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ON THE LANGLANDS TYPE DISCRETE GROUPS III. THE CONTINUOUS COHOMOLOGY

DO NGOC DIEP

Dedicated to Hoang Tuy on the occasion of his seventieth birthday

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$$U/\Gamma\cap U \rightarrowtail {}^\circ P \twoheadrightarrow M/\Gamma_M$$

converges and the cohomology group $H^*(K_M\backslash {}^\circ P/\Gamma\cap P;V)$ is isomorphic to the direct sum of E_2-terms. Every cohomology class of this type can be represented by an V-valued automorphic form. The restriction map sends the cohomology classes at infinity of $H^*(\Gamma;V)$, represented by singular values of the associated Eisenstein series to the cohomology classes of the boundary $\partial(\bar{X}_{cusp}/\Gamma)$, compatible with its weight decomposition. All together these give us a decomposition of the cohomology of Langlands type discrete groups into the cuspidal and Eisenstein parts.

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INTRODUCTION

For a fixed reductive algebraic (Lie) $\mathbb{Q}$-group G, the subgroup $G_{\mathbb{Z}}$ of the integral points, under some natural conditions (see [R1]) admits an interesting property: *Every finite dimensional representation of $G_{\mathbb{Z}}$ restricted to a subgroup of finite index can be extended to the whole group G as a rational representation* (for more details, see [R1], [R2]). Arithmetically defined subgroups $\Gamma \subset G$ are, by definition, just the subgroups,

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whose images $\rho(\Gamma)$ under some algebraic faithful representation (ρ, V) are commensurable with $\rho(G_{\mathbf{Z}})$. In particular, V is a bimodule over the maximal compact subgroup K and the center $Z(\mathcal{G}_{\mathbf{C}})$ of the universal enveloping algebra $U(\mathcal{G}_{\mathbf{C}})$. Therefore one considers the local K-finitely generated and local $Z(\mathcal{G})$-finite spectrum submodule ${}^{\circ}L^2_V(G/\Gamma)$ of the automorphic forms in the twisted regular representation

$$L^2_V(G/\Gamma) = L^2(G) \otimes_\Gamma V.$$

The problem of decomposing this regular twisted representation into irreducible components requires a theory of V-valued Eisenstein series. Really, the theory of Eisenstein series uses only some definitive properties needed for the reduction theory. Abstracting these properties, R. P. Langlands introduced a larger than arithmetically defined class of discrete groups. We call them the Langlands type discrete groups, see [La].

So, following R. P. Langlands, in Part II of this series we considered the theory of V-valued Eisenstein series for Langlands type discrete groups, where V can be any unitary Γ-module of (in-)finite dimension, but with the following two basic properties: *K-finiteness and $Z(\mathcal{G})$-finiteness of spectrum.* We say in this case that V has *finite spectrum.* In general the theory is related to the spectral decomposition of $L^2_V(G/\Gamma)$ as a G-module.

As remarked in Part I, it is easy to see that Γ can be supposed to be acting freely on the smooth contractible quotient $X = K \setminus G$ of G by a maximal compact subgroup K, and hence X/Γ is the Eilenberg - MacLane space $K(\Gamma, 1)$. The cohomology $H^*(\Gamma; V)$ is then isomorphic to the sheaf cohomology $H^*(X/\Gamma; \mathcal{F}_V)$, where $\mathcal{F}_V$ is the local coefficient sheaf associated to the representation (σ, V) of Γ.

Following A. Borel and J.-P. Serre, in Part I we constructed the compactification $\bar{X}_{cusp}/\Gamma$ of X/Γ, the boundary $\partial(\bar{X}_{cusp}/\Gamma)$ of which is homomorphic to the quotient by Γ of the cuspidal part in the Tits building.

We show in this third paper (Theorem 1) that for a fixed percuspidal subgroup $P = MAU$, the spectral sequence of the corresponding fiberation

$$U/\Gamma_U \rightarrowtail {}^{\circ}P/\Gamma \cap P \twoheadrightarrow M/\Gamma_M$$

converges and the cohomology $H^*(K_M \setminus {}^{\circ}P/\Gamma \cap P; V)$ is isomorphic to the direct sum of the E_2-terms. Every cohomology class of this type can be represented by an V-valued automorphic form, with which then associates some Eisenstein series [D2].

