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QUASIDIFFERENTIABLE FUNCTIONS AND PAIRS OF CONVEX COMPACT SETS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. In the theory of optimization several types of piecewise differentiable functions occur in a quite natural way. As a typical example for such nondifferentiable functions we mention the finite max-min combinations of differentiable functions. A more general class are the quasidifferentiable functions, which are investigated in detail by V. F. Demyanov and A. M. Rubinov (see for instance [1]).

The directional derivatives of these functions can be represented as a difference of two sublinear functions. Since a sublinear function is uniquely described by its subdifferential in the origin, there exists a natural correspondence between the directional derivatives and the set of pairs of convex compact sets. In this paper we give a comprehensive representation of the results in [5], [6] and [7]. Moreover we show that there exists a natural definition for the difference between pairs of convex compact sets.

1. Introduction

Let U be an open subset of $\mathbf{R}^n$ and let $f_i : U \longrightarrow \mathbf{R}$, $i \in \{1, \dots, m\}$, continuously differentiable functions. Then a typical example for a piecewise differentiable function is given by:

$$f : U \longrightarrow \mathbf{R}, \quad \text{with} \quad f(x) = \max_{i \in \{1,\dots,k\}} \min_{j \in M_i} f_j(x),$$

where $M_i \subseteq \{1, \dots, m\}$ for each $i \in \{1, \dots, k\}$. For a given point $x_0 \in U$ the *essential active* index sets are given by $\hat{M}_i(x_0) = \big\{ j \in M_i \mid f_j(x_0) = \min_{l \in M_i} f_l(x_0) \big\}$ for $i \in \{1, \dots, k\}$ and by $\hat{I}(x_0) = \big\{ i \in \{1, \dots, k\} \mid f(x_0) = \max_{i \in \{1,\dots,k\}} \min_{j \in \hat{M}_i(x_0)} f_j(x_0) \big\}$.

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For the directional derivative of f at $x_0 \in U$ in the direction $g \in \mathbf{R}^n$ the following formula $\left.\frac{df}{dg}\right|_{x_0} = \max_{i \in \hat{I}(x_0)} \min_{j \in \hat{M}_i(x_0)} \langle \left.\nabla f_j\right|_{x_0}, g \rangle$ holds, since every max-min combination of linear functions is representable as the difference of two sublinear functions, namely:

$$\left.\frac{df}{dg}\right|_{x_0} = \max_{i \in \hat{I}(x_0)} \Big\{ \sum_{\substack{k \in \hat{I} \\ k \neq i}} \max_{j \in \hat{M}_k(x_0)} \langle -\nabla f_j\big|_{x_0}, g \rangle \Big\} - \sum_{k \in \hat{I}(x_0)} \max_{j \in \hat{M}_k(x_0)} \langle -\nabla f_j\big|_{x_0}, g \rangle.$$

A more general class of nondifferentiable functions has been considered by V. F. Demyanov and A. M. Rubinov in [1]. They consider the following situation:

Let $(X, \|\cdot\|)$ be a real normed vector space, let X^* be its topological dual, and let $U \subseteq X$ be an open subset of X. The dual norm of X will be denoted by $\|\cdot\|^*$. Moreover, let

$$\langle \cdot, \cdot \rangle \ : \ X^* \times X \to \mathbf{R}$$

be the dual pairing given by

$$\langle \, v, x \, \rangle \ := v(x).$$

Definition 1.1. *A continuous real-valued function $f : U \to \mathbf{R}$ is said to be* quasidifferentiable *at $x_0 \in U$ if the following two conditions are satisfied:*

(a) For every $g \in X \setminus \{0\}$ the directional derivative

$$\left.\frac{df}{dg}\right|_{x_0} = \lim_{t \to 0+} \frac{f(x_0 + tg) - f(x_0)}{t}$$

exists.

(b) There exist two sets $\underline{\partial} f|_{x_0}$, $\overline{\partial} f|_{x_0} \in \mathcal{K}(X^*)$ such that

$$\left.\frac{df}{dg}\right|_{x_0} = \max_{v \in \underline{\partial} f|_{x_0}} \langle v, g \rangle + \min_{w \in \overline{\partial} f|_{x_0}} \langle w, g \rangle.$$

Here $\mathcal{K}(X^*)$ denotes the collection of all nonempty weak-*-compact convex subsets of X^*. We remark that, by Theorem of Alaoglu (cf. [9], p. 228) the elements of $\mathcal{K}(X^*)$ are bounded in the dual norm.

A real-valued function $p : X \to \mathbf{R}$ is called *sublinear* if

(i) $p(tx) = tp(x)$ for all $x \in X$ and $t \in \mathbf{R}_+ := \{t \in \mathbf{R} \mid t \geq 0\}$,

(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

It was shown by L. Hörmander [3] that a sublinear function $p : X \to \mathbf{R}$ is continuous if and only if its subdifferential at the origin

$$\partial p|_0 := \{v \in X^* \mid \langle v, x \rangle \leq p(x),\ x \in X\}$$

is an element of $\mathcal{K}(X^*)$. Since every continuous sublinear function $p : X \to \mathbf{R}$ can be expressed as

$$p(x) = \max_{a \in A} \langle a, x \rangle \tag{1}$$

for a set $A := \partial p|_0 \in \mathcal{K}(X^*)$, the condition (b) is equivalent to the requirement that the directional derivative as a function of the direction g can be expressed as the difference of two sublinear functions. By

$$\mathcal{P}(X) := \{p : X \to \mathbf{R} \mid p \text{ is sublinear and continuous }\}$$

we denote the convex cone of all real-valued sublinear functions defined on X and by

$$\mathcal{D}(X) := \{\varphi = p - q \mid p, q \in \mathcal{P}(X)\}$$

the real vector space of differences of sublinear functions. This space is a lattice with respect to the pointwise max- and min-operations (cf. [1], p. 74).

