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A REMARK ON VECTOR-VALUED EQUILIBRIA AND GENERALIZED MONOTONICITY

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. It is the purpose of this note to show that existence results for vector-valued equilibria, of the type considered recently in [1] or [2], can be deduced in a straightforward way from corresponding results about scalar-valued equilibria. We shall proceed as follows. First we prove an existence result for scalar-valued equilibria, employing a certain notion of generalized monotonicity, and from this we deduce several results for the vector-valued case, by using appropriate gauge functions.

1. The vectorial equilibrium problem

The setting for the vectorial equilibrium problem is as follows:

X is a real topological vector space;

$K \subseteq X$ is a convex, nonempty set;

Z is a locally convex topological vector space;

$P \subseteq Z$ is a closed convex cone, with $\operatorname{int} P \neq \emptyset$ and $P \neq Z$.

On Z a vectorial ordering is defined by means of

$$z \preceq 0 :\Longleftrightarrow z \in -P, \qquad z \succeq 0 :\Longleftrightarrow z \in P,$$
$$z \prec 0 :\Longleftrightarrow z \in -\operatorname{int} P, \quad z \succ 0 :\Longleftrightarrow z \in \operatorname{int} P.$$

Furthermore, a mapping $F : K \times K \to Z$ is given. The vectorial equilibrium problem, as considered in [1] or [2], consists in finding $x^* \in X$ such that

$$(1) \qquad x^* \in K, \quad F(x^*, y) \not\prec 0, \quad \forall y \in K.$$

We shall also consider the related problem of finding $x^* \in X$ such that

$$(2) \qquad x^* \in K, \quad F(x^*, y) \succeq 0, \quad \forall y \in K.$$

Both problems (1) and (2) can be reduced to the scalar equilibrium problem, which we consider next.

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2. The scalar equilibrium problem

Here X and K are as before, $Z := \mathbf{R}$ and $P := \mathbf{R}_+$. Then $\not\prec 0$ and $\succeq 0$ coincide with ≥ 0. We are given a function $f : K \times K \to \mathbf{R}$, and we consider the problem

$$x^* \in K, \quad f(x^*, y) \geq 0, \quad \forall y \in K. \tag{3}$$

It is convenient to introduce the following definition for $f, g : K \times K \to \mathbf{R}$: f is *g-monotone* iff, for all $x, y \in K$,

$$f(x, y) \geq 0 \Longrightarrow g(y, x) \leq 0; \tag{4}$$

f is *maximal g-monotone* iff f is g-monotone and, for all $x, y \in K$,

$$\big(g(\xi, x) \leq 0, \quad \forall \xi \in \,]x, y]\,\big) \Longrightarrow f(x, y) \geq 0. \tag{5}$$

We have the following existence result for problem (3).

Lemma 1. *Let the functions $f, g : K \times K \to \mathbf{R}$ satisfy the following conditions:*

(i) *for all $x \in K$, $f(x, x) \geq 0$;*

(ii) *for all $y \in K$, $S(y) := \{\xi \in K \mid g(y, \xi) \leq 0\}$ is closed in K;*

(iii) *for all $x \in K$, $W(x) := \{\xi \in K \mid f(x, \xi) < 0\}$ is convex;*

(iv) *f is maximal g-monotone;*

(v) *there exist $D \subseteq K$ compact and closed in K, and $y^* \in D$ such that $f(\xi, y^*) < 0$ for all $\xi \in K \setminus D$ (coercivity).*

Then there exists $x^* \in K$ such that $f(x^*, y) \geq 0$ for all $y \in K$.

Proof. For all $y \in K$ let

$$T(y) := \mathrm{cl}_K\{x \in K \mid f(x, y) \geq 0\}.$$

Then $T(\cdot)$ is a KKM map, i.e., for every finite subset $\{y_1, \ldots, y_n\} \subseteq K$ there holds $\mathrm{conv}\,\{y_1, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} T(y_i)$. Indeed, suppose to the contrary that $x \in \mathrm{conv}\,\{y_1, \ldots, y_n\}$, but $x \notin T(y_i)$ for all $i = 1, \ldots, n$. Then $y_i \in W(x)$ for all $i = 1, \ldots, n$. From condition (iii), $W(x)$ is convex, hence $x \in W(x)$, a contradiction with condition (i). Since $T(\cdot)$ is a KKM map with closed values, and since $T(y^*)$ is contained in the compact set D by condition (v), it follows from the KKM Lemma (see [5]) that there exists $x^* \in D$ such that $x^* \in T(y)$ for all $y \in K$. The sets $S(y)$ are closed in K, by condition (ii), and from the g-monotonicity of f – see (4) – it follows then that $T(y) \subseteq S(y)$. Therefore we obtain that $x^* \in S(y)$ for all $y \in K$, i.e.,