Following G. Harder [H], J. Schwermer [S], K. F. Lai [L], we prove (Theorem 2) in this paper that the preimage

$$H^*_{inf}(\Gamma; V) := \mathrm{res}^{-1}(\mathrm{Im\ \ res}),$$

(the so called cohomology at infinity) of the restriction map

$$H^*(X/\Gamma; V) \xrightarrow{\text{res}} H^*(\partial(\bar{X}_{cusp}/\Gamma); V)$$

consists entirely of classes which can be represented by special values of V-valued Eisenstein series, and that the image of these classes is compatible with the weight decomposition of the boundary cohomology

$$H^*(\partial(\bar{X}_{cusp}/\Gamma); V) = \oplus_{w \in W} \oplus_{\lambda > 0} H^*_{w\lambda-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V).$$

All together these give us a decomposition of the cohomology of Langlands type discrete groups into the cuspidal and Eisenstein parts.

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1. Cohomology of discrete cuspidal subgroups

Recall that G is a reductive Lie group, K a maximal compact subgroup of G, Γ a fixed Langlands type discrete subgroup and (σ, V) some finite spectrum unitary Γ-module of any (finite or infinite) dimension. Let $P = MAU$ be a cuspidal subgroup of G,

$$\Gamma_M := \Gamma \cap M, \quad \Gamma_P := \Gamma \cap P, \quad \Gamma_U := \Gamma \cap U, \quad X := K \setminus G,$$
$$X_M := (K \cap M) \setminus M, \quad X/\Gamma := K \setminus G/\Gamma, \quad {}^\circ X := x_0.{}^\circ P,$$

where $x_0 = K.e \in X$. We have therefore a fiberation

$$U/\Gamma_U \rightarrowtail {}^\circ X/\Gamma_P \twoheadrightarrow X_M/\Gamma_M.$$

Theorem 1. *For the cohomology*

$$H^*(\Gamma_P; V) = H^*({}^\circ X/\Gamma_P; V) = H^*(X/\Gamma \cap P; V)$$

there exists a convergent spectral sequence of Hochschild-Serre type such that if X_M/Γ_M is compact, the map

$$i: \quad \Omega^p(X_M/\Gamma_M; \mathbf{H}^q(\mathfrak{U}; V)) \longrightarrow \Omega^{p+q}({}^\circ X/\Gamma \cap P; V)$$

induces an isomorphism of the cohomology groups

$$i^* : \bigoplus_{p+q=m} H^p(X/\Gamma_M; \mathbf{H}^q(\mathfrak{U}; V)) \xrightarrow{\simeq} H^m({}^\circ X/\Gamma \cap P; V),$$

where $\mathbf{H}^*(\mathfrak{U}; V)$ *denote the Lie algebra cohomology of Lie algebra* $\mathfrak{U} = \operatorname{Lie} U$ *with coefficients in* V*, represented by harmonic forms.*

Proof. From the properties of Langlands type discrete groups, one deduces that Γ_M is also a Langlands type discrete group in M. With the fiberation

$$U/\Gamma_U \rightarrowtail {}^\circ X/\Gamma \cap P \twoheadrightarrow M/\Gamma_M$$

there exists a spectral sequence

$$H^p(X_M/\Gamma_M; H^q(U/\Gamma_U; V)) \Longrightarrow H^{p+q}({}^\circ X/\Gamma \cap P; V).$$

Lemma 1.1 (Van Est; see, for example, [BW]).

$$H^*(U/\Gamma_U; V) \cong H^*(\mathfrak{U}; V) \cong \mathbf{H}^*(\mathfrak{U}; V).$$

Lemma 1.2. *There exists a convergent spectral sequence with* E_2*-terms*

$$H^p(X_M/\Gamma_M; \mathbf{H}^q(\mathfrak{U}; V)) \Longrightarrow H^m({}^\circ X/\Gamma \cap P; V),$$

with $p + q = m$.

Proof. Essentially, this lemma was proved in [H] for the finite dimensional Γ-modules V. We consider here an arbitrary K-finite and $Z(\mathcal{G})$-finite spectrum Γ-module V, or shortly finite spectrum Γ-module V. The cohomology of X_M/Γ_M with coefficients in the local system of $\mathbf{H}^*(\mathfrak{U}; V)$, determined by the operation of the fundamental group Γ_M. This action coincides with the restriction of the action of M on $\mathbf{H}^*(\mathfrak{U}; V)$. The last action of M is defined by the action of ${}^\circ P$ on the cochain complex defining $\mathbf{H}^*(\mathfrak{U}; V)$. We have a diagram

$$\begin{array}{ccc}
{}^\circ X/(\Gamma \cap P) & \xrightarrow{\pi} & X_M/\Gamma_M \\
\uparrow & & \uparrow \pi \\
Y & \xrightarrow{\pi'} & {}^\circ X/(\Gamma \cap P),
\end{array}$$

where Y is the pull-back of the fiberation π over itself. The fiber bundle

$$Y \xrightarrow{\pi'} {}^\circ X/(\Gamma \cap P)$$

is induced by the action of $\Gamma \cap P$ on U/Γ_U. For this fiberation we have the well-known Hochschild-Serre spectral sequence, which converges to $H^*({}^\circ X/(\Gamma \cap P); V)$. The lemma is proved.