Observe that every difference of sublinear functions is representable by a difference of two non-negative sublinear functions. This follows immediately from Theorem of Hahn-Banach. Namely, let

$$\varphi(h) := \inf_{v \in \overline{\partial} f\big|_{x_0}} \langle\, v, g \,\rangle - \sup_{v \in -\overline{\partial} f\big|_{x_0}} \langle\, v, h \,\rangle.$$

Then we can add to the first summand a continuous linear functional $f_1 \in X^\star$ with

$$\inf_{v \in \overline{\partial} f\big|_{x_0}} \langle\, v, g \,\rangle \geq f_1(h)$$

and, analogously, to the second summand a $f_2 \in X^\star$ with

$$\sup_{v \in -\overline{\partial} f\big|_{x_0}} \langle\, v, h \,\rangle \geq f_2(h).$$

Now we take a representation:

$$\varphi(h) = (\inf_{v \in \overline{\partial} f\big|_{x_0}} \langle\ v, g\ \rangle - f_1(h) + \max\{f_1(h) - f_2(h), 0\})$$

$$-(\sup_{v \in -\overline{\partial} f\big|_{x_0}} \langle\ v, h\ \rangle - \min\{f_2(h) - f_1(h), 0\}).$$

This technique suggests that different representations of a given function as a difference of sublinear functions arise from adding and subtracting suitable sublinear functions. But the situation is more complicated, as the following example shows.

Example 1.2. There exists an element $\varphi \in \mathcal{D}(X)$ with two different representations as a difference of sublinear functions, which do not connected by adding and subtracting suitable sublinear functions, namely

$$\varphi : \mathbf{R}^2 \to \mathbf{R},$$

with

$$\begin{aligned}\varphi(x_1, x_2) &= \max\{0, x_1, x_2, x_1 + x_2\} - \max\{x_1, x_2, x_1 + x_2\} \\ &= \max\{0, x_1, x_2\} - \max\{x_1, x_2\}.\end{aligned}$$

For a more detailed investigation we use the representation (1) of a sublinear function which yields a representation of $\varphi \in \mathcal{D}(X)$ in terms of a pair of compact convex sets $(A, B) \in \mathcal{K}(X^*) \times \mathcal{K}(X^*)$. Observe that we have

$$\max_{a \in A} \langle a, x\rangle - \max_{b \in B} \langle b, x\rangle \ = \ \max_{c \in C} \langle c, x\rangle - \max_{d \in D} \langle d, x\rangle$$

if and only if $A + D = B + C$, where $+$ denotes the usual Minkowski addition, i.e. $A+B = \{x \in X^* \mid x = a+b,\ a \in A,\ b \in B\}$. This motivates the introduction of the following equivalence relation $\sim$ on $\mathcal{K}(X^*)\times\mathcal{K}(X^*)$:

$$(A, B) \sim (C, D) \text{ if and only if } A + D = B + C$$

which will be studied in the next section.

2. The Minkowski-Rådström-Hörmander lattice

In this section, we will consider pairs of nonempty convex compact sets. Let $X = (X, \tau)$ be a topological Hausdorff vector space and let $\mathcal{K}(X)$ be the collection of all nonempty convex compact subsets of X. On $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$ the equivalence relation

$$(A, B) \sim (C, D) \quad \Longleftrightarrow \quad A + D \;=\; B + C$$

is introduced and $[A, B] \in \mathcal{K}^2(X)_{/\sim}$ denotes the equivalence class which is represented by $(A, B) \in \mathcal{K}^2(X)$.

We recall some notations:

If (X, τ) is locally convex, then we denote for a continuous linear functional $f \in X^*$ by

$$H_f(A) := \{z \in \; A \mid f(z) \;=\; \max_{y \in A} f(y)\}$$

the face of $A \in \mathcal{K}(X)$ with respect to f. For the sum of the faces of two nonempty compact convex sets $A, B \subseteq X$ with respect to $f \in X^*$ the following identity holds:

$$H_f(A \;+\; B) = H_f(A) \;+\; H_f(B).$$

For $A \in \mathcal{K}(X)$ we denote by $\mathcal{E}(A)$ the set of extremal points, and by $\mathcal{E}_0(A)$ the set of exposed points. For two compact convex sets $A, B \subseteq X$ in a topological vector space X we will use the notation

$$A \vee B := \operatorname{conv}(A \cup B),$$

where the operation "conv" denotes the convex hull. For a set $A \subseteq X$ we denote by $\bar{A}$ the closure of A. If $A, K \subset X$ are nonempty compact convex sets then A is said to be a *summand* of K if and only if there exists a nonempty compact convex set $B \subset X$ with $A \;+\; B \;=\; K$ for the Minkowski sum. An essential role in $\mathcal{K}(X)$ is played by the

Order Cancellation Law (see [3], [11]). *Let X be a topological vector space and $A, B, C \subseteq X$ compact convex subsets. Then the inclusion $A \;+\; B \subseteq A \;+\; C$ implies $B \subseteq C$.*

The following identity for compact convex sets was first observed by A. Pinsker, namely:

Pinsker's Identity (see [8]). *For $A, B, C \in \mathcal{K}(X)$ in a topological vector space X we have:*

$$(A \; + \; C) \vee (B \; + \; C) = C \; + \; (A \vee B).$$

This identity can be proved as follows: Every $x \in (A \vee B) + C$ can be represented as $x = \alpha \cdot a + (1 - \alpha) \cdot b + c$ with $a \in A,\; b \in B,\; c \in C$ and $0 \leq \alpha \leq 1$. Now $x = \alpha \cdot a + (1-\alpha) \cdot b + c \; = \; \alpha \cdot (a+c) \; + \; (1-\alpha) \cdot (b+c)$ and hence $C + (A \vee B) \subseteq (A \; + \; C) \vee (B \; + \; C)$.