$$g(y, x^*) \leq 0, \quad \forall y \in K.$$

Now fix $y \in K$ arbitrarily. Then $]x^*, y] \subseteq K$ and therefore

$$g(\xi, x^*) \leq 0, \quad \forall \xi \in \left]x^*, y\right].$$

From the maximal g-monotonicity of f – see (5) – it follows then that

$$f(x^*, y) \geq 0.$$

Since $y \in K$ was arbitrary, the claimed result follows. □

We observe that the coercivity condition (v) is vacuously satisfied, if K is compact, by choosing $D := K$.

In the applications to follow we need only two special cases of Lemma 1, which we single out as theorems. We let $f \wedge g := \min\{f, g\}$.

Theorem 1. *Let the functions $f, g : K \times K \to \mathbf{R}$ be such that, for all $x, y \in K$, the following conditions are satisfied:*

(i) $(f \wedge g)(x, x) \geq 0$*;*

(ii) $\{\xi \in K \mid g(y, \xi) \leq 0\}$ *is closed in* K*;*

(iii) $\{\xi \in K \mid f(x, \xi) < 0\}$ *is convex;*

(iv) $f(x, y) \geq 0 \Longrightarrow g(y, x) \leq 0$*;*

(v) $\forall u \in]x, y[,\ \big(g(u, x) \leq 0,\ f(u, y) < 0\big) \Longrightarrow \big((f \wedge g)(u, \xi) < 0\ \forall \xi \in \left]x, y\right]\big)$;

(vi) $\{\xi \in [x, y] \mid f(\xi, y) \geq 0\}$ *is closed in* $[x, y]$*;*

(vii) *there exist $D \subseteq K$ compact and closed in K, and $y^* \in D$ such that $f(\xi, y^*) < 0$ for all $\xi \in K \setminus D$.*

Then there exists $x^* \in K$ such that $f(x^*, y) \geq 0$ for all $y \in K$.

Proof. It only remains to show that f is maximal g-monotone. By (iv), f is g-monotone. Let $g(\xi, x) \leq 0$ for all $\xi \in \left]x, y\right]$, and assume, for contradiction, that $f(u, y) < 0$ for some $u \in]x, y[$. From (v) we obtain $(f \wedge g)(u, \xi) < 0$ for all $\xi \in]x, y]$, in particular $(f \wedge g)(u, u) < 0$, which contradicts (i). Hence there holds $f(u,y) \geq 0$ for all $u \in]x,y[$. From (vi) follows then $f(x,y) \geq 0$. □

Remark 1. If $g \geq f$, then (iii) and (v) are satisfied, if, for all $x, y, u \in K$,

$$\big(f(u,x) \leq 0,\ f(u,y) < 0\big) \Longrightarrow \big(f(u,\xi) < 0\ \forall \xi \in]x,y]\ \big).$$

Theorem 2. *Let the function $f : K \times K \to \mathbf{R}$ be such that, for all $x, y \in K$, the following conditions are satisfied:*

(i) $f(x,x) \geq 0$*;*

(ii) $\{\xi \in K \mid f(\xi,y) \geq 0\}$ *is closed in* K*;*

(iii) $\{\xi \in K \mid f(x,\xi) < 0\}$ *is convex;*

(iv) *there exist $D \subseteq K$ compact and closed in K, and $y^* \in D$ such that $f(\xi, y^*) < 0$ for all $\xi \in K \setminus D$.*

Then there exists $x^* \in K$ such that $f(x^*, y) \geq 0$ for all $y \in K$.

Proof. We set

$$g(y,x) := -f(x,y).$$

Then, obviously, f is maximal g-monotone, and Lemma 1 gives at once the claimed result. □

Theorem 2 is a classical result due to Fan [5]. In connection with Lemma 1 an interesting case occurs, if $f(x,y) = a(x,y) + b(x,y)$, $g(x,y) = a(x,y) - b(y,x)$. Suppose that f is g-monotone, $a(x,x) = 0$ and $b(x,x) = 0$ on K, $a(x,\cdot)$ and $b(x,\cdot)$ are convex, $a(\cdot,y)$ is upper semicontinuous along line segments in K. Then f is maximal g-monotone. The proof is a replica of the proof of Lemma 3 in [3]. Here we shall not pursue this case further.