Next we consider for each fixed q an embedding i_q which commutes with differentials d, d_M and operators $* \circ \#, *_M \circ \#_M$

$$\begin{array}{ccc} \Omega^p(X_M/\Gamma_M; H^q(\mathfrak{U}; V)) & \xrightarrow{i_q} & \Omega^{p+q}({}^\circ X/\Gamma \cap P; V) \\ {\scriptstyle d_M}\downarrow & & \downarrow{\scriptstyle d} \\ \Omega^{p+1}(X_M/\Gamma_M; H^q(\mathfrak{U}; V)) & \xrightarrow{i_q} & \Omega^{p+q+1}({}^\circ X/\Gamma \cap P; V) \end{array}$$

$$\begin{array}{ccc} \Omega^p(X_M/\Gamma_M; \mathbf{H}^q(\mathfrak{U}; V)) & \xrightarrow{i_q} & \Omega^{p+q}({}^\circ X/\Gamma \cap P; V) \\ {\scriptstyle (-1)^{(n-p)q} *_M \circ \#_M}\downarrow & & \downarrow{\scriptstyle * \circ \#} \\ \Omega^{n-p}(X_M/\Gamma_M; \mathbf{H}^{s-q}(\mathfrak{U}; V^*)) & \xrightarrow{i_{s-q}} & \Omega^{s+n-p-q}({}^\circ X/\Gamma \cap P; V^*), \end{array}$$

where $n = \dim X_M$, $s = \dim \mathfrak{U}$. From this commutative diagrams we deduce that for each q, i_q interwines the Laplacian Δ_M on X_M/Γ_M and Δ on ${}^\circ X/(\Gamma \cap P)$,

$$i_q \circ \Delta_M = \Delta \circ i_q.$$

The first diagram gives us a homomorphism i^* on cohomology

$$i^* : \bigoplus_{p+q=m} H^p(X_M/\Gamma_M; H^q(\mathfrak{U}; V)) \Longrightarrow H^m({}^\circ X/(\Gamma \cap P); V).$$

The Hodge decomposition gives us therefore injectivity and subjectivity of i^*. The theorem is proved.

Remark 1.4. Since V has finite spectrum on each irreducible component, Δ acts as a scalar multiplication and with finite multiplicity. Thus the Hodge decomposition for Δ is the same as in the case, where V is of finite dimension.

Recall that for any finite spectrum unitary Γ-module (σ, V), the space $\mathcal{A}_V(G/\Gamma, \sigma, \chi)$ of V-valued automorphic forms of type (σ, χ) has also a finite spectrum (see [D2], Theorem 2).

For any percuspidal subgroup $P = MAU$ we have the relationship for the corresponding Lie algebras

$$\mathcal{M} = \mathcal{K}_M \oplus \mathcal{P}_M, \quad \mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{U},$$

$$^\circ\mathcal{P} = \mathcal{M} \oplus \mathcal{U}, \quad {}^\circ\mathcal{R} = \mathcal{P}_M \oplus \mathcal{U},$$

$$\mathcal{R} = {}^\circ\mathcal{R} \oplus \boldsymbol{a}, \quad \wedge^{m\circ}\mathcal{R} = \bigoplus_{p+q=m} (\wedge^p \mathcal{P}_M \otimes \wedge^q \mathcal{U}).$$

Then each cohomology class

$$[\varphi] \in H^m({}^\circ X/\Gamma \cap P; V)$$

can be represented by a unique harmonic representative

$$\varphi \in \mathbf{H}({}^\circ X/\Gamma \cap P; V) \subset \Omega^m({}^\circ X/\Gamma \cap P; V)$$

which can be viewed as a function

$$\varphi : {}^\circ P/\Gamma \cap P \to \mathrm{Hom}(\wedge^m \mathcal{R}_M; V),$$

such that

$$\varphi(kpu) = \wedge^m Ad_{{}^\circ\mathcal{R}}(k) \otimes \sigma(k)(\varphi(p)).$$

We have K_M-invariant embedding

$$\mathrm{Hom}(\wedge^{m\circ}\mathcal{R}; V) \hookrightarrow \mathrm{Hom}(\wedge^m \mathcal{R}; V).$$

We can extend φ to a function

$$\varphi_\Lambda : G/(\Gamma \cap P) \to \mathrm{Hom}(\wedge^m \mathcal{R}; V),$$

$$\varphi_\Lambda(ktp) = \wedge^m Ad^*_{\mathcal{R}}(k) \otimes \sigma(k)\varphi(p) \exp -(\rho + \Lambda)(\ln t)$$

and define the Eisenstein series (see [D2])

$$E(\varphi, \Lambda)(x) := \sum_{\gamma \in \Gamma/(\Gamma \cap P)} \varphi_\Lambda(x\gamma).$$