The converse inclusion can be seen as follows: Let $x \in (A{+}C) \vee (B{+}C)$. Then we have: $x = \alpha \cdot (a+c_1) + (1-\alpha) \cdot (b+c_2)$ with $a \in A,\; b \in B,\; c_1, c_2 \in C$ and $0 \leq \alpha \leq 1$. Now $x = \alpha \cdot (a+c_1) + (1-\alpha) \cdot (b+c_2) = \alpha \cdot a + (1-\alpha) \cdot b + \alpha \cdot c_1 + (1-\alpha) \cdot c_2$. Hence the converse inclusion $(A{+}C) \vee (B{+}C) \subseteq C + (A \vee B)$ also holds.

We will use the abbreviation $A + B \vee C$ for $A + (B \vee C)$ and $C + d$ for $C + \{d\}$ for compact convex sets A, B, C and a point d. Moreover we will write $[a, b]$ instead of $\{a\} \vee \{b\}$.

A. G. Pinsker [8] introduced the following partial order on $\mathcal{K}(X)_{/\sim}$

$$[A, B] \preceq [C, D] \quad \Longleftrightarrow \quad A + D \; \subseteq \; B + C.$$

This partial ordering is independent of a special choice of representants. Namely, for $(A', B') \in [A, B]$ and $(C', D') \in [C, D]$ the inclusion $A + D \subseteq B + C$ implies $A' + D' \subseteq B' + C'$, since:

$$\begin{array}{rll} B + C \supset & A + D; & \\ B + C + C' \supset & A + D + C' = A + C + D'; & \text{since } (C, D) \sim (C', D') \\ B + C' \supset & A + D'; & \text{from cancelling } C \\ B + C' + B' \supset & A + D' + B' = B + A' + B'; & \text{since } (A, B) \sim (A', B') \\ C' + B' \supset & D' + A'; & \text{from cancelling } A \end{array}$$

Thus from the algebraic point of view the set $\mathcal{K}(X)$ of all nonempty compact convex subsets of a real topological vector space X is a *commutative semi-ring with cancellation property* endowed with the "*addition* $\oplus$ " given by

$$A \; \oplus \; B := A \vee B$$

and the "*multiplication* $\star$" given by

$$A \star B := A \; + \; B.$$

Within this context, the elements of $\mathcal{K}^2(X)$ (with respect to the relation $\sim$) can be considered as *fractions.*

The operation of an addition can be extended to the elements of $\mathcal{K}^2(X)_{/\sim}$ as follows:

$$\mathcal{K}^2(X)_{/\sim} \times \mathcal{K}^2(X)_{/\sim} \longrightarrow \mathcal{K}^2(X)_{/\sim}$$

with

$$[A,B] \oplus [C,D] = [(A+D) \vee (C+D), B+D].$$

This operation is independent from a special choice of representants by using the above technique. Moreover, if $[A,B] \preceq [C,D]$, i.e. $A+D \subseteq B+C$, then we define a difference $[E,F]$ by the equation

$$[A,B] \oplus [E,F] \;=\; [C,D],$$

and we will write $[E,F] = [C,D] \ominus [A,B]$. The existence of such a difference follows from

Proposition 2.1. *Let (X,τ) be a topological vector space and $[A,B]$, $[C,D] \in \mathcal{K}^2(X)_{/\sim}$. If $[A,B] \preceq [C,D]$, i.e. $A+D \subseteq B+C$, then the difference $[E,F] = [C,D] \ominus [A,B]$ exists.*

Proof. We have to solve the equation

$$[A,B] \oplus [E,F] \;=\; [C,D],$$

which is equivalent to the equation

$$(A+F) \vee (B+E) + D \;=\; B+C+F.$$

This equation can be solved as follows. Let us put

$$E \;=\; A+U \quad \text{and} \quad F \;=\; B+V,$$

for some elements $U, V \in \mathcal{K}(X)$, which we consider for a moment as unknown variables. If we fix the first variable V, for instance $V \;=\; A+D$ then we can find a solution for the second variable $U \in \mathcal{K}(X)$ which satisfies $U \vee V \;=\; B+C$. This can be seen as follows

$$[A,B] \oplus [E,F] \;=\; [C,D] \Longleftrightarrow (A+F) \vee (B+E) + D \;=\; B+C+F,$$

i.e.

$$\begin{aligned}(A+F)\vee(B+E)+D &= B+C+F\\ &\Updownarrow\\ (A+B+V)\vee(B+A+U)+D &= B+C+B+V\\ &\Updownarrow\\ A+B+D+(U\vee V) &= B+B+C+V\\ &\Updownarrow\\ A+D+(U\vee V) &= B+C+V.\end{aligned}$$

Any $U\in\mathcal{K}(X)$ with $U\vee V = B+C$ is a solution, for instance $U = B+C$ since $A+D\subseteq B+C$. □

Corollary 2.2. *Let (X,τ) be a topological vector space and $A,B\in\mathcal{K}(X)$ with $A\subseteq B$. Then every $E\in\mathcal{K}(X)$ with $A\vee E = B$ gives a solution of*

$$[A,\{0\}]\oplus[E,\{0\}] = [B,\{0\}].$$

Proof. This follows immediately from the equation

$$(A+\{0\})\vee(E+\{0\})+\{0\} = B+\{0\}$$ □

Thus for elements $A,B\in\mathcal{K}(X)$ of a topological vector space X with $A\subseteq B$ we define the *convex complement* by $E = B\hat{\ominus}A$ as the minimal inclusion $E\in\mathcal{K}(X)$ with $A\vee E == B$. Let us observe, that by the Krein-Milman Theorem this difference is uniquely determined.

The multiplication between elements in $\mathcal{K}(X)$ can be extended to $\mathcal{K}^2(X)_{/\sim}$ by

$$[A,B]\star[C,D] = [A+C,B+D] \text{ for } [A,B],[C,D]\in\mathcal{K}^2(X)_{/\sim}.$$

It is easy to see that this extension of $\star$ is independent of a special choice of representants. The neutral element for the multiplication is $[\{0\},\{0\}]$ and the multiplicative inverse of $[A,B]\in\mathcal{K}^2(X)_{/\sim}$ is given by $[B,A]\in\mathcal{K}^2(X)_{/\sim}$.

