lambda^+$ as $t \to +\infty$ and $\Lambda^+ \subset \Big\{x \ : \ \eta \le x_1 = x_2 = x_3 \le 1\Big\}$, (2.3) follows

$$\Big|\frac{d}{dt}(W(t) - W^*(t))\Big| \ge \frac{|\gamma|}{2} > 0, \quad \text{for} \quad t \ge t_2.$$

Since this contradicts the boundeness of $W(t) - W^*(t)$ on $[t_0, \infty)$, it follows that if (1) holds then $\lim\limits_{t\to+\infty}(x_1(t) - x_1^*(t)) = 0$.

If (2) holds, for each $n \ge 1$ we take a number $\tau_n \in [S_n, S_{n+1}]$ such that

$$|W(\tau_n) - W^*(\tau_n)| = \max_{S_n \le t \le S_{n+1}} |W(t) - W^*(t)|. \tag{2.4}$$

Since $(W - W^*)'(S_n) = 0$ for $n \ge 1$, it follows that $(W - W^*)'(\tau_n) = 0$ for $n \ge 1$. Therefore, by (2.3),

$$W(\tau_n) - W^*(\tau_n) = \alpha(\tau_n)\Big(\frac{x_2(\tau_n)}{x_1(\tau_n)} - 1\Big) + \beta(\tau_n)\Big(\frac{x_3(\tau_n)}{x_1(\tau_n)} - 1\Big).$$

Since $\dfrac{x_i(t)}{x_1(t)} \to 1$ as $t \to +\infty$ $(i = 1, 2, 3)$ and $\alpha(t)$, $\beta(t)$ are bounded, it follows that

$$\lim_{n\to\infty}\big(W(\tau_n) - W^*(\tau_n)\big) = 0. \tag{2.5}$$

Since $S_n \to +\infty$ as $n \to \infty$, it follows from (2.4) and (2.5) that $W(t) - W^*(t) \to 0$ as $t \to +\infty$. Therefore, if (2) holds we have $\lim_{t\to+\infty}(x_1(t) - x_1^*(t)) = 0$. Since the possibilities (1) and (2) are exhaustive, the theorem is proved.

Remark. As a consequence of the global attractivity of the solution $x^*(t)$, Theorem 2.2 also implies the uniqueness of a positive T-periodic solution of (0.1).

In the next theorem we do not assume periodic conditions on the coefficients $\alpha(t)$ and $\beta(t)$.

Theorem 2.3. *Suppose that $\alpha(t)$ and $\beta(t)$ are continuous, nonegative and bounded above. If $\inf_{t\in\mathbf{R}}(\alpha(t)+\beta(t)) > 2$, then $\lim_{t\to+\infty} dist(x(t), \partial\mathbf{R}^3_+ \cap \partial M_2) = 0$ for any solution $x(t)$ of (0.1) with $x(t_0) \in \mathbf{R}^3_+ \setminus L$, $t_0 \in \mathbf{R}$.*

Proof. Suppose $x(t)$ is a solution of (0.1) with $x(t_0) \in \mathbf{R}^3_+ \setminus L$. Denote by Λ^+ the ω-limit set of $x(t)$. By the attractivity of M_2 we may assume, without loss of generality, that $x(t_0) \in \operatorname{int}(M_2) \setminus L$. We also define here a Liapunov function

$$V(x) = x_1x_2x_3(x_1+x_2+x_3)^{-3}$$

on M_2. Hence

$$\begin{aligned} D^+V(x) &= \frac{x_1x_2x_3}{(x_1+x_2+x_3)^4}\left(1 - \frac{\alpha(t)+\beta(t)}{2}\right)\times \\ &\quad \left[(x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\right] \\ &\le \left(1 - \frac{1}{2}\min_{0\le t\le T}\left(\alpha(t)+\beta(t)\right)\right)x_1x_2x_3(x_1+x_2+x_3)^{-4}\times \\ &\quad \left[(x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\right] =: -\psi(x) \end{aligned}$$

By Theorem 2.1 and using the same argument as in the proof of Theorem 2.2 we can conclude that $\Lambda^+ \subset \partial\mathbf{R}^3_+ \cap \partial M_2$. Since Λ^+ is bounded, it follows

$$\lim_{t\to+\infty} \operatorname{dist}\left(x(t), \partial\mathbf{R}^3_+ \cap \partial M_2\right) = 0.$$

The theorem is proved.