The series converges absolutely and uniformly on any compact $\Omega \times \omega \subset G \times (\mathcal{A}^*_{\mathbb{C}})^+$. Its sum is a function of class $C^\infty(G \times (\mathcal{A}^*_{\mathbb{C}})^+)$, homomorphic on $\Lambda \in (\mathfrak{A}^*_{\mathbb{C}})^+$, right Γ-invariant on the variable x, a $\wedge^m Ad^*_{\mathfrak{K}}(k) \otimes \sigma(k)$-function on x satisfying equations

$$Z.E(\varphi, \Lambda) = \chi(\mu_\Lambda(Z))E(\varphi, \Lambda),$$

where

$$(\mathfrak{A}^*)^+ := \{\Lambda \in \mathfrak{A}^*; \langle \Lambda - \rho, \alpha\rangle > 0, \forall \alpha \in \Delta(P|A)\}$$
$$(\mathfrak{A}^*_{\mathbb{C}})^+ := \{\Lambda \in \mathfrak{A}^*_{\mathbb{C}}; \mathrm{Re}\Lambda \in (\mathfrak{A}^*)^+\}.$$

The constant term $E^{P_2}(\varphi, \Lambda)(x)$ can be expressed as

$$E^{P_2}(\varphi, \Lambda)(x) = \sum_{s\in W(\mathfrak{A}_1,\mathfrak{A}_2)} (c(s;\Lambda)\varphi)(x)\exp(-s\Lambda - \rho_{P_2})(H_{P_2}(a(x)))$$
$$= \sum_{s\in W(\mathfrak{A}_1,\mathfrak{A}_2)} (c(s,\Lambda)\varphi)_{s\Lambda}(x)$$

(see [D2], Theorem 4).

The weight decomposition

$$\mathbf{H}^*(\mathfrak{U}, V) = \oplus_{w\in W} \oplus_{\lambda\in\Delta^+} H^*_{w\lambda-\rho}(\mathfrak{U}; V)$$

provides us a corresponding decomposition for

$$H^m({}^\circ X/(\Gamma \cap P); V) = \oplus_{w\in W} \oplus_{\lambda>0} \oplus_{p+q=m} \mathbf{H}^p(X_M/\Gamma_M; \mathbf{H}^q_{w\lambda-\rho}(\mathfrak{U}; V)).$$

We refer to the elements of $\sum_{p+q=m} \mathbf{H}^p_{w\lambda-\rho}(.; \mathbf{H}^q_{w\lambda-\rho}(.; V))$ as a class of weight $w\lambda - \rho$.

Lemma 1.5. *If $[\varphi] \in H^m({}^\circ X/(\Gamma \cap P); V)$ is of weight $w\lambda - \rho$, then*

$$d\varphi_\Lambda = \sum_{i=1}^{l} -(w\lambda_i + \Lambda_i) + \frac{dt_i}{t_i} \wedge \varphi_\Lambda,$$

$$\delta\varphi_\Lambda = 0,$$
$$\Delta\varphi_\Lambda = (\langle\lambda, \lambda\rangle - \langle\Lambda, \Lambda\rangle)\varphi_\Lambda.$$

Proof. We identify $\mathfrak{A} \cong (\mathbf{R}^*_+)^l$. On the i^{th} copy $\mathbf{R}^*_+$ choose the differential form $\frac{dt_i}{t_i}$. We have

$$d\varphi_\Lambda = \sum_{i=1}^{l} \frac{dt_i}{t_i} \wedge \frac{\partial}{\partial t_i}\varphi_\Lambda.$$

From Lemma 3.1 of [H], we have

$$\frac{\partial}{\partial t_i}\varphi_\Lambda = (-\frac{\alpha_i}{2} - \Lambda_i - w\lambda_i + \frac{\alpha_i}{2})\varphi_\Lambda.$$

The other statements can be proved in the same maner as in [H].

2. Boundary $\partial(\bar{X}_{cusp}/\Gamma)$ and the cohomology at infinity

For a fixed weight $w\lambda-\rho$ cohomology class $[\varphi] \in H^m({}^\circ X/(\Gamma\cap P); V)$, we now consider the corresponding Eisenstein series $E(\varphi,\Lambda)$ and its Fourier constant term $E^P(\varphi,\Lambda)$.