From the Pinsker formula follows, that the distributivity law holds also in $\mathcal{K}^2(X)_{/\sim}$, i.e., for all $[A,B],[C,D],[E,F]\in\mathcal{K}^2(X)_{/\sim}$ we have

$$[A,B]\star([C,D]\oplus[F,G]) = ([A,B]\star([C,D])\oplus([A,B]\star[F,G])$$

All together we have

Theorem 2.3. *Let* (X, τ) *be a topological vector space. Then* $(\mathcal{K}^2(X)_{/\sim}, \star, \oplus)$ *is a partially ordered semi-ring with cancellation property, such that for every* $[A, B] \in \mathcal{K}^2(X)_{/\sim}$ *the multiplicative inverse exists and for all* $[A, B], [C, D] \in \mathcal{K}^2(X)_{/\sim}$*, with* $[A, B] \preceq [C, D]$ *the difference.*

3. Minimal pairs of compact convex sets

For a nonempty compact convex set $A \subset X$ we consider a set $\mathcal{S} \subseteq X^* \setminus \{0\}$ such that

$$\overline{\operatorname{conv}(\bigcup_{f \in \mathcal{S}} H_f(A))} \;=\; A.$$

Such a set $\mathcal{S} \subset X^* \setminus \{0\}$ is called a *shape of* A and will be denoted by $\mathcal{S}(A)$. For a shape $\mathcal{S}(A)$ we consider subsets

$$\mathcal{S}_p(A) := \{f \in \mathcal{S}(A) \mid \operatorname{card}(H_f(A)) = 1\},$$

which may be empty and

$$\mathcal{S}_l(A) := \mathcal{S}(A) \setminus \mathcal{S}_p(A).$$

In the sequel we will state some typical sufficient conditions for minimality: The criteria presented here are of two different types: The first type of criteria uses conditions which ensure that two compact convex sets are in a certain "general position", while the second type of criteria uses information about exposed points of the Minkowski sum of compact convex sets. For a more detailed presentation of this topic we refer to ([5], [6], [7]).

We start with a criterium for minimality which is of the first type.

Theorem 3.1. *Let* (X, τ) *be a real locally convex topological vector space and let* $A, B \subset X$ *be nonempty compact convex sets. Assume that there is a shape* $\mathcal{S}(A)$ *of* A *satisfying the following conditions:*

(i) *for every* $f \in \mathcal{S}(A)$, $\operatorname{card}(H_f(B)) = 1$,

(ii) *for every* $f \in \mathcal{S}_l(A)$ *and every* $b \in B$*, the condition*

$$H_f(A) \;+\; (b \;-\; H_f(B)) \;\subseteq\; A \text{ implies } b \;=\; H_f(B),$$

(iii) *for every* $f \in \mathcal{S}_p(A)$, $H_f(A) - H_f(B) \in \mathcal{E}(A - B)$, *or conversely, by interchanging A and B. Then the pair $(A,B) \in K^2(X)$ is minimal.*

Proof. Assume, that $A^{'} \subseteq A$ and $B^{'} \subseteq B$ are nonempty compact convex sets such that

$$A^{'} \; + \; B \; = \; A \; + \; B^{'}.$$

Choose an element $f \in \mathcal{S}(A)$. Since

$$H_f(A) \; + \; H_f(B^{'}) \; = \; H_f(B) \; + \; H_f(A^{'})$$

and $H_f(B) = \{b\}$, this can be written as:

$$H_f(A) \; + \; H_f(B^{'}) \; = \; \{b\} \; + \; H_f(A^{'}).$$

Now we choose an element $b^{'} \in H_f(B^{'})$ and determine for every extremal point $e \in \mathcal{E}(H_f(A))$ an element $a_e \in H_f(A^{'})$ such that

$$e \; + \; b^{'} \; = \; b \; + \; a_e.$$

The following two cases are possible:

• **case p):** Assume that $f \in \mathcal{S}_p(A)$. Then $e - b = a_e - b^{'}$. Since, by condition iii), $e - b \in \mathcal{E}(A - B)$, we have $a_e = e$ and $b^{'} = b$. Hence $H_f(B^{'}) = H_f(B) = \{b\}$ and therefore

$$H_f(A) \; = \; H_f(A^{'}).$$

• **case l):** Now we assume that $f \in \mathcal{S}_l(A)$. In this case we choose finitely many extremal points $e_1, ..., e_n \in \mathcal{E}(H_f(A))$ which lead to the system of equations

$$e_i \; + \; b^{'} \; = \; b \; + \; a_{e_i} \; , \quad i \in \{1, ..., n\}.$$

Clearly,

$$\bigvee_{i=1}^{n} (e_i + b^{'}) = \bigvee_{i=1}^{n} (b + a_{e_i}).$$

Hence by the result of A. G. Pinsker ([8]), this implies

$$(\bigvee_{i=1}^{n} (e_i) + b^{'} = b + (\bigvee_{i=1}^{n} a_{e_i}),$$

which gives

$$\bigvee_{i=1}^{n}(e_i + b^{'}) \subseteq b + H_f(A^{'}).$$

Since $e_1, ..., e_n \in \mathcal{E}(H_f(A)$ are arbitrarily chosen, it follows from the theorem of Krein-Milman (cf. [9], p. 239) that

$$H_f(A) + b^{'} \subseteq b + H_f(A^{'}).$$

By condition ii) this gives $b = b^{'}$ and hence

$$H_f(A^{'}) = H_f(A).$$

Thus, for all $f \in \mathcal{S}(A)$ we have

$$H_f(A^{'}) = H_f(A).$$

Therefore

$$A^{'} \supseteq \overline{\operatorname{conv}(\bigcup_{f\in\mathcal{S}(A)} H_f(A'))} \;=\; \overline{\operatorname{conv}(\bigcup_{f\in\mathcal{S}(A)} H_f(A))} \;=\; A\ ,$$

i.e., $A^{'} = A$.