3. Reduction of the vectorial case to the scalar case

We return to the vectorial case. Let Z^* denote the topological dual space of Z, and let $P^* := \{\lambda \in Z^* \mid \langle \lambda, z \rangle \geq 0\ \forall z \in P\}$ denote the polar cone of P. We emphasize that P does not have to be pointed. Since $\operatorname{int} P \neq \emptyset$ and $P \neq Z$, P^* has a weak* compact base, i.e., there exists $B \subseteq P^*$, B convex, weak* compact, such that $0 \notin B$ and $P^* = \bigcup_{t \geq 0} tB$. We may choose for instance $B := \{\lambda \in P^* \mid \langle \lambda, c \rangle = 1\}$, where $c \in \operatorname{int} P$ is arbitrary (see [6], p. 539). We fix such a base B and let

$$\psi(z) := \max_{\lambda \in B} \langle \lambda, z \rangle, \quad \varphi(z) := \min_{\lambda \in B} \langle \lambda, z \rangle.$$

The function ψ is sublinear and lower semicontinuous on Z, φ is superlinear and upper semicontinuous on Z, and $\varphi \leq \psi$. For all $z \in Z$ there holds:

$$\begin{aligned} z \preceq 0 \Longleftrightarrow z \in -P &\Longleftrightarrow \langle \lambda, z \rangle \leq 0, \quad \forall \lambda \in P^* \\ &\Longleftrightarrow \langle \lambda, z \rangle \leq 0, \quad \forall \lambda \in B \Longleftrightarrow \psi(z) \leq 0; \\ z \prec 0 \Longleftrightarrow z \in -\operatorname{int} P &\Longleftrightarrow \langle \lambda, z \rangle < 0, \quad \forall \lambda \in P^* \setminus \{0\} \\ &\Longleftrightarrow \langle \lambda, z \rangle < 0, \quad \forall \lambda \in B \Longleftrightarrow \psi(z) < 0. \end{aligned}$$

Likewise there holds

$$z \succeq 0 \Longleftrightarrow \varphi(z) \geq 0, \quad z \succ 0 \Longleftrightarrow \varphi(z) > 0.$$

Now let $F : K \times K \to Z$ be given and set

$$\Psi(x,y) := \psi(F(x,y)), \quad \Phi(x,y) := \varphi(F(x,y)).$$

Then $\Phi \leq \Psi$ and, for all $x, y \in K$,

$$\begin{aligned} \Psi(x,y) \leq 0 \Longleftrightarrow F(x,y) \preceq 0, &\quad \Psi(x,y) < 0 \Longleftrightarrow F(x,y) \prec 0, \\ \Phi(x,y) \geq 0 \Longleftrightarrow F(x,y) \succeq 0, &\quad \Phi(x,y) > 0 \Longleftrightarrow F(x,y) \succ 0. \end{aligned}$$

Problem (1) now takes the form of problem (3), namely

$$x^* \in K, \quad \Psi(x^*, y) \geq 0, \quad \forall y \in K. \tag{6}$$

Likewise problem (2) takes the form

$$x^* \in K, \quad \Phi(x^*, y) \geq 0, \quad \forall y \in K. \tag{7}$$

In order to obtain existence results for problems (1) or (2) we simply have to apply Theorem 1 or Theorem 2 to (6) or (7). The decisive point is that all hypotheses and the conclusion of these theorems are formulated in terms of $f(x,y) \geq 0$, ≤ 0, < 0, and $g(x,y) \geq 0$, ≤ 0, < 0. Hence, identifying f and g with Ψ or Φ, they can be rewritten in terms of the vectorial inequalities $F(x,y) \prec 0$, $\preceq 0$, $\succ 0$, $\succeq 0$ and their negations, and thus are independent of the selected base B.

We first turn to Theorem 1. We let $f := \Psi$ and $g := \Phi$. Then $f \wedge g = \Phi$. Therefore from Theorem 1 we obtain

Corollary 1. *Let $F : K \times K \to Z$ be such that, for all $x, y \in K$, the following conditions hold:*

(i) $F(x,x) \succeq 0$;

(ii) $\{\xi \in K \mid F(y,\xi) \not\succ 0\}$ *is closed in* K;

(iii) $\{\xi \in K \mid F(x,\xi) \prec 0\}$ *is convex;*

(iv) $F(x,y) \not\prec 0 \Longrightarrow F(y,x) \not\succ 0$;

(v) $\forall u \in \,]x,y[,\ \big(F(u,x) \not\succ 0,\ F(u,y) \prec 0\big) \Longrightarrow \big(F(u,\xi) \not\succeq 0\ \forall \xi \in \,]x,y]\,\big)$;

(vi) $\{\xi \in [x,y] \mid F(\xi,y) \not\prec 0\}$ *is closed in* $[x,y]$;

(vii) *there exist* $D \subseteq K$ *compact and closed in* K, *and* $y^* \in D$ *such that* $F(\xi,y^*) \prec 0$ *for all* $\xi \in K \setminus D$.