Lemma 2.1. $dE(\varphi,\pm\lambda) = 0 \iff dE^P(\varphi,\pm\lambda) = 0.$

Proof. For each cusp form $\omega \in \Omega^{m+1}(X/\Gamma; V)$, which is a χ-eigenvector for $\mathcal{Z}(\mathcal{G}_{\mathbb{C}}) = \text{cent}\, U(\mathcal{G}_{\mathbb{C}})$,

$$*\circ\#\omega \in {}^\circ\mathcal{A}_V(G/\Gamma,\sigma,\chi),$$

on one hand. On the other hand, we have

$$dE(\varphi,\pm\lambda) \in \mathcal{A}_V(G/\Gamma,\sigma).$$

Then by [HC Lemma 15], the following integral converges

$$\langle dE(\varphi,\lambda),\omega\rangle := \int_{X/\Gamma} (dE(\varphi,\lambda), \overline{*\circ\#\omega}).$$

This integral is equal to

$$\langle E(\varphi,\lambda),\delta\omega\rangle.$$

Since $\delta\omega$ is also a cusp form and the Eisenstein series are orthogonal to the cusp forms, we conclude that

$$\langle dE(\varphi,\lambda),\omega\rangle = \int_{\partial(X/\Gamma)} (E(\varphi,\lambda), \overline{*\circ\#\delta\omega}) = 0,$$

if $dE^P = 0$. Now due to Langlands' Theorem 3 [D2], we see that in this case $dE = 0$.

Lemma 2.2. *If $E(\varphi, \lambda)$ is a closed form, $E(\varphi, \lambda)$ and $E^P(\varphi, \lambda)$ represent the same cohomology class of the boundary, symbolically*

$$[E(\varphi, \lambda)]_{\partial(\bar{X}_{cusp}/\Gamma)} = [E^P(\varphi, \lambda)]_{\partial(\bar{X}_{cusp}/\Gamma)}.$$

Proof. Since for any cuspidal subgroup $P = MAU$ we have

$$H^m({}^\circ X/(\Gamma \cap P); V) \cong \bigoplus_{p+q=m} \mathbf{H}^p(X_M/\Gamma_M; \mathbf{H}^q(\mathfrak{U}; V))$$

and in virtue of the Langlands assumption, for every cuspidal subgroup, there exists a finite set of percuspidal ones

$$P_1, P_2, \ldots, P_s,$$

such that the associated Siegel domains

$$\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_s$$

recover M/Γ_M, i.e.

$$M/\Gamma_M = \bigcup_{i=1}^{s} \mathbf{S}_i \Gamma_{M_i}.$$

We have also a convergent spectral sequence, such that $H^*(\partial(\bar{X}_{cusp}/\Gamma); V)$ is isomorphic to the direct sum of E_2-terms. It rests to recall that

$$\mathbf{H}^q(\mathfrak{U}; V) \cong \mathbf{H}^q(U/\Gamma_U; V).$$

These ones are invariant under forming Fourier constant terms. The lemma is proved.

Recall that the boundary cohomology admits also a weight decomposition

$$H^*(\partial(\bar{X}_{cusp}/\Gamma); V) = \bigoplus_{w \in W} \bigoplus_{\lambda > 0} H^*_{w\lambda - \rho}(\partial(\bar{X}_{cusp}/\Gamma); V),$$

following the weight decomposition

$$\mathbf{H}^*(\mathfrak{U}; V) = \bigoplus_{w \in W} \bigoplus_{\lambda > 0} \mathbf{H}_{w\lambda - \rho}(\mathfrak{U}; V).$$

Following the weight decomposition of

$$\mathrm{Hom}(\wedge^m \mathcal{R}; V) = \sum_{\mu} \sum_{p+q=m} \wedge^p \mathcal{A} \otimes \mathrm{Hom}_\mu(\wedge^q \mathcal{R}, V),$$

we have

$$c(s,\Lambda)\varphi = \sum_{\mu} \sum_{p+q=m} \frac{dt_{\alpha_1}}{t_{\alpha_1}} \wedge \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge c_q^\mu(s,\Lambda)\varphi$$

Lemma 2.3.

$$\begin{aligned} dE^P(\varphi,\Lambda) = \sum_{s\in W} \sum_{\mu} \sum_{p+q=m} & -s(\lambda_i + \Lambda_i)\frac{dt_{\alpha_i}}{t_{\alpha_i}} \wedge \ldots \\ & \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge \frac{dt_i}{t_i} \wedge (c_q^\mu(s,\Lambda)\varphi)_{s\Lambda} \\ & + \sum_{\mu} \sum_{p+q=m} (\frac{dt_{\alpha_1}}{t_{\alpha_1}} \wedge \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge dc_q^\mu(s,\Lambda)\varphi)_{s\Lambda}. \end{aligned}$$