Then, by cancellation law, the equality

$$A \;+\; B^{'} \;=\; B \;+\; A^{'}$$

implies that

$$B^{'} = B,$$

which completes the proof. □

The next criterium for minimality is based on a sufficient condition on the indecomposability of a nonempty compact convex set and is formulated in terms of exposed points of this set.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a real Banach space, and let $(A, B) \in \mathcal{K}^2(X)$. If for every exposed point $a + b \in \mathcal{E}_0(A + B)$ with $a \in \mathcal{E}_0(A)$, $b \in \mathcal{E}_0(B)$ there exists $b_1 \in \mathcal{E}_0(B)$ or $a_1 \in \mathcal{E}_0(A)$ such that $a + b_1 \in \mathcal{E}_0(A + B)$ and $a - b_1 \in \mathcal{E}(A - B)$ or $a_1 + b \in \mathcal{E}_0(A + B)$ and $a_1 - b \in \mathcal{E}(A - B)$, then (A, B) is minimal.*

Proof. Let $(A,B) \in \mathcal{K}^2(X)$ and $f \in X^*$. Then

$$H_f(A+B) = H_f(A) + H_f(B).$$

This implies the unique representation of every exposed point of $A+B$ as a sum of exposed points of A and B.

Let us show that the pair $(A,B) \in \mathcal{K}^2(X)$ is minimal. To do this, we choose a pair $(A^{'}, B^{'}) \in \mathcal{K}^2(X)$ such that $A^{'} \subseteq A$, $B^{'} \subseteq B$ and $A + B^{'} = B + A^{'}$. Let $a+b \in \mathcal{E}_0(A+B)$. Without loss of generality we can assume that for $a \in \mathcal{E}_0(A)$ there exists $b_0 \in \mathcal{E}(B)$ such that $a + b_0 \in \mathcal{E}_0(A+B)$ and $a - b_0 \in \mathcal{E}(A-B)$. Hence there exists a continuous linear functional $f_0 \in X^*$ such that

$$H_{f_0}(A+B) = \{a + b_0\}.$$

By the above formula for faces we have

$$H_{f_0}(A) = \{a\} \quad \text{and} \quad H_{f_0}(B) = \{b_0\}.$$

Since

$$A \ + \ B^{'} \ = \ B \ + \ A^{'} \ =: K,$$

it follows that

$$H_{f_0}(A) + H_{f_0}(B^{'}) \ = \ H_{f_0}(B) + H_{f_0}(A^{'}).$$

Hence there exist elements $a^{'} \in H_{f_0}(A^{'}) \subseteq A$ and $b^{'} \in H_{f_0}(B^{'}) \subseteq B$ such that

$$a \ + \ b^{'} \ = \ b_0 \ + \ a^{'}.$$

Since $a - b_0 \in \mathcal{E}(A-B)$, it follows that $a = a^{'}$, $b_0 = b^{'}$. The equality implies

$$B \ + \ a \ \subseteq B \ + \ A^{'} \ = \ K.$$

Hence $a + b \in K$. Since $a+b \in \mathcal{E}_0(A+B)$, by a modification of V. Klee ([4]) of Krein-Milman Theorem, it follows that

$$A \ + \ B \ = \ K.$$

Hence the cancellation law implies

$$A \ + \ B^{'} \ = \ B \ + \ A^{'} \ , \quad \text{i.e.} \quad A = A^{'},$$

and

$$A \ + \ B^{'} \ = \ B \ + \ A^{'} \ , \quad \text{i.e. } B = B^{'}.$$

Therefore $(A, B) \in \mathcal{K}^2(X)$ is minimal. □

4. Reduction techniques

We now present a technique for the reduction of pairs of compact convex sets by cutting hyperplanes or by excision which is based on the following observation:

Lemma 4.1. *Let X be a real topological vector space and $A, B, S \in K(X)$. Then*

$$A \vee B + S \subseteq (A \vee S) + (B \vee S).$$

Proof. Given any $x \in A \vee B$ and $s \in S$, then $x = \alpha a + \beta b$ for some elements $a \in A$, $b \in B$ and numbers $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Hence

$$\begin{aligned} x + s = \alpha a + \beta b + s &= \alpha a + \beta b + (\alpha + \beta) s \\ &= (\alpha a + \beta s) + (\alpha s + \beta b) \in A \vee S + B \vee S. \end{aligned}$$

□

Remark 4.2. i) If $S = A \cap B \neq \phi$ then $A \vee B + A \cap B \subseteq A + B$.

ii) If for a compact convex set $S \neq \emptyset$ the equation $A + B = A \vee B + S$ holds then $S = A \cap B$.

Part i) is obvious. To prove ii) observe that $A + B = A \vee B + S$ implies both $A + B \supseteq B + S$ and $A + B \supseteq A + S$ and hence $S \subset A \cap B$.

From Lemma 4.1 follows

$$A \vee B + A \cap B \subseteq A + B \subseteq A \vee B + S.$$

Hence $A \cap B \subseteq S$ and therefore $S = A \cap B$.

This lemma leads to the following definition

Let X be a real topological vector space and $A, B, S \in K(X)$. Then S is *separating* the sets A and B if and only if for every $a \in A$ and $b \in B$ we have $[a, b] \cap S \neq \emptyset$, with $[a, b] = \{a\} \vee \{b\}$.