Then there exists $x^* \in K$ *such that* $F(x^*,y) \not\prec 0$ *for all* $y \in K$.

Remark 2. Let $F : K \times K \to Z$ be given, and suppose that, for all $x,y,u \in K$ and $c \not\succeq 0$,

$$\{\xi \in K \mid F(u,\xi) \preceq c\} \text{ is convex;} \tag{8}$$

$$\big(F(u,x) \not\succ 0,\ F(u,x) \succ F(u,y)\big) \Longrightarrow \big(F(u,x) \succ F(u,\xi)\ \forall \xi \in \,]x,y]\,\big). \tag{9}$$

Then conditions (iii) and (v) of Corollary 1 hold. Indeed, (iii) follows from (8) by the fact that, given $a \prec 0$ and $b \prec 0$, there exists $c \prec 0$ such that $a \preceq c$ and $b \preceq c$. To prove (v), let $F(u,x) \not\succ 0$, $F(u,y) \prec 0$. Assume, for contradiction, that $F(u,\xi) \succeq 0$ for some $\xi \in \,]x,y]$. If $F(u,x) \succeq 0$, then $F(u,x) \succ F(u,y)$, and from condition (9) follows $F(u,x) \succ F(u,\xi) \succeq 0$, a contradiction with $F(u,x) \not\succ 0$. On the other hand, if $F(u,x) \not\succeq 0$, then by Lemma 2.2 of [2] there exists $c \not\succeq 0$ such that $c \succeq F(u,x)$ and $c \succeq F(u,y)$. From condition (8) follows $c \succeq F(u,\xi) \succeq 0$, a contradiction with $c \not\succeq 0$. □

Thus one sees that Corollary 1 includes Theorem 3.1 of [2]. A mapping F which satisfies condition (iv) of Corollary 1 was termed pseudomonotone in [2]. Conditions (8) and (9) are automatically satisfied if, for all $u \in K$, $F(u,\cdot)$ is affine.

Now we let $f := \Phi$ and $g := \Psi$. Again $f \wedge g = \Phi$, and we obtain from Theorem 1

Corollary 2. *Let* $F : K \times K \to Z$ *be such that, for all* $x,y \in K$, *the following conditions hold:*

(i) $F(x,x) \succeq 0$;

(ii) $\{\xi \in K \mid F(y,\xi) \preceq 0\}$ *is closed in* K;

(iii) $\{\xi \in K \mid F(x,\xi) \not\succeq 0\}$ *is convex;*

(iv) $F(x,y) \succeq 0 \Longrightarrow F(y,x) \preceq 0$;

(v) $\forall u \in]x,y[,\ \big(F(u,x) \preceq 0,\ F(u,y) \not\succeq 0\big) \Longrightarrow \big(F(u,\xi) \not\succeq 0,\ \forall \xi \in]x,y]\ \big)$;

(vi) $\{\xi \in [x,y] \mid F(\xi,y) \succeq 0\}$ *is closed in* $[x,y]$;

(vii) *there exist* $D \subseteq K$ *compact and closed in* K, *and* $y^* \in D$ *such that* $F(\xi,y^*) \not\succeq 0$ *for all* $\xi \in K \setminus D$.

Then there exists $x^* \in K$ *such that* $F(x^*,y) \succeq 0$ *for all* $y \in K$.

Remark 3. Let $F : K \times K \to Z$ be given, and suppose that, for all $x,y,u \in K$ and $c \not\succeq 0$,

$$\{\xi \in K \mid F(u,\xi) \preceq c\} \text{ is convex;} \tag{10}$$

$$\big(F(u,x) \preceq 0,\ F(u,x) \not\preceq F(u,y)\big) \Longrightarrow \big(F(u,x) \not\preceq F(u,\xi),\ \forall \xi \in]x,y]\big). \tag{11}$$

Then condition (v) of Corollary 2 holds. Indeed: Let $F(u,x) \preceq 0$ and $F(u,y) \not\succeq 0$, and assume, for contradiction, that $F(u,\xi) \succeq 0$ for some $\xi \in]x,y]$. Then $F(u,x) \preceq F(u,\xi)$, and from condition (11) follows $F(u,x) \preceq F(u,y)$. Thus, with $c := F(u,y)$, there holds $c \not\succeq 0$, $F(u,x) \preceq c$, $F(u,y) \preceq c$. From (10) follows $F(u,\xi) \preceq c$. Since $F(u,\xi) \succeq 0$, this contradicts $c \not\succeq 0$. □

Conditions (10) and (11) are automatically satisfied if, for all $u \in K$, $F(u,\cdot)$ is affine.

We turn now to Theorem 2. We let $f := \Psi$. Then we obtain from Theorem 2

Corollary 3. *Let* $F : K \times K \to Z$ *be such that, for all* $x,y \in K$, *the following conditions hold:*

(i) $F(x,x) \not\prec 0$;

(ii) $\{\xi \in K \mid F(\xi,y) \not\prec 0\}$ *is closed in* K;

(iii) $\{\xi \in K \mid F(x,\xi) \prec 0\}$ *is convex;*

(iv) *there exist* $D \subseteq K$ *compact and closed in* K, *and* $y^* \in D$ *such that* $F(\xi,y^*) \prec 0$ *for all* $\xi \in K \setminus D$.

Then there exists $x^* \in K$ *such that* $F(x^*,y) \not\prec 0$ *for all* $y \in K$.

Finally we let $f := \Phi$. Then we obtain from Theorem 2 the following result.

Corollary 4. *Let* $F : K \times K \to Z$ *be such that, for all* $x,y \in K$, *the following conditions hold:*

(i) $F(x,x) \succeq 0$;

(ii) $\{\xi \in K \mid F(\xi,y) \succeq 0\}$ *is closed in* K;

(iii) $\{\xi \in K \mid F(x,\xi) \not\succeq 0\}$ *is convex;*

(iv) *there exist* $D \subseteq K$ *compact and closed in* K, *and* $y^* \in D$ *such that* $F(\xi, y^*) \not\succeq 0$ *for all* $\xi \in K \setminus D$.

Then there exists $x^* \in K$ *such that* $F(x^*, y) \succeq 0$ *for all* $y \in K$.

4. Extension

Under more restrictive assumptions about the space Z it is possible to extend the approach proposed above to the case when the mapping F is multivalued. We indicate briefly how this can be done.

Within the same framework as before, let $F : K \times K \to Z$ be a multivalued mapping. We consider the problem of finding $x^* \in X$ such that

$$x^* \in K, \quad F(x^*, y) \not\subseteq -\operatorname{int} P, \quad \forall y \in K. \tag{12}$$

We assume that the space Z is a Banach space, provided with its norm topology. The dual space Z^* is equipped with the weak* topology $\sigma(Z^*, Z)$. Again B is a weak* compact base of the polar cone P^*. Then the canonical bilinear form $\langle \cdot, \cdot \rangle$ is continuous on $B \times Z$. In fact, B is norm-bounded ([4], Corollaire II.4), and given a net (λ_i, z_i) in $B \times Z$ converging to (λ, z), we have $\|\lambda_i\| \le c$ for all i; so $|\langle \lambda_i, z_i \rangle - \langle \lambda, z \rangle| \le c\,\|z_i - z\| + |\langle \lambda_i - \lambda, z \rangle|$, which shows that $\langle \lambda_i, z_i \rangle$ converges to $\langle \lambda, z \rangle$.

We assume that, for all $x, y \in K$, $F(x,y)$ is norm-compact and nonempty. Then we can define

$$f(x,y) := \max_{\substack{\lambda \in B \\ z \in F(x,y)}} \langle \lambda, z \rangle = \max_{z \in F(x,y)} \psi(z),$$

$$g(x,y) := \min_{\substack{\lambda \in B \\ z \in F(x,y)}} \langle \lambda, z \rangle = \min_{z \in F(x,y)} \varphi(z),$$

where ψ, φ are as in the preceding section. It is clear that the inequalities

$$f(x,y) \ge 0, \quad f(x,y) \le 0, \quad f(x,y) < 0$$

are equivalent with

$$F(x,y) \not\subseteq -\operatorname{int} P, \quad F(x,y) \subseteq -P, \quad F(x,y) \subseteq -\operatorname{int} P,$$

respectively, and the inequalities

$$g(x,y) \geq 0, \quad g(x,y) \leq 0, \quad g(x,y) < 0$$

are equivalent with

$$F(x,y) \subseteq P, \quad F(x,y) \not\subseteq \operatorname{int} P, \quad F(x,y) \not\subseteq P,$$

respectively. In particular, if $x^* \in K$ and $f(x^*,y) \geq 0$ $\forall y \in K$, then x^* is a solution of (12). Hence, with this choice of f and g, Theorems 1 and 2 give sufficient conditions for the solvability of (12) The details can surely be left to the reader.

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