ACTA MATHEMATICA VIETNAMICA
Volume 22, Number 1, 1997, pp. 183–198

PIVOTING ALGORITHMS BASED ON BOOLEAN VECTOR LABELING

HANS VAN MAAREN

Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. We reconsider complementary pivoting algorithms on the base of a modified form of Scarf's primitive sets and Tuy's abstract pivotal method. Here, the pivots are to be controled by a new type of labeling. The analysis of the limiting stage provides a new variety of fixed point results, where the usual convexity assumptions are relaxed drastically. The results mentioned also connect separation and fixed point theory in a natural way and are even applicable outside the domain of linear spaces. Moreover, they focus on the differences between integer and vector labeling in a very accurate manner.

Introduction

The aim of this paper is to introduce a new type of labeling in the performance of complementary pivoting algorithms. This so called *Boolean vector labeling* can be considered as intermediate between *integer* and *vector* labeling. The possibility of such a labeling is included (at least for a substantial part) in Scarf [5]. The analysis of the limiting stage however has not yet been carried out sofar. It is precisely the result of this analysis which justifies a renewed attention to the pivoting algorithms based on Scarf's notion of *primitive sets*.

We shall show that these algorithms, whenever modified according to [3], constitute the base for a variety of new intersection and fixed point theorems. These theorems do not assume the presence of a *linear space* but are applicable in a context where only *orderings* are available. Consequently, the usual convexity assumptions which are present in most (constructive-) fixed point results are relaxed drastically. Another feature of this new type of theorems is the linkage of *separation* theory and *fixed point theory.*

Received October 26, 1996
Key words. Complementary pivoting, labeling rules, fixed points, convexity, separation.

Perhaps the most interesting outcome of our analysis is the way it focuses on the possibility of approximating Kakutani-type fixed points by means of *integer* labeling. It is known that an algorithm using vector labeling might be artificially replaced by one using integer labeling, but only at the cost of an enormous increase of the dimension. This has been noticed for instance by Van der Heyden [8] and by Fan [2]. Our results imply that the use of primitive sets in modified form makes such an increase unnecessary. This feature shall be discussed in Section 4.

To obtain our results we first extend the notion of a primitive set in a manner used by Scarf in [6] and then proceed, according to Tuy [7], to explore orderings in its description rather than coordinates. Further we use Boolean vector labeling to control the pivot steps. The modification of the topological arguments calls for the introduction of two boundary concepts defined in terms of orderings. Finally the convexity restrictions are eliminated and replaced by conditions on the action of the orderings on the underlying space.

1. Preliminaries

1.1. A *multiply ordered space* X is a topological space together with a set $\mathcal{O}$ of continuous, complete orderings on X. That is, for each $\preceq \in \mathcal{O}$ we have

(i) $\preceq$ is transitive and reflexive,

(ii) $x \preceq y$ or $y \preceq x$, for each $x, y \in X$,

(iii) $\{x \in X | a \preceq x\}$ and $\{x \in X | x \preceq a\}$ are closed sets for each $a \in X$.

If $x \preceq y$ and $y \preceq x$, for some $\preceq \in \mathcal{O}$ x and y are called *indifferent* with respect to $\preceq$, written as $x \approx y$. If $x \succeq y$ and $x \not\approx y$ we shall write $x \succ y$. In order to distinguish between different orderings we sometimes prefer to use an index set P for the orderings involved. Thus $\mathcal{O} = \{\preceq_p | p \in P\}$. In this paper P is assumed to be finite. A nonempty subset of P is called a *coalition.*

1.2. For $a \in X$ and $Q \subset P$ then *open Q-cone on a* is the set

$$C^0_Q(a) = \{x \in X \mid x \succ_q \ a \quad \text{for all } q \in A\},$$

while the *closed Q-cone on a* is

$$C_Q(a) = \{x \in X \mid x \succeq_q a \quad \text{for all } q \in Q\}.$$

1.3. Now two types of combinatorial boundaries of X are considered. For $Q \subset P$ the *weak Q-Pareto boundary* of X is the set

$$\partial_Q^0(X) = \{a \in X \mid C_Q^0(a) = \emptyset\},$$

and the *strong Q-Pareto boundary* of X is given by

$$\partial_Q(Z) = \{a \in X \mid C_Q(a) = \{a\}\}.$$

1.4. As a standard example consider the n-dimensional Euclidean simplex with $\mathcal{O}$ being the set of orderings induced by the barycentric coordinates: $x \preceq_i y$ iff $x_i \leq y_i$ $(i \in P = \{1, \ldots, n+1\})$.