Proof. We have

$$E^P(\varphi,\Lambda) = \sum_{s\in W} (c(s,\Lambda)\varphi)_{s\Lambda},$$

$$dE^P(\varphi,\Lambda) = \sum_{s\in W} d(c(s,\Lambda)\varphi)_{s\Lambda}.$$

Thus, on one hand we have

$$\begin{aligned} d(c(s,\Lambda)\varphi)_{s\Lambda} = \sum_{\mu} \sum_{p+q=m} & s(\lambda_i + \Lambda_i)\frac{dt_{\alpha_1}}{t_{\alpha_1}} \wedge \ldots \\ & \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge \frac{dt_i}{t_i} \wedge (c_q^\mu(s,\Lambda)\varphi)_{s\Lambda} \\ & + \sum_{\mu} \sum_{p+q=m} (\frac{dt_{\alpha_1}}{t_{\alpha_1}} \wedge \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge dc_q^\mu(s,\Lambda)\varphi)_{s\Lambda}. \end{aligned}$$

On the other hand, for each automorphic form

$$\frac{dt_{\alpha_1}}{t_{\alpha_1}} \wedge \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge \varphi : {}^\circ P/(\Gamma \cap P) \to \wedge^p \mathcal{A} \wedge \mathrm{Hom}(\wedge^q({}^\circ\mathcal{R}), V)$$

we can define $E(\frac{dt_{\alpha_1}}{t_{\alpha_1}} \wedge \cdots \wedge \frac{dt_{\alpha_p}}{t_{\alpha_p}} \wedge \varphi, \Lambda)$.

Lemma 2.4. $$dE^P(\varphi, \Lambda) = \sum_{i=1}^{l} -(w\lambda_i + \Lambda_i)E^P(\frac{dt_i}{t_i} \wedge \varphi, \Lambda).$$

Proof. It is easy to use repeatedly the parabolic rank one case [H] for each $t_1, t_2, \ldots, t_l$ and remark that

$$dE^P = \sum_{i=1}^{l} d_i E^P,$$

where for each t_i, d_i is the corresponding differential in fixing the others. The lemma is well proved .

3. Spectral decomposition

Recall that if $[\varphi] \in H^m_{w\lambda-\rho}({}^\circ X/(\Gamma \cap P); V)$, i. e. is of weight $w\lambda - \rho$, then $E(\varphi, \pm(\rho + \lambda))$ is a harmonic form, but has an isolate pole at such a point. We define

$$\mathrm{Res}_{\Lambda=w\lambda} E(\varphi, \Lambda) := \lim_{\Lambda \to w\lambda} \sum (\Lambda_i - w\lambda_i) E^P(\frac{dt_i}{t_i} \wedge \varphi, \Lambda).$$

Lemma 3.1. *$E'(\varphi, w\lambda) = Res_{\Lambda=w\lambda} E(\varphi, \Lambda)$ is a closed form.*

Proof. In vitue of Lemma 2.1, we have

$$dE(\varphi, \Lambda) = 0 \iff dE^P(\varphi, \Lambda) = 0.$$

Posing

$$c^\mu_q = \mathrm{Res}_{\Lambda=w\lambda} c^\mu_q(s, \Lambda),$$

we have

$$\begin{aligned} \left\|E'(\varphi, w\lambda)\right\|_2^2 &= c. < \varphi, C_{w\lambda-\rho}\varphi > \\ &= \int_{{}^\circ X/(\Gamma\cap P)} (\varphi, \overline{*_1 \circ \#_1 C_{w\lambda-\rho}\varphi}). \end{aligned}$$

The others are orthogonal each to other. Then $E'(\varphi, w\lambda)$ has a WP-ellipticity property. So $E'(\varphi, w\lambda)$ is a *closed form* by the Andreoti-Vesentini theorem. The lemma is proved.

We use now the Laplacian-Hodge decomposition for $C_{\lambda-\rho}$, $\lambda > 0$,

$$C_{\lambda-\rho}\varphi = \tilde{C}_{\lambda-\rho}\varphi + H,$$

where H is the sum of the eigenvectors to nonzero eigenvalues of Δ_M.

$$[C_{\lambda-\rho}\varphi] = [\tilde{C}_{\lambda-\rho}\varphi]$$

and

$$\langle\varphi, \tilde{C}_{\lambda-\rho}\varphi\rangle = \langle\varphi, C_{\lambda-\rho}\rangle.$$

We define

$$\tilde{C}_{\lambda-\rho}[\varphi] := [\tilde{C}_{\lambda-\rho}\varphi].$$

It is a strictly positive self-adjoint linear operator, acting on cohomology groups. We have therefore