Now we have

Lemma 4.3. *Let X be a real topological vector space, $A, B, S \in K(X)$ such that S is separating A and B. Then*

$$A + B \subseteq A \vee B + S.$$

Proof. Let $a \in A$, $b \in B$. Then there exists $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ such that

$$\alpha a + \beta b \in S.$$

Hence

$$\begin{aligned} a + b &= (\alpha a + \beta a) + (\alpha b + \beta b) \\ &= (\beta a + \alpha b) + (\alpha a + \beta b) \in A \vee B + S. \end{aligned}$$

□

Next we have

Proposition 4.4. *Let X be a real topological vector space, $A, B \in K(X)$ such that $A \cap B \neq \emptyset$. Then*

i) if $A \cap B$ separates the sets A and B, then

$$A \vee B + A \cap B = A + B.$$

ii) if X is locally convex then $A \vee B + A \cap B = A + B$ implies that $A \cap B$ separates A and B.

Proof. i) Let us write $S := A \cap B$. Then from Remark 4.2 i) it follows that

$$A \vee B + A \cap B \subseteq A + B.$$

Moreover from Lemma 4.3 we have that

$$A + B \subseteq A \vee B + A \cap B.$$

Hence

$$A + B = A \vee B + A \cap B.$$

ii) Now let us assume that X is locally convex and that $A \vee B + A \cap B = A + B$ holds. If $A \cap B \neq \emptyset$ does not separate the sets A and B, then there exist points $a \in A$ and $b \in B$ such that

$$[a, b] \cap (A \cap B) = \emptyset.$$

Since X is locally convex there exists a continuous linear functional $f \in X^*$ such that

$$\max(f(a), f(b)) \leq \min_{z \in A \cap B} f(z).$$

Now choose elements $a_0 \in A$, $b_0 \in B$ such that

$$f(a_0) = \min_{a \in A} f(a), \quad f(b_0) = \min_{b \in B} f(b).$$

Since

$$\max(f(a_0), f(b_0)) \leq \max(f(a), f(b)) < \min_{z \in A \cap B} f(z),$$

it follows that

$$[a_0, b_0] \cap (A \cap B) = \emptyset.$$

Since by assumption $A + B = A \vee B + A \cap B$, there exist elements $a_1 \in A$, $b_1 \in B$, $z_1 \in A \cap B$, and numbers $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, such that

$$a_0 + b_0 = \alpha a_1 + \beta b_1 + z_1.$$

Hence

$$f(a_0) + f(b_0) = \alpha f(a_1) + \beta f(b_1) + f(z_1).$$

Since

$$\max(f(a_0) - f(z_1),\ f(b_0) - f(z_1)) < 0,$$

this implies

$$f(a_0) > \alpha f(a_1) + \beta f(b_1) \quad \text{and} \quad f(b_0) > \alpha f(a_1) + \beta f(b_1).$$

Hence $f(a_0) > f(b_1)$ and $f(b_0) > f(a_1)$ which leads to the contradiction

$$f(a_0) > f(b_1) \geq f(b_0) > f(a_1).$$

□

Theorem 4.5. *Let X be a real topological vector space, $A \in K(X)$ a nonempty compact convex set. Moreover let us assume that there exists a nonempty compact convex subset $C \subsetneq A$ such that $A \setminus C$ is nonempty and convex. Then the pairs*

$$(A, C),\ (\overline{A \setminus C}, C \cap \overline{(A \setminus C)}) \in K^2(X)$$

are equivalent.

Proof. Put

$$S = C \cap (\overline{A \setminus C}).$$

Then it is obvious that S separates $\overline{A \setminus C}$ and C. Hence by Proposition 4.4 i) we have

$$(\overline{A \setminus C}) \vee C + S = \overline{A \setminus C} + C.$$

Since

$$(\overline{A \setminus C}) \vee C = A,$$

we get

$$A + S = (\overline{A \setminus C}) + C,$$

which means

$$(A, C) \sim (\overline{A \setminus C}, C \cap (\overline{A \setminus C})).$$

□

In the case where X is a real locally convex topological vector space, the assumption that the sets C and $A \backslash C$ are convex is equivalent to the existence of a point $z \in A$ and a continuous linear functional $f \in X^*$ such that

$$\overline{A \setminus C} = A^+_{f,z} := \{x \in A \mid f(x) \geq f(z)\},$$

and

$$C = A^-_{f,z} := \{x \in A \mid f(x) \leq f(z)\}.$$

Observe that

$$(\overline{A \setminus C}) \cap C = A_{f,z} := \{x \in A \mid f(x) = f(z)\}.$$

The above result now leads us to a theorem on the reduction of pairs of nonempty compact convex sets by cutting hyperplanes.

Theorem 4.6. *Let X be a real locally convex topological vector space, $A, B \in K(X)$ nonempty compact convex sets and let us assume that there exists an element $z \in A \cap B$ and a continuous linear functional $f \in X^*$ such that $A^+_{f,z} = B^+_{f,z}$ and $A_{f,z} = B_{f,z}$. Then the pairs*

$$(A, A^-_{f,z}) \ , \ (B, B^-_{f,z}) \in K^2(X)$$

are equivalent.

Proof. By the above Theorem 4.5 we have $(A, A^-_{f,z}) \sim (A^+_{f,z}, A_f)$. Since by assumption $A^+_{f,z} = B^+_{f,z}$ and $A_{f,z} = B_{f,z}$, it follows that

$$(A, A^-_{f,z}) \sim (B, B^-_{f,z}).$$

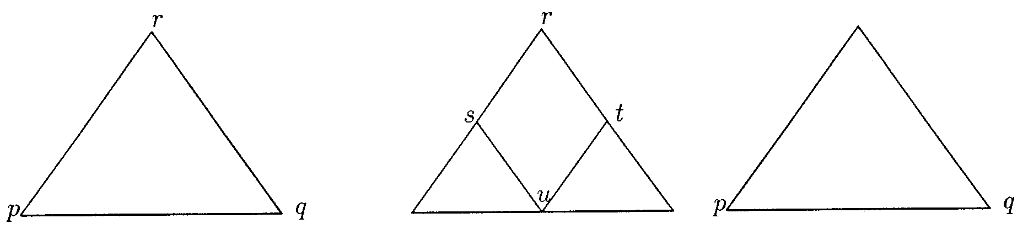
□

For further results about pairs of compact convex sets we refer to [2], [10], [7] and [12].