In this example $\partial_Q^0(X) = \partial_Q(X) = \{a \in X \mid a_i = 0 \text{ for } i \notin Q\}$, the convex hull of $\{e_q \mid q \in Q\}$. Here e_q denotes the n-dimensional $0, 1$ - vector with a one only at the q^{th} coordinate.

Notice that if we reverse the orderings involved we obtain $\partial_q^0(X) = \{a \in X \mid a_q = 0\}$, while $\partial_q(X) = \emptyset$, for any $q \in P$.

1.5. A multiply ordered space $(X, \mathcal{O})$ satisfies the *nondegeneracy assumption on the Pareto-boundaries*, whenever

$$(ND_1) \quad \partial_Q^0(X) = \partial_Q(X) \quad \text{for all } Q \subset P.$$

1.6. We say that $(X, \mathcal{O})$ satisfies the *nondegeneracy assumption on the cones*, if

(ND_2) a is a closure point of $C_Q^0(a)$ whenever this last set is nonempty, for each a and $Q \subset P$.

Notice that the standard example satisfies (ND_2) because of convexity of $C_Q^0(a) \cup \{a\}$. In [4] ND_2 is referred to as "coalitionwise convexity".

As a second degenerate example consider $X = \{a, b\}$ supplied with two orderings $\preceq_1$ and $\preceq_2$ satisfying $a \prec_1 b$ and $b \prec_2 a$. It is easily verified that (ND_1) holds. However $C_1^0(a) = \{b\} \neq \emptyset$, but a is no closure point of $\{b\}$ since any subset of X is closed. In this example therefore, (ND_2) fails.

In Section 4 we shall give a second nondegenerate example.

In [4] (page 420) a typical nondegenerate nonconvex example is presented.

1.7. A multiply ordered space is called nondegenerate whenever both the nondegeneracy assumptions hold.

1.8. A *continuously integer labeled* multiply ordered space X is a multiply ordered space which is labeled by a mapping $\mathcal{L} : X \to C(P)$, the set of coalitions, satisfying

$$\left.\begin{array}{l} x = \lim x_m \\ q \in \mathcal{L}(x_m) \text{ for all } m \end{array}\right\} \Rightarrow q \in \mathcal{L}(x).$$

1.9. We can now state the main result of [4], the generalized Knaster-Kuratowski-Mazurkiewicz lemma:

(GKKM) Let X be a separable, sequentially compact topological space which is nondegererately multiply ordered by $\mathcal{O}$. Suppose X is continuously integer labeled by $\mathcal{L}$. Then coalition Q and $a \in X$ exist with

(i) $a \in \partial_Q(X)$,

(ii) $Q \subset \mathcal{L}(a)$.

1.10. The above result is established by a modification of the methods of [6]:

For given finite $A \subset X$ an *agreement* for a coalition Q is defined to be a mapping $\alpha : Q \to A$ satisfying

(i) $\alpha(q)$ is the minimum of $\alpha(Q)$ with respect to ordering $\preceq_q$, for each $q \in Q$ and

(ii) the system of inequalities $\alpha(q) \prec_q x$ $(q \in Q)$ has no solution in A.

The above notion generalizes the concept of a primitive set in order to meet the context of multiply ordered spaces. Notice that there is no a priori relation between the number of orderings in $\mathcal{O}$ and whatsoever topological dimension of X.

The intuitive idea behind an agreement for a coalition Q is that the points $\alpha(q)$ $(q \in Q)$ approximate $\partial_Q(X)$ whenever grid A is sufficiently fine, and hence approximate Pareto optimal points for coalition Q.

For a given (integer) labeling $\ell : A \to P$ Scarf's primitive set algorithm is now modified to obtain coalition Q_A and agreement $\alpha : Q_A \to A$ such that $Q_A = \ell \circ \alpha(Q_A)$ which leaves us with a discrete version of 1.9. More or less standard topological arguments are used to obtain GKKM.

In [4] it is explicitly described how the nondegeneracy assumptions are used to obtain one single point a with the desired properties.

In Figure 1 we give an example.

