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LEXICOGRAPHICAL CHARACTERIZATION OF THE FACES OF CONVEX SETS

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

ABSTRACT. We prove that a nonempty proper subset of a convex set C is a face if and only if it is the set of points where a linear mapping achieves its lexicographical maximum over C.

1. Introduction

A basic notion in the theory of convex sets is that of a face. Given a convex set $C \subseteq \mathbf{R}^n$, one says that $F \subseteq C$ is a face of C if every closed line segment in C with a relative interior point F has both endpoints in F. For instance, the set of points where a linear function attains its maximum over C is a face of C. A face obtained in this way is called an exposed face. Not all faces of a convex set are necessarily exposed. As observed in [2], if C is the convex hull of a torus and D is one of the two closed disks forming the sides of C, each relative boundary point of D is a face of C but not an exposed face; however, these points are exposed faces of D, which is in turn an exposed face of C.

The aim of this paper is to show that all faces of a convex set are exposed in a lexicographical sense, namely, we will prove that for each nonempty proper face F of a convex set C there exists a linear mapping the set of whose lexicographical maximum points is F. As a consequence of this result, it turns out taking the exposed faces of a convex set C, the exposed faces of the exposed faces, and so on, one obtains all the nonempty proper faces of C.

Let us recall now some notions and notation which we shall use in the next section. The elements of $\mathbf{R}^n$ will be considered column vectors and

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the superscript T will mean transpose. One says that $x = (\xi_1, \ldots, \xi_n)^T \in \mathbf{R}^n$ is lexicographically less than $y = (\eta_1, \ldots, \eta_n)^T \in \mathbf{R}^n$ (in symbols, $x <_L y$) if $x \neq y$ and if for $k = \min\{i \in \{1, \ldots, n\} \mid \xi_i \neq \eta_i\}$ we have $\xi_k < \eta_k$. We write $x \leq_L y$ if either $x <_L y$ or $x = y$. The notation $\max_L$ will stand for lexicographical maximum, i.e., given $S \subset \mathbf{R}^n$, we write $z = \max_L S$ if $z \in S$ and $z' \leq_L z$ for all $z' \in S$. Notice that, in case such a point z exists, it is unique since the lexicographical order is a total order relation. The symbols $\subseteq$, $\subset$, and $\supset$ will denote inclusion, strict inclusion and strict containment, respectively.

Our characterization of faces of convex sets will be based on the following lexicographical separation theorem for two sets.

Theorem 1 (1, Theorem 2.1). *Let C and D be two nonempty subsets of $\mathbf{R}^n$. The convex hulls of C and D are disjoint if and only if there exists a square real matrix A such that $Ax <_L Ay$ for all $x \in C$ and $y \in D$.*

2. The characterization theorem

We next present the main result of this paper.

Theorem 2. *Let C be a convex subset of $\mathbf{R}^n$ and $\emptyset \neq F \subset C$. Then F is a face of C if and only if there exists a $k \times n$ matrix A, for some $k \in \{1, \ldots, n\}$, such that*

$$F = \{y \in C \mid Ay = \max_L\{Ax \mid x \in C\}\}. \tag{1}$$

Proof. If (1) holds then the set $AF = \{Ay \mid y \in F\}$ is a singleton, say $AF = \{b\}$ for some $b \in \mathbf{R}^n$. Let $x, x' \in C$, $\lambda \in (0, 1)$ and suppose that $(1 - \lambda)x + \lambda x' \in F$, i.e., $(1 - \lambda)Ax + \lambda Ax' = b$. Since $Ax \leq_L b$ and $Ax' \leq_L b$, it follows that $Ax = b = Ax'$, whence $x, x' \in F$. This proves that F is a face of C.

Conservely, assume that F is a face of C. Then the sets $C \setminus F$ and F are convex and disjoint. Hence, by Theorem 1, there is a $n \times n$ matrix $B = (b_1, \ldots, b_n)^T$ such that

$$Bx <_L By \qquad (x \in C \setminus F,\ y \in F). \tag{2}$$

Let $B_j = (b_1, \ldots, b_j)^T$ for $j = 1, \ldots, n$ and $\ell = \min\{j \mid B_j x <_L B_j y$ for all $x \in C \setminus F$ and $y \in F\}$. If $B_\ell F = \{B_\ell y \mid y \in F\}$ is a singleton then (1) holds with $A = B_\ell$. Let us consider the case when $B_\ell F$ is not a singleton, i.e., there exist $y_1, y_2 \in F$ with $B_\ell y_1 <_L B_\ell y_2$. If $\ell = 1$ then, taking $x_0 \in C \setminus F$