5. Examples

In this section, we will illustrate the reduction technique and the two criteria for minimality by simple examples of compact convex sets in the plane.

Example 5.1. We begin with an example for the convex complement. Let us consider the sets $A = \{p\} \vee \{q\} \vee \{r\}$ and $B = \{r\} \vee \{s\} \vee \{t\} \vee \{u\}$ as indicated in the following figure.



The convex complement is $E = A \hat{\ominus} B = \{p\} \vee \{q\}$.

Example 5.2. The two different representations of

$$\varphi : \mathbf{R}^2 \to \mathbf{R},$$

with

$$\begin{aligned}\varphi(x_1, x_2) &= \max\{0, x_1, x_2, x_1 + x_2\} - \max\{x_1, x_2, x_1 + x_2\} \\ &= \max\{0, x_1, x_2\} - \max\{x_1, x_2\}\end{aligned}$$

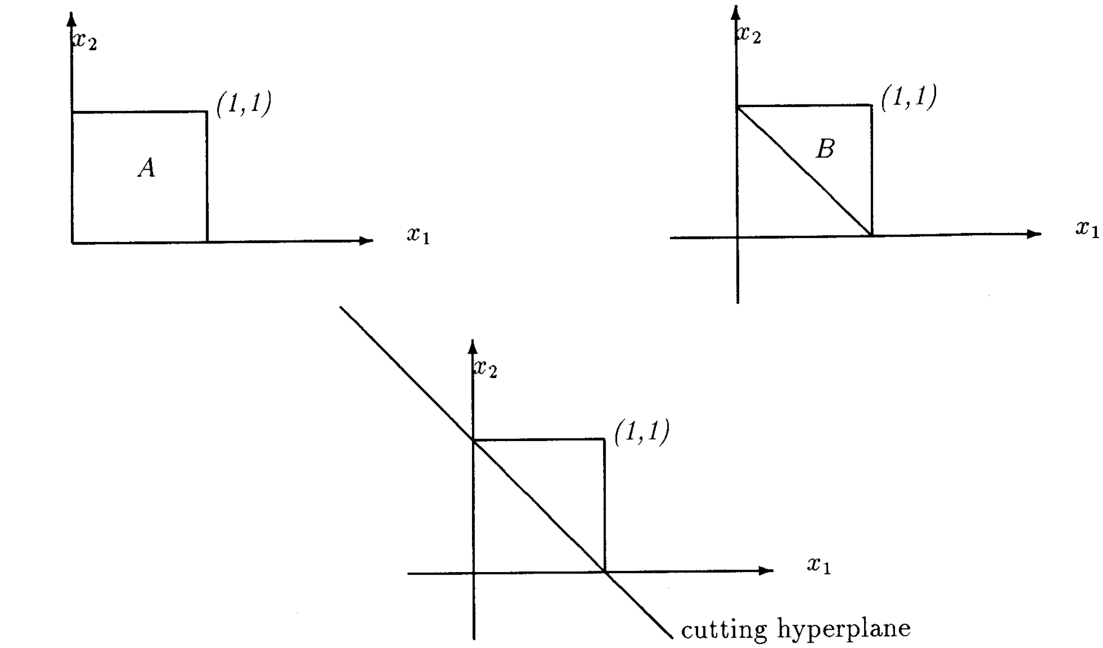
given in Example 1.2, can be explained by the reduction technique.
If we put

$$p_A(x_1, x_2) = \max\{0, x_1, x_2, x_1 + x_2\}, \quad p_B(x_1, x_2) = \max\{x_1, x_2, x_1 + x_2\}$$

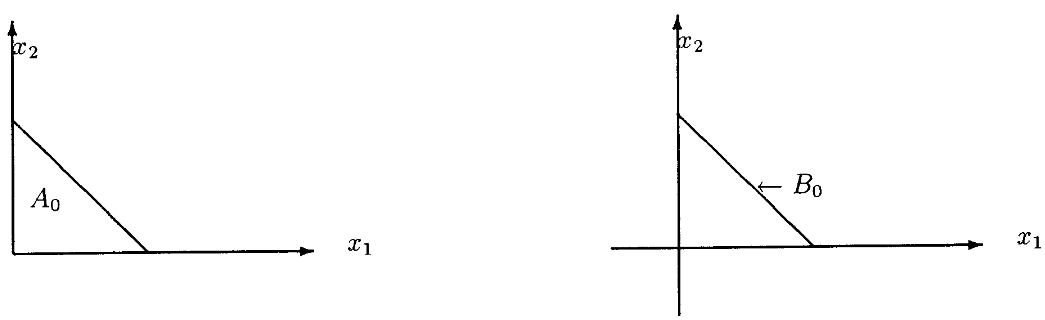
and

$$p_{A_0}(x_1, x_2) = \max\{0, x_1, x_2\}, \quad p_{B_0}(x_1, x_2) = \max\{x_1, x_2\},$$

then the reduction of the pair (A, B) to (A_0, B_0) which follows from Theorem 4.6 is now illustrated by the following figure:



This gives the reducted pair (A_0, B_0) :



Example 5.3. Let $R > 0$ be given and define a linear map

$$T : \mathbf{R}^2 \to \mathbf{R}^2,$$

by

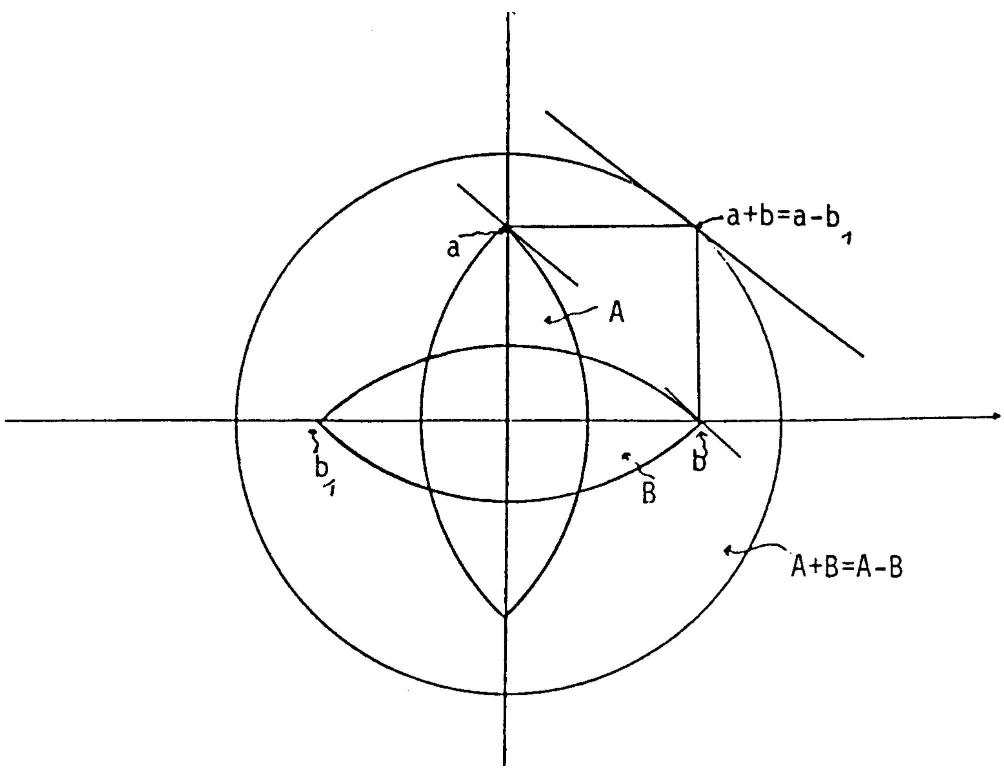
$$T(x_1, x_2) := (-x_1, x_2).$$

Furthermore, put $u := (\frac{R}{\sqrt{2}}, 0)$ and consider the balls

$$K_1 := B(u; R) := \{z \in \mathbf{R}^2 \mid \|z - u\| \leq R\} \text{ and } K_2 := B(-u; R)$$

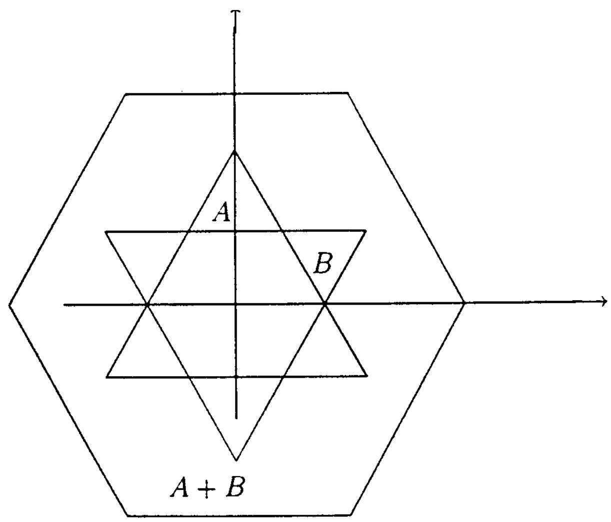
in the Euclidean norm. Put

$$A := K_1 \cap K_2 \text{ and } B := T(A).$$



Then $A + B = A - B = B(0; R)$ is the ball with radius R at the origin $0 = (0, 0) \in \mathbf{R}^2$. It is easy to see that the condition stated in Theorem 3.2 gives the minimality of the pair (A, B).

Example 5.4. Let $R > 0$ be given. Put $x := \frac{R}{\sqrt{2}}$, $y := \frac{R}{2}$. Furthermore, put $a_1 := (0, R)$, $a_2 := (x, -y)$, $a_3 := (-x, -y)$, $A := a_1 \vee a_2 \vee a_3$ and $B := -A$.



Then Theorem 3.1 gives the minimality of the pair (A, B).

References

1. V. F. Demyanov and A. M. Rubinov, *Quasidifferential Calculus*, Optimization Software Inc., Publications Division, New York, 1986.
2. J. Grzybowski, *Minimal pairs of compact convex sets,* Archiv der Mathematik **63** (1994), 173-181.
3. L. Hörmander, *Sur la fonction d' appui des ensembles convexes dans un espace localement convex,* Arkiv för Matematik **3** (1954), 181-186.
4. V. Klee, *Extremal structure of convex sets II,* Math. Z. **69** (1958), 90-104.
5. D. Pallaschke, S. Scholtes and R. Urbański, *On minimal pairs of compact convex sets,* Bull. Acad. Polon. Sci. Sér. Math. **39** (1991), 1-5.
6. D. Pallaschke and R. Urbański, *Some criteria for the minimality of pairs of compact convex sets,* Zeitschrift für Operations Research **37** (1993), 129-150.
7. D. Pallaschke and R. Urbański, *Reduction of quasidifferentials and minimal representations,* Math. Programming, (Series A) **66** (1994), 161-180.
8. A. G. Pinsker, *The space of convex sets of a locally convex space,* Trudy Leningrad Engineering-Economic Institute, **63** (1966), 13-17.
9. S. Rolewicz, *Metric Linear Spaces*, PWN and D.Reidel Publ. Company, Warszawa-Dordrecht, 1984.

10. S. Scholtes, *Minimal pairs of convex bodies in two dimensions,* Mathematika **39** (1992), 267-273.

11. R. Urbański, *A generalization of the Minkowski-Rådström-Hörmander Theorem,* Bull. Acad. Polon. Sci. Sér. Math. **24** (1976), 709-715.

12. R. Urbański, *On minimal convex pairs of convex compact sets,* to appear in Archiv der Mathematik (1996).

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