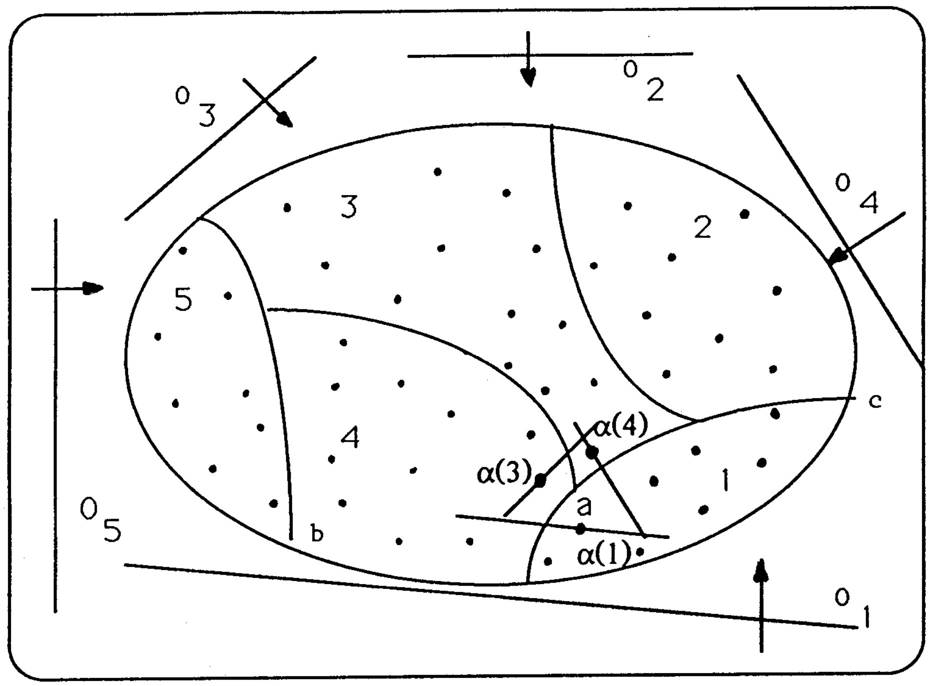

Figure 1

Generalized KKM. $P = \{1, 2, 3, 4, 5\}$.

Agreement $\alpha : Q = \{1, 2, 3\} \to A$ satisfies $\mathcal{O} = \ell \circ \alpha(Q)$; $a \in \partial_{1,3,4}(X)$ with $\{1, 3, 4\} \subset \mathcal{L}(a)$;

b and c are the only other solutions, with $Q = \{4, 5\}$ and $Q = \{1, 2\}$ respectively.

1.11. Remarks on GKKM:

(a) As presented in the above form, GKKM actually generalizes Scarf's intersection lemma: Suppose $\{F_i | i \le n+1\}$ constitutes a closed covering of the n-dimensional unit simplex (our standard example). Let $\mathcal{L}(x) = \{i \mid x \in F_i\}$ and apply GKKM. We obtain $Q \subset \{1, \ldots, n+1\}$ and $a \in \{x \mid x_i = 0 \text{ for } i \notin Q\}$ with $\mathcal{L}(a) \supset Q$. Now if we add the boundary conditions

$$\{x \mid x_i = 0\} \subset F_i, \quad i \le n+1,$$

as is done in Scarf's intersection lemma, we conclude $\{1, \ldots, n+1\} \subset \mathcal{L}(a)$, whence $a \in \bigcap_{i \le n+1} F_i$.

(b) KKM cannot be derived from GKKM in a similar straightforward fashion. However, as is noticed by Scarf in [5] (page 192), KKM is easily derived from Scarf's intersection lemma by embedding a simplex, together with a closed covering satisfying the KKM-type boundary conditions, into a large simplex, thereby extending the given covering to a closed covering satisfying the Scarf-type boundary conditions.

(c) The open version of KKM however can be derived from GKKM in a straightforward manner:

Suppose $\{U_i \mid i \leq n+1\}$ is an open covering of the n-dimensional unit simplex X satisfying

$$\mathrm{Conv}\{e_i \mid i \in I\} \subset \bigcup_{i \in I} U_i \quad (I \subset \{1, \ldots, n+1\}).$$

Let $\mathcal{L}(x) = \{i \mid x \notin U_i\}$. If $\mathcal{L}(x) \neq \emptyset$ for a any x we may apply GKKM obtaining $Q \subset \{1, \ldots, n+1\}$ and $a \in \partial_Q(X) = \mathrm{Conv}\,\{e_q \mid q \in Q\}$ with $\mathcal{L}(a) \supset Q$. The latter means $a \notin \bigcup_{i \in Q} U_i$, a contradiction. We conclude that x exists with $\mathcal{L}(x) = \emptyset$, implying $x \in \bigcap_{i \leq n+1} U_i$.