and $\lambda \in (0,1)$ close to 1, we would obtain a point $(1-\lambda)x_0 + \lambda y_2 \in C$ satisfying $b_1^T y_1 < b_1^T((1-\lambda)x_0 + \lambda y_2)$. But, by (2), this implies that $(1-\lambda)x_0 + \lambda y_2 \in F$, which is impossible (as F is a face of C). Thus, $\ell > 1$. Let

$$k = \max\{j \in \{1, \dots, n\} \mid B_j F \text{ is a singleton}\}.$$

We will next prove that $A = B_k$ satisfies (1). Let us suppose that one has $B_k F = \{B_k x\}$ for some $x \in C \setminus F$. Then we have $k < n$ since otherwise x would violate (2). As $B_{k+1}x \leq_L B_{k+1}y$ for all $y \in F$ and $B_{k+1}F$ is not a singleton, there exist $y_1, y_2 \in F$ with $B_{k+1}x \leq_L B_{k+1}y_1 <_L B_{k+1}y_2$. Therefore, by $B_k x = B_k y_1 = B_k y_2$ we deduce $b_{k+1}^T x \leq b_{k+1}^T y_1 < b_{k+1}^T y_2$. Hence, for $\lambda \in (0,1)$ close to 1, we have $B_k y_1 = B_k((1-\lambda)x + \lambda y_2)$ and $b_{k+1}^T y_1 < b_{k+1}^T((1-\lambda)x + \lambda y_2)$, which implies that $B_{k+1}y_1 <_L B_{k+1}((1-\lambda)x = \lambda y_2)$. It follows that $By_1 <_L B((1-\lambda)x + \lambda y_2)$. But this, together with (2), implies that $(1-\lambda)x + \lambda y_2 \in F$, which contradicts that F is a face of C. Thus, we must have $B_k F \neq \{B_k x\}$ for all $x \in C \setminus F$ and hence $A = B_k$ satisfies (1). □

Based on Theorem 2, we can introduce the following notion of the degree of non-exposedness of a face of a convex set.

Definition 3. Let C be a convex subset of $\mathbf{R}^n$ and let $F \subset C$ be a nonempty face of C. The degree of non-exposedness of F relative to C is

$$d_C(F) = \min\{k \mid \text{there exiss a } k \times n \text{ matrix A satisfying (1)}\} - 1.$$

This definition is justified by the fact that one has $d_C(F) = 0$ if and only if F is an exposed face of C. According to Theorem 2, the degree of non-exposedness of a face of a convex set in $\mathbf{R}^n$ satisfies $0 \leq d_C(F) \leq n-1$. More precisely, one has:

Proposition 4. *Let C and F be as in Definition 3. Then*

$$d_C(F) \leq \dim C - \dim F - 1,$$

where dim *denotes dimension of the affine hull.*

Proof. The inequality follows from the following observation. Let $\dim C = p$ and let $M = \{x \in \mathbf{R}^n \mid a_i^T x = \alpha_i \ (i = 1, \dots, n-p)\}$ be the affine hull of C. If $A = (b_1, \dots, b_k)^T$ is a matrix satisfying (1) with $k = d_C(F) + 1$ then the vectors $a_1, \dots, a_{n-p}, b_1, \dots, b_k$ are linearly independent. To prove this assertion, observe first that the vectors $a_1, \dots, a_{n-p}$

are linearly independent (by $\dim C = p$). Assume, by contradiction, that $a_1, \dots, a_{n-p}, b_1, \dots, b_k$ are linearly dependent and denote by ℓ the smallest index j such that $a_1, \dots, a_{n-p}, b_1, \dots, b_j$ are linearly dependent. We will next prove that the matrix $\tilde{A}$ obtained from A by deleting its row b_ℓ^T also satisfies (1), thus contradicting the equality $k = d_C(F)+1$. Indeed, let $y \in F$ and $x \in C$. By (1), we have $Ax \leq_L Ay$ whence also $A_{\ell-1}x \leq_L A_{\ell-1}y$, with $A_\ell = \{a_1, \dots, a_{\ell-1})^T$. If $A_{\ell-1}x <_L A_{\ell-1}y$ then, obviously, $\tilde{A}x \leq_L \tilde{A}y$. If, instead, $A_{\ell-1}x = A_{\ell-1}y$ then, by $a_i^T x = \alpha_i = a_i^T y$ $(i = 1, \dots, n-p)$ and the fact that b_ℓ is a linear combination of $a_1, \dots, a_{n-p}, b_1, \dots, b_{\ell-1}$, we have $b_\ell^T x = b_\ell^T y$. Therefore, by $Ax \leq_L Ay$ we also have $\tilde{A}x \leq_L \tilde{A}y$. We have thus proved that $F \subseteq \{y \in C \mid \tilde{A}y = \max_L\{\tilde{A}x \mid x \in C\}\}$. To prove the opposite inclusion, let y be a point belonging to the right hand side and let $x \in C$. One has $\tilde{A}x \leq_L \tilde{A}y$. If we had $Ay <_L Ax$ then we should have $A_{\ell-1}x = A_{\ell-1}y$ and $b_\ell^T y < b_\ell^T x$, which is impossible because b_ℓ is a linear combination of $a_1, \dots, a_{n-p}, b_1, \dots, b_{\ell-1}$, and $a_i^T x = \alpha_i = a_i^T$ $(i = 1, \dots, n-p)$. Thus $Ax \leq_L Ay$, which concludes the proof of our assertion.