Lemma 3.3. *If $\varphi \in \mathrm{Ker}(\tilde{C}_{\lambda-\rho})$, then $E(\varphi, \Lambda)$ is holomorphic at the point λ,*

$$H^m(\partial(\bar{X}_{cusp}/\Gamma); V) = \mathrm{Ker}\tilde{C}_\lambda \oplus \mathrm{Im}\tilde{C}_\lambda.$$

Proof. We have

$$0 = dE^P(\varphi, \lambda) = \sum_\mu (\mu - \lambda + \rho)\frac{dt_i}{t_i} \wedge (C^\mu_{m-1}\varphi)_{s\Lambda} + \cdots .$$

Thus, if $\mu = \lambda - \rho$, then $C_\mu\varphi$ is a coboundary, and hence

$$[(E')^P(\varphi, \lambda)]|_{\partial(\bar{X}_{cusp}/\Gamma)} = \tilde{C}_{\lambda-\rho}[\varphi].$$

The lemma is proved.

Lemma 3.4. *If $\lambda \neq 0$ is nonpositive, i.e. does not belong to the Weyl chamber, there exists w such that $w\lambda > 0$ and then*

$$\mathbf{H}^{N-m-1}_{-\rho-w\lambda}(\partial(\bar{X}_{cusp}/\Gamma); V) = \mathrm{Ker}\tilde{C}_{-\rho-w\lambda} \oplus \mathrm{Im}\ \tilde{C}_{-\rho-w\lambda}.$$

If $\varphi \in F_\lambda := *_1 \circ \#_1\mathrm{Ker}\tilde{C}_{-\rho-w\lambda}$, the form $E(\varphi, w\lambda)$ is closed and

$$[E(\varphi, w\lambda)]|_{\partial(\bar{X}_{cusp}/\Gamma)} = \sum_s [C_0^{-\rho-w\lambda}(s, w\lambda)\varphi].$$

Proof. If $\varphi \in F_\lambda$, $*_1 \circ \#_1\varphi \in \mathrm{Ker}\tilde{C}^{-\rho-w\lambda}$. We have

$$dE^P(*_1 \circ \#_1\varphi, \Lambda) = \sum_{i=1}^{l} -(w\lambda_i + \Lambda_i)E^P(\varphi, \Lambda).$$

Thus $\frac{1}{-(w\lambda+\Lambda)} dE^P(*_1 \circ \#_1\varphi, \Lambda)$ is holomorphic at $-w\lambda$, see [H] for details .

Remark 3.5. It was proved for SL_n and Sp_{2n} that

$$c(s,\Lambda) \equiv 0, \forall s \neq 1, c(1,\lambda) \equiv 1.$$

Thus

$$[E(\varphi, w\lambda)]|_{\partial(\bar{X}_{cusp}/\Gamma)} = [\varphi].$$

Consider the last case $\lambda = 0$. We see that $E(\varphi, \Lambda)$ is holomorphic at $\Lambda = 0$. We have

$$*_1 \circ \#_1 : H^m_\rho(\partial(\bar{X}_{cusp}/\Gamma); V) \to H^{N-m-1}_\rho(\partial(\bar{X}_{cusp}/\Gamma); V).$$

Always $dE(\varphi, 0) = 0$, because we have

$$* \circ \#dE^P(\varphi, \Lambda) = -\sum_i (\Lambda_i - w\lambda_i)E^P(.,.) \equiv 0,$$

$$\begin{array}{ccc} H^m(\partial(\bar{X}_{cusp}/\Gamma); V) & \xrightarrow{c(s,0)} & \mathcal{H}^{(m)} = \sum \mathcal{A}(^\circ X/\Gamma_{M_i}, \wedge^m Ad^*_{\mathbb{R}} \otimes \sigma, \lambda(c_M)) \\ \downarrow c(s,0) & & \downarrow \\ \mathcal{H}^{(m)} & \xrightarrow{c(s,0)} & \mathcal{H}^{(m)} \end{array}$$

So from the functional equations for $c(s,\Lambda)$, we have $c(s,0)^2 = Id$ and

$$\mathcal{H}^{(m)} = \sum_{s \in W} \mathcal{H}^{(m)}_{\varepsilon(s)},$$

where $\varepsilon(s) \in \mathbf{Z}/(2)^{|W|}$.

Denote $P^\pm_s$ the projectors on eigenspaces with eigenvalues ± 1 of $c(s,0)$. So

$$\begin{aligned} \mathcal{H}^{(m)} &= \oplus P^+_{s_1} \dots P^+_{s_l} P^-_{s_{l+1}} \dots P^-_{s_{|W|}} \mathcal{H}^{(m)} \\ &= \prod_{s \in W} (P^+_s + P^-_s)\mathcal{H}^{(m)}. \end{aligned}$$

We denote $P^+_{s_1}\dots P^+_{s_l}P^-_{s_{l+1}}\dots P^-_{s_{|W|}}\mathcal{H}^{(m)}$ by $\mathcal{H}^{(m)}_{\varepsilon(s)}$, where $\varepsilon(s_i)=\pm 1$ and

$$\varepsilon(s)=\varepsilon(s_1)\dots\varepsilon(s_{|W|}), s=(s_1,\dots,s_{|W|})$$

and $l(s):=\#\{s_i;\varepsilon(s_i)=1\}$.