The above argument even shows:

1.12. Corollary (GKKM, open version). Let X be a separable, sequentially compact topological space, which is nondegenerately multiply ordered by $\mathcal{O} = \{\preceq_1, \ldots, \preceq_N\}$. Let $\{U_i \mid i \leq N\}$ be a set of open sets satisfying

$$\partial_Q(X) \subset \bigcup_{i \in Q} U_i \quad (Q \subset \{1, \ldots, N\}).$$

Then $\bigcap_{i \leq N} U_i \neq \emptyset$.

1.13. By embedding a closed covering $\{F_i \mid i \leq N\}$ into an open covering $\{F_i(\varepsilon) | i \leq N\}$ in such a way that $F_i = \bigcap_{\varepsilon > 0} F_i(\varepsilon)$, a closed version of Corollary 1.12 can be obtained for a large class of topological spaces, including Euclidean space.

2. Boolean vector labeling

2.1. In the results above the pivot steps leading from one agreement to another one are steered by integer labeling. That is, given the original labeling $\mathcal{L} : X \to C(P)$ and a specific grid $A \subset X$ a selection $\ell : A \to P$ of

$\mathcal{L}$ is used to perform the modified primitive set algorithm. We now show that it is possible to use all information of $\mathcal{L}$ to obtain a much stronger result. Indeed, we believe that this so-called Boolean vector labeling is the best substitute for vector labeling in situations where the latter procedure is troublesome or even impossible due to the lack of vectorial structure.

In the above we refer to both $\mathcal{L}$ and selection ℓ as being a labeling. We shall do the same in case of Boolean vector labeling.

2.2. A *Boolean vector labeling* on a multiply ordered space $(X, \mathcal{O})$ is a mapping Π which assigns to each $x \in X$ a non empty set of vertices of the cube $[0,1]^{\#P}$. Thus $\Pi(x)$ is a set of $0,1$-vectors whose coordinates are indexed by P. We let e_p be the "vertex" with a one at the p^{th} coordinate and zero at the other coordinates. The vertex with only ones is denoted by e, and the vertex with only zeros simply by 0.

A Boolean vector labeling π is called *continuous* if it satisfies

$$\left.\begin{array}{l} x = \lim x_m \\ \sigma \in \Pi(x_m) \text{ for all } m \end{array}\right\} \Rightarrow \sigma \in \Pi(x).$$

2.3. Now for a given grid $A \subset X$ and a fixed selection π of Π we start with the system $\varepsilon = \sum\limits_{p \in P} e_p$ and perform the extended primitive set algorithm, again in modified form (see [3]), to obtain a coalition Q and agreement $\alpha : Q \to A$ satisfying

either (i) $0 \in \pi(\alpha(Q))$,

or (ii) $e = \sum\limits_{q \in Q} \lambda_q \pi(\alpha(q)) + \sum\limits_{p \notin Q} \mu_p e_p$ for some coefficients λ_q, $\mu_q \geq 0$.

As far as the generated sequence of agreements concerns we refer to [3]. In that paper the e_p ($p \notin Q$) correspond to the so called passive orderings of the agreement involved. The e_q ($q \in Q$) correspond to the active orderings.

Here, we only discuss the LP pivots of the above procedure. These steps perform the exchange of *a* $\#P$-dimensional $0,1$-vector with a vector from a set of $\#P$ such vectors. To resolve degeneracy (which is rather structural here) one might *perturb* the starting system $e = \sum\limits_{p \in P} e_p$ into $e + \varepsilon = \sum\limits_{p \in P} \mu_p e_p$, for an appropriate choice of $\varepsilon \approx (0, 0, \ldots, 0)$ and $\mu_p \approx 1$ ($p \in P$). Of course, it is also possible to perform the LP pivots with $0,1$-vectors, using lexicographic pivoting.

2.4. Lemma. *If* $(X, \mathcal{O})$ *is a nondegenerate multiply ordered, separa-*

ble and sequentially compact space and Π *is a continuous Boolean vector labeling on* X, *then coalition* Q *and* $a \in X$ *exist with*

(i) $a \in \partial_Q(X)$ *and*

(ii) *Either* $0 \in \Pi(a)$ *or* $e = \sum_{q \in Q} \lambda_q \sigma_q + \sum_{p \notin Q} \mu_p e_p$ *for some coefficients* λ_q, $\mu_p \geq 0$ *and* $\sigma_q \in \Pi(a)$.