The proof of the inequality in the statement follows from the linear independence of $a_1, \dots, a_{n-p}, b_1, \dots, b_k$ and the fact that

$$F \subseteq \{x \in \mathbf{R}^n \mid a_i^T x = \alpha_i \ (i = 1, \dots, n-p), \ Ax = \max_L\{Ay \mid y \in C\}\}. \square$$

As it is well known, if F is a face of C and F' is a face of F then F' is a face of C. Next proposition establishes a relation between $d_C(F)$, $d_F(F')$ and $d_C(F')$.

Proposition 5. *Let C be a convex subset of $\mathbf{R}^n$, $F \subset C$ a face of C and $F' \subset F$ a nonempty face of F. Then*

$$d_C(F) + d_F(F') \geq d_C(F') - 1.$$

Proof. Let $k = d_C(F) + 1$, $k' = d_F(F') + 1$ and let A and A' be $k \times n$ and $k' \times n$ matrices, respectively, such that $F = \{x \in C | Ax = \max_L\{Ay | y \in C\}\}$ and $F' = \{x \in F \mid A'x = \max_L\{Ay \mid y \in F\}\}$. One can easily check that, for $\tilde{A} = \begin{pmatrix} A \\ A' \end{pmatrix}$, one has

$$F' = \{x \in C \mid \tilde{A}x = \max_L\{\tilde{A}y \mid y \in C\}\},$$

whence by Definition 3

$$d_C(F') \leq k + k' - 1 = d_C(F) + d_F(F') - 1.$$

□

Our last result, which is an immediate consequence of Theorem 2, states that by iterating the operation of taking exposed faces one obtains all faces of a convex set.

Proposition 6. *Let C and F be as in Definition 3 and let $k = d_C(F) > 0$ Then there are faces of C, $F_0 \supset F_1 \supset \cdots \supset F_k \supset F_{k+1}$, with $F_0 = C$ and $F_{k+1} = F$, such that*

$$d_{F_j}(F_i) = i - j - 1 \qquad (0 \le j < i \le k+1).$$

In particular, F_i is an exposed face of F_{i-1} for each $i = 1, \dots, k$.

Proof. Take a $(k+1) \times n$ matrix $A = (a_1, \dots, a_{k+1})^T$ satisfying (1) and let $A_{ji} = (a_{j+1}, \dots, a_i)^T$ $(0 \le j < i \le k+1)$. Define

$$F_i = \{x \in C \mid A_{0i}x = \max_L\{A_{0i}y \mid y \in C\}\} \quad (i = 1, \dots, k+1).$$

One can easily verify that

$$F_i = \{x \in F_j \mid A_{ji}x = \max_L\{A_{ji}y \mid y \in F_j\}\} \quad (0 \le j < i \le k+1),$$

whence $d_{F_j}(F_i) \le i - j - 1$. On the other hand, $d_C(F_i) \le i - 1$ and, by Proposition 5,

$$k = d_C(F) = d_C(F_{k+1}) \le d_C(F_i) + d_{F_i}(F_{k+1}) + 1 \le d_C(F_i) + k - i + 1;$$

therefore, $d_C(F_i) = i - 1$. Using again Proposition 5, we obtain

$$d_{F_j}(F_i) \ge d_C(F_i) - d_C(F_j) - 1 = i - j - 1$$

and hence we get $d_{F_j}(F_i) = i - j - 1$.

References

1. J. E. Martinez-Legaz and L. Singer, *Lexicographical separation in* $\mathbf{R}^n$, Linear Algebra and Its Application **90** (1987) , 147-163.
2. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

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