Remark. In the case that $\alpha(t)$ and $\beta(t)$ are T-periodic, Theorem 2.3 and Lemma 1.2 imply the uniqueness of a positive T-periodic solution of (0.1).

3. The case of almost periodic coefficients

In this section we assume that $\alpha(t)$ and $\beta(t)$ are continuous, nonnegative and almost periodic instead of being T-periodic. Given a function $g(t)$ defined on $(-\infty,+\infty)$, we let g_L and g_M denote $\inf_{t\in\mathbf{R}}\{g(t)\}$ and $\sup_{t\in\mathbf{R}}\{g(t)\}$, respectively. Suppose that $f=(f^1,\dots,f^n):\mathbf{R}\to\mathbf{R}^n$ is continuous. Let us recall that f is almost periodic if for each $\varepsilon>0$ there exists a positive number $\ell=\ell(\varepsilon)$ such that each interval $(\alpha,\alpha+\ell)$, $(\alpha\in\mathbf{R})$, contains at least a number $\tau=\tau(\varepsilon)$ satisfying

$$\sup_{t\in\mathbf{R}}\|f(t+\tau)-f(t)\|\le\varepsilon,\quad\text{where}\quad \|f(t)\|=\max_{1\le i\le n}\{|f^i(t)|\}.$$

We recall Bochner's criterion for almost periodicity: *$f(t)$ is almost periodic if and only if for every sequence of numbers $\{\tau_m\}_1^\infty$, there exists a subsequence $\{\tau_{m_k}\}_{k=1}^\infty$ such that the sequence of translates $\{g(t+\tau_{m_k})\}_{k=1}^\infty$ converges uniformly on* $(-\infty,+\infty)$ (see, for example [3]).

Note that a continuous almost periodic function is always bounded.

First of all we need the following lemma which was given by S. Ahmad [1].

Lemma 3.1 (see [1]). *Let $a,b:\mathbf{R}\to\mathbf{R}$ be continuous and bounded above and below by positive constants. Then the logistic equation*

$$y'(t)=y(t)\big(a(t)-b(t)y(t)\big), \tag{3.1}$$

has a unique solution $y^0(t)$ defined on $(-\infty,+\infty)$ such that

$$\delta<y^0(t)<\Delta\quad for\quad t\in(-\infty,+\infty),$$

where δ, Δ are any positive numbers satisfying $\delta<\dfrac{a_L}{b_M}$, $\Delta>\dfrac{a_M}{b_L}$.

Lemma 3.2. *Suppose that $a(t)$, $b(t)$ are as in Lemma 3.1. If, in addition, $a(t)$, $b(t)$ are almost periodic then the solution $y^0(t)$ in Lemma 3.1 is almost periodic.*

Proof. Take $\varepsilon'>0$. By Bochner's criterion, it follows that $\big(a(t),b(t)\big)$ is almost periodic. Therefore there exists a positive number $\ell=\ell(\varepsilon')$ such that each interval $(\alpha,\alpha+\ell)$, $\alpha\in\mathbf{R}$, contains at least a number τ such that

$$\sup_{t\in\mathbf{R}}\big|a(t+\tau)-a(t)\big|<\varepsilon',\quad \sup_{t\in\mathbf{R}}\big|b(t+\tau)-b(t)\big|<\varepsilon'. \tag{3.2}$$

Let us fix τ as above. Define $W(t) = \frac{1}{y^0(t)}$. From (3.1) it follows that

$$\begin{aligned}\frac{d}{dt}\big[W(t) - W(t+\tau)\big] &= b(t) - b(t+\tau) - a(t)\big[W(t) - W(t+\tau)\big]\\ &= \big[a(t+\tau) - a(t)\big]W(t+\tau).\end{aligned} \tag{3.3}$$

Consider the following equation

$$z'(t) = -a(t)z(t) + b(t) - b(t+\tau) + \big[a(t+\tau) - a(t)\big]W(t+\tau). \tag{3.4}$$

Since $a_L > 0$, it is not hard to prove that if $z(t)$ is a bounded solution of (3.4) defined on $(-\infty, +\infty)$, then

$$\begin{aligned}&\inf_{t\in\mathbf{R}}\left\{\frac{b(t) - b(t+\tau) + \big[a(t+\tau) - a(t)\big]W(t+\tau)}{a(t)}\right\} \le z(t)\\ &\le \sup_{t\in\mathbf{R}}\left\{\frac{b(t) - b(t+\tau) + \big[a(t+\tau) - a(t)\big]W(t+\tau)}{a(t)}\right\} \quad (t \in \mathbf{R}).\end{aligned} \tag{3.5}$$

Therefore, from (3.2) and (3.5) we have

$$-\frac{\varepsilon'\left(1 + \frac{1}{y_L^0}\right)}{a_L} \le z(t) \le \frac{\varepsilon'\left(1 + \frac{1}{y_L^0}\right)}{a_L}, \quad \text{for any } t \in \mathbf{R}. \tag{3.6}$$

Since $\frac{1}{y^0(t)} - \frac{1}{y^0(t+\tau)}$ is a bounded solution of (3.4) defined on $(-\infty, +\infty)$, it follows that

$$\left|\frac{1}{y^0(t)} - \frac{1}{y^0(t+\tau)}\right| \le \varepsilon' \,\frac{1 + \frac{1}{y_L^0}}{a_L},$$

consequently,

$$\big|y^0(t) - y^0(t+\tau)\big| \le \varepsilon' \,\frac{1 + \frac{1}{y_L^0}}{a_L}\,\left(y_M^0\right)^2.$$

Therefore if

$$\varepsilon = \varepsilon' \,\frac{1 + \frac{1}{y_L^0}}{a_L}\left(y_M^0\right)^2$$

then $\left|y^0(t+\tau) - y^0(t)\right| < \varepsilon$ and we can take $\ell(\varepsilon) = \ell(\varepsilon')$. This proves that $y^0(t)$ is almost periodic. The lemma is proved.

By Lemma 3.2, the equation (1.1) has a unique almost periodic solution defined on $(-\infty, +\infty)$ which is bounded above and below by positive constants. Therefore, the system (0.1) has a unique almost periodic solution $\tilde{x}(t)$ in L defined on $(-\infty, +\infty)$ whose components are bounded above and below by positive constants. In the proof of Theorem 2.2, the periodicity of $x^*(t)$ was not used. Therefore we have the following

Theorem 3.3. *Suppose that $\alpha(t)$, $\beta(t)$ are continuous, nonnegative almost periodic and* $\sup_{t\in\mathbf{R}} \big(\alpha(t)+\beta(t)\big) < 2$. *Then $x(t) - \tilde{x}(t) \to 0$ as $t \to +\infty$ for any solution $x(t)$ of* (0.1) *with $x(t_0) \in int(\mathbf{R}^3_+)$, $t_0 \in \mathbf{R}$.*

Remark. If $\alpha(t)$, $\beta(t)$ are only assumed to be continuous and nonnegative then Theorem 3.3 is also true with $\tilde{x}(t)$ being a unique solution in L, defined on $(-\infty, +\infty)$, whose components are bounded above and below by positive constants. In fact, this follows from Lemma 3.1 and the proof of Theorem 2.2.

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