Proof. We construct a sequence A_m of grids ($A_m \subset A_{m+1}$) such that $\bigcup A_m$ is dense. For each m we obtain a coalition Q_m and agreement α_m satisfying condition 2.3 (i) or condition 2.3 (ii). Since there are only finitely many coalitions, one specific Q must appear infinitely many times in this sequence Q_m. we select the subsequence involved. Now the nondegeneracy assumptions on $(X, \mathcal{O})$ together with the sequential compactness of X ensure the existence of a subsequence such that the points $\alpha_m(q)$ ($q \in Q$) converge to one single point $a \in \partial_Q(X)$. This is shown in [4].

If in this last sequence 2.3(i) holds infinitely many times we obtain $0 \in \Pi(a)$, by continuity of Π.

If (2.3)(ii) holds infinitely many times we select the subsequence involved. Now, for each m we have a feasible basis for 2.3(ii) of the form $\{\pi(\alpha_m(q)) \mid q \in Q\} \cup \{e_p \mid p \notin Q\}$. Since there are only finitely many of such bases, one particular basis appears infinitely many times. The continuity of Π now finishes the proof.

2.5. Now we shall analyze the above lemma in detail.

We assume that Q and $a \in X$ are given such that the second statement of 2.4(ii) is valid.

Fix a specific $p \notin Q$. Considering only the p^{th} coordinate of the equation we conclude:

$$1 = \sum_{q \in Q} \lambda_q (\sigma_q)_p + \mu_p.$$

If all $(\sigma_q)_p = 1$ we have $\sum_{q \in Q} \lambda_q + \mu_q = 1$. Considering only the coordinates indexed by Q we conclude that in this case the vector with all ones is a convex combination of other $0,1$-vectors, which is only possible if at least one of the σ_q has only ones at the coordinates indexed by Q. Summarized we have:

2.6. (*) there exists $\rho \in \Pi(a)$ with $\rho_q = 1$ for all $q \in Q$, or for each $p \notin Q$ we have a $\xi \in \Pi(a)$ with $\xi_p = 0$.

2.7. We proceed with fixing a $\overline{q} \in Q$.

First consider only the $\overline{q}^{th}$ coordinate of the equation:

$$1 = \sum_{q \in Q} \lambda_q (\sigma_q)_{\overline{q}}.$$

Evidently, not all $(\sigma_q)_{\overline{q}}$ can be zero. Thus

(**) there exists $\sigma \in \Pi(a)$ with $\sigma_{\overline{q}} = 1$.

Suppose now that no $\tau \in \Pi(a)$ has $\tau_{\overline{q}} = 0$. Then clearly $\sum_{q \in Q} \lambda_q = 1$ and, again considering only the coordinates indexed by Q, we find that the all-one vector is a convex combination of other $0,1$-vectors. Hence

(***) there exists $\rho \in \Pi(a)$ with $\rho_q = 1$ for all $q \in Q$ or for some $\tau \in \Pi(a)$ we have $\tau_{\overline{q}} = 0$.

Combining the above conclusions we obtain:

2.8. Theorem. *Suppose $(X, \mathcal{O})$ is a nondegenerate multiply ordered, separable and sequentially compact space. If Π is a continuous Boolean vector labeling on X then there exists a coalition Q and $a \in \partial_Q(X)$ satisfying at least one of the following statements:*

(i) $0 \in \Pi(a)$,

(ii) *there exists $\rho \in \Pi(a)$ with $\rho_p = 1$ for all $q \in Q$,*

(iii) *for each $p \notin Q$ there exists $\xi \in \Pi(a)$ with $\xi_p = 0$ and for each $q \in Q$ there are $\sigma, \tau \in \Pi(a)$ with $\sigma_q = 1$ and $\tau_q = 0$.*

2.9. We now show how Lemma 2.4 can be formulated as a generalized Knaster-Kuratowski-Mazurkiewicz-Shapley lemma. As in the case of GKKM this generalization is presented in the typical context of multiply ordered spaces, thus appearing in a form which is in some sense the dual of the classical formulation, without boundary conditions on the covering.

Corollary (GKKMS). *Let X be a separable and sequentially compact space which is nondegenerately multiply ordered by $\mathcal{O} = \{\preceq_1, \dots, \preceq_N\}$. Let $\{F_P \mid P \subset N, P \neq \emptyset\}$ be a closed covering of X. Then there exists $Q \subset N$ and $\mathcal{S} \subset \{P \subset N \mid P \neq \emptyset\}$ such that*

(i) $\partial_Q(X) \cap \bigcap_{S \in \mathcal{S}} F_{\mathcal{S}} \neq \emptyset$,

(ii) $\mathcal{S} \cup \{\{p\} \mid p \notin Q\}$ *is balanced.*

Proof. For $T \subset N$ let $e_T = \sum_{t \in T} e_t$.

Let $\Pi(x) = \{e_T \mid x \in F_T\}$. Then Π is continuous, $\Pi(x) \neq \emptyset$ and $0 \notin \Pi(x)$ for all $x \in X$. We obtain Q and $a \in \partial_Q(X)$, and $\{\sigma_q \mid q \in Q\} \subset \Pi(a)$ such that the e_p ($p \notin Q$) and the σ_q ($q \in Q$) form a basis for the system in 2.4 (ii). Now $\sigma_q = e_{T(q)}$ for some $T(q) \subset N$.

Clearly $\{\{p\} \mid p \notin Q\} \cup \{T(q) \mid q \in Q\}$ is balanced. Also it follows that for any q we have $a \in F_{T(q)}$.

This finishes the proof.

3. Approximation of fixed points

3.1. To a multifunction $F : X \rightarrow> X$ we associate a Boolean vector labeling according to

$$\sigma \in \Pi(x) \Leftrightarrow \exists y \in F(x) \text{ with } \begin{cases} x \preceq_p y & \text{if } \sigma_p = 1, \\ x \succeq_p y & \text{if } \sigma_p = 0. \end{cases}$$

It is easy to verify that the above labeling Π is continuous whenever X is (sequentially) compact and F has a closed graph.

3.2. In order to explain the result of 2.8 in terms of multifunctions we first introduce some terminology.

A set $S \subset X$ *dominates* $T \subset X$ with *respect to ordering* $\preceq$ whenever $t \prec s$ for any $t \in T$ and $s \in S$, which we shall write as $T \prec S$.

Ordering $\preceq$ is said to *separate* T and S if either $T \prec S$ or $S \prec T$.

3.3. We now return to Theorem 2.8 again and we suppose that the labeling Π originates from a multifunction $F : X \rightarrow> X$ as described in 3.1.

The condition $0 \in \Pi(a)$ clearly means that $F(a)$ dominates a with respect to none of the orderings involved.

Statement (ii) means that for some $y \in F(a)$ we have $a \preceq_q y$ for all $q \in Q$. This clearly implies $y \in C_Q(a)$. Since $a \in \partial_Q(X)$ and the nondegeneracy assumptions are assumed to be valid we conclude that $y = a$ in this case. Thus statement (ii) implies that a is a fixed point of F. Also in this case we obtain that a is dominated by $F(a)$ with respect to *none* of the orderings in $\mathcal{O}$.

Statement (iii) provides, for any $p \notin Q$, the existence of $y \in F(a)$ satisfying $y \preceq_p a$, while to each $q \in Q$ there are $x, z \in F(a)$ with $a \preceq_q x$ and $a \succeq_q z$.

Again we conclude that $F(a)$ dominates a with respect to *none* of the orderings in $\mathcal{O}$. Moreover, in this case, *none* of the orderings in Q separates a and $F(a)$.

We summarize the above in the next theorem.

3.4. Theorem. *Suppose X is a separable, sequentially compact topological space and $F : X \rightarrow> X$ is a multifunction with closed graph and non empty images. Then for any set $\mathcal{O}$ of orderings which multiply orders X in a nondegenerate manner there exists $a \in X$ which is dominated by $F(a)$ with respect to none of the orderings involved. Moreover, there exists a coalition Q with $a \in \partial_Q(X)$ while none of the orderings indexed by Q separates a from $F(a)$.*

4. Remarks on the Euclidean case

4.1. We now consider a compact set X in Euclidean space and a multifunction $F : X \rightarrow> X$ with non empty images and closed graph. If we also assume the images $F(x)$ to be convex we know by the Hahn-Banach theorem that for $a \notin F(a)$ a linear functional exists which separates a from $F(a)$. There is indeed even a set of such functionals which is open in the dual space.

Now any linear functional u obviously defines a continuous complete ordering on X by $x \preceq y$ if $u(x) \leq u(y)$, in fact any continuous functional does.