Lemma 3.6.

$$H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)=\bigoplus_{s\in W}H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)_{\varepsilon(s)}.$$

The image of the restriction map

$$res: H^m(X/\Gamma;V)\to H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)$$

is

$$\sum_{s\in W, l(s)\neq |W|/2} H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)_{\varepsilon(s)}.$$

For $[\varphi]\in\mathcal{H}^{(m)}_{\varepsilon(s)}\cap H^{(m)}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)$, $[E(\varphi,0)]|_{\partial(\bar{X}_{cusp}/\Gamma)}=[\varphi]$.

Proof. We consider the pairing

$$H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)\times H^{N-m-1}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V^*)\to \mathbf{C}$$

$$\langle[\varphi],[\psi]\rangle=\int_{\partial(\bar{X}_{cusp}/\Gamma)}(\varphi,\psi).$$

If $[\varphi],[\psi]$ are the restrictions of classes on X/Γ, then

$$\langle[\varphi],[\psi]\rangle=0.$$

We introduce

$$R:=\mathrm{Image}\{r: H^m_{-\rho}(X/\Gamma;V)\to H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)\},$$

$$S:=\mathrm{Image}\{r: H^{N-m-1}_{-\rho}(X/\Gamma;V)\to H^{N-m-1}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)\}.$$

Then R and S are orthogonal each to another. So for each irreducible component E we have

$$\begin{aligned}\mathrm{mult}_E R+\mathrm{mult}_E S &\le \mathrm{mult}_E H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V)\\ &=\mathrm{mult}_E H^{N-m-1}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma);V).\end{aligned}$$

Certainly, for every $\varphi \in H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V)$, $E(\varphi, 0) \in R$ and for every

$$\psi \in H^{N-m-1}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V), E(\psi, 0) \in S.$$

We have also

$$\mathrm{Ker}\{\sum_{s\in W} c(s,0) : H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V) \to F := \sum_{l(s)=|W|/2} \mathcal{H}^{(m)}_{\varepsilon(s)} \cap H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V)\},$$

$$\mathrm{Ker}\{\sum_{s\in W} c(s,0) : H^{N-m-1}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V) \to H,$$

$$H := \sum_{l(s)=|W|/2} \mathcal{H}^{(N-m-1)}_{\varepsilon(s)} \cap H^{(N-m-1)}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V)\},$$

$$\mathrm{mult}_E F + \mathrm{mult}_E H \geq \mathrm{mult}_E H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V),$$

$$\#_1 \circ *_1(F) \subset \sum_{l(s)=|W|/2} \mathcal{H} H^{(N-m-1)}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V) \subset S$$

$$\#_1 \circ *_1(H) \subset \sum_{l(s)=|W|/2} \mathcal{H}^{(m)}_{\varepsilon(s)} \cap H^{(m)}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V) \subset R.$$

Hence,

$$\mathrm{mult}_E R + \mathrm{mult}_E S = \mathrm{mult}_E H^{(m)}_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V).$$

Thus we have

$$R = \sum_{l(s)\neq|W|/2} \mathcal{H}^{(m)}_{\varepsilon(s)} \cap H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V),$$

$$S = \sum_{l(s)=|W|/2} \mathcal{H}^{(m)}_{\varepsilon(s)} \cap H^m_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V).$$

The lemma is proved.

From the last three lemmas we have our main result about spectral decomposition

Theorem 2. (i) *The image of the restriction map*

$$\mathrm{res} : H^*(X/\Gamma; V) \to H^*(\partial(\bar{X}_{cusp}/\Gamma); V)$$

is compatible with the decomposition

$$\bigoplus_{w\in W}\bigoplus_{\lambda>0}\{H^*_{w\lambda-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V)\oplus H^*_{w\lambda-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V)\}$$
$$\bigoplus_{s\in W} H^*_{-\rho}(\partial(\bar{X}_{cusp}/\Gamma); V)_{\epsilon(s)}.$$

(ii) *For every* $\omega \in Im(res)$*, there exists a cohomology class* $\tilde{\omega} \in H^*(X/\Gamma; V)$ *such that* $\tilde{\omega}$ *can be represented by singular values of an Eisenstein series, which is harmonic and* $res(\tilde{\omega}) = \omega$.

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