In the next we assume that $\mathcal{O}$ is a set of orderings induced by linear functionals. Notice that, because of convexity of the sets $\{a\} \cup C_Q^0(a)$ the nondegeneracy assumption on the cones is trivially fulfilled, in case X is convex. Even in this case, the nondegeneracy assumption on the Pareto boundaries might be violated however. This heavily depends on the shape of X and the mutual dependence of the functionals involved. The troubles caused by this mutual dependence however, can be overcome by a slight perturbation of these functionals. Intuitively it is clear that a "randomly" chosen set of linear functionals will do.

The next lemma provides the argument.

Lemma. *If $\mathcal{O} \subset \mathbf{R}^n$ is such that any subset of less than $n+1$ elements consists of linearly independent vectors, then $(\mathbf{R}^n, \mathcal{O})$ is multiply ordered nondegenerately.*

Proof. We need only show the implication

$$C_Q^0(0) = \emptyset \Rightarrow C_Q(0) = \{0\}.$$

Let $Q = \{u_1, \dots, u_N\}$. Since $C_Q^0(0) = \emptyset$, the system $xu_i > 0$ $(i \leq N)$ is infeasible, whence by Gordan's Transposition theorem 0 is a nontrivial nonnegative combination of the u_i $(i \leq N)$:

$$\sum_{i \leq N} \lambda_i u_i = 0; \quad \lambda \geq 0; \ \lambda \neq 0.$$

After rearrangement of indices we obtain

$$\sum_{i \leq M} \lambda_i u_i = 0; \quad \lambda_i > 0 \quad (i \leq M),$$

for some $M \leq N$. Because of our assumptions on $\mathcal{O}$ we have $M \geq n+1$. Now for $y \in C_Q(0)$ we have

$$\begin{cases} yu_i \geq 0 \quad (i \leq M), \\ \sum_{i \leq M} \lambda_i y u_i = 0; \ \lambda_i > 0 \quad (i \leq M), \end{cases}$$

leaving us with $yu_i = 0$ $(i \leq M)$. Since $\{u_i, \dots, u_M\}$ must contain a basis of $\mathbf{R}^n$ we conclude $y = 0$.

Remark. It is left to the reader to show that a set $\mathcal{O}$ satisfying the conditions of the above lemma nondegenerately multiply orders the n-dimensional sphere.

4.2. By now it is seen that for a given set S the set of points s dominated by S with respect to none of the orderings in $\mathcal{O}$ approximates the closed convex hull of S, whenever the density of $\mathcal{O}$ increases.

4.3. We now show that our results simply that a fixed point of a graph closed multifunction F with non empty convex images can be approximated by Boolean vector labeling, in the nondegenerate case.

Let $\mathcal{O}$ be a set of linear functionals which is countable and dense in the dual space. Suppose $\mathcal{O} = \cup \mathcal{O}_m$ with $\mathcal{O}_m$ finite and $\mathcal{O}_m \subset \mathcal{O}_{m+1}$. We assume that each $\mathcal{O}_m$ multiply orders X nondegenerately. For each $\mathcal{O}_m$ we select a grid $A_m \subset X$ in such a way that for any agreement $\alpha : Q \to A_m$ the points in the set $\{x \in X \mid \alpha(q) \preceq_q x \text{ for all } q \in Q\}$ have mutual distance at most $\dfrac{1}{m}\cdot$ Here the nondegeneracy assumptions are essential.

In [4] it is shown that for a fixed Q and an increasing sequence G_k of grids with a dense union the $\#(Q)$ points of the agreements on Q become arbitrarily close when k increases. This is carried out in the abstract topological setting of [4]. In our situation this means that for $Q \subset \mathcal{O}_m$ a grid $G_m(Q)$ exists such that the $\#(Q)$ points of the agreements on Q have mutual distance at most $\frac{1}{m}\cdot$ This property remains true for a refinement of $G_m(Q)$. Now let A_m be the common refinement of the $G_m(Q)$ $(Q \subset \mathcal{O}_m)$.

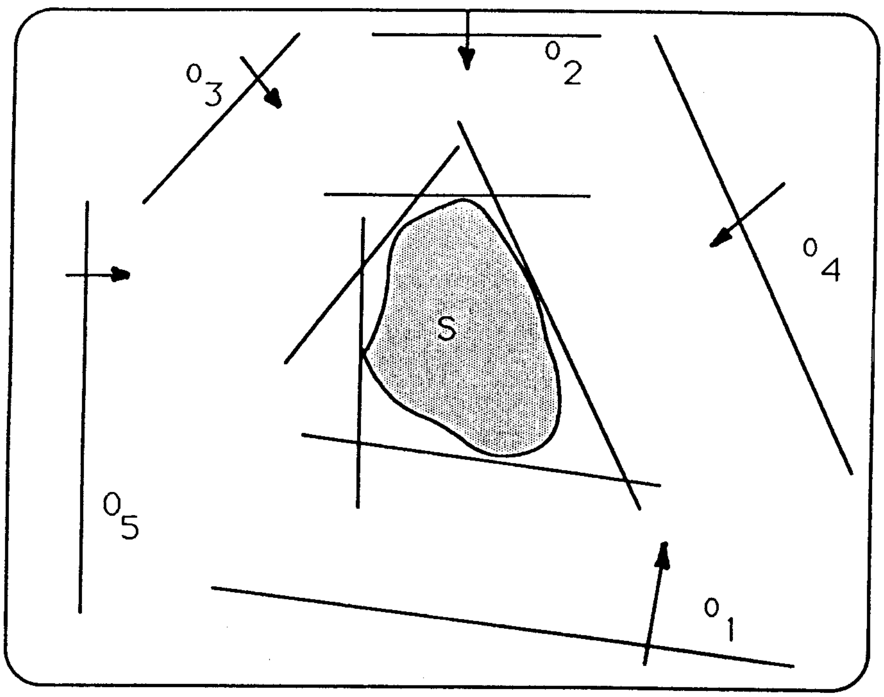

Figure 2
$\{s \in \mathbf{R}^2 | \; \forall \preceq \in \mathcal{O} \; \exists y \in S \; y \preceq s\}$ approximates the closed convex hull of S.

Next we notice that the analysis of the equation in 2.4 (ii) which was carried out in 2.5, 2.6 and 2.7 is also valid for equation 2.3 (ii) with only changing $\Pi(a)$ into $\pi(\alpha(Q))$.

Consequently, we obtain for each m a coalition $Q_m \subset \mathcal{O}_m$ and agreement $\alpha_m : Q_m \to A_m$ with the property that

either (*) there exists $z_m \in \alpha_m(Q_m)$ and $Z_m \in F(z_m)$ with $z_m \preceq_q Z_m$ for all $q \in Q_m$.

or (**) for each $p \in \mathcal{O}_m$ there exists $y_m \in \alpha_m(Q_m)$ and $Y_m \in F(y_m)$ with $Y_m \preceq_p y_m$.

Next we select for each m some $x_m \in \alpha_m(Q_m)$. Since a converging subsequence of (x_m) exists we may assume that $\lim x_m = \overline{x}$. We shall show that $\overline{x} \in F(\overline{x})$.

First assume that (*) is the case infinitely many times. Considering the corresponding subsequence we conclude $\overline{x} = \lim x_m = \lim Z_m$ and hence $\overline{x} \in F(\overline{x})$.

Now suppose that (**) holds infinitely many times. If $\overline{x} \notin F(\overline{x})$ there is a linear functional $p \in \mathcal{O}$ such that $F(\overline{x})$ dominates $\overline{x}$ with respect to ordering $\preceq_p$. Since $p \in \mathcal{O}_m$ for sufficiently large m we might as well assume that $p \in \mathcal{O}_m$ for all m. Now apply (**) for this specific p and use standard topological arguments to obtain the contradicting fact that $F(\overline{x})$ cannot possibly dominate $\overline{x}$ with respect to ordering $\preceq_p$.

4.4. We emphasize that the above procedure is only considered to point out the possibilities of Boolean vector labeling. From the viewpoint of efficiency there are serious troubles to deal with. First, as is known, computation with primitive sets causes difficulties with respect to storage. Also, the pivot steps, leading from one agreement to another, become more complicated (considered as a combinatorial problem, see also [6]) with increasing number of functionals involved.

A surprising feature however is the fact that the LP-pivot steps are independent of the multifunction:

These steps only perform the exchange of an n-dimensional $0, 1$*-vector with a vector from a set of n such vectors* (here n denotes the number of functionals involved).

The above observation is of some computational theoretic interest: if we assume that, for fixed n, all possible pivots of this form have been carried out once and hence are known in advance an actual run of our algorithm only requires function evaluations to control the pivots on the set of agreements, precisely as in the case of integer labeling.

As mentioned earlier in the introduction, an algorithm using vector labeling might be *artificially* replaced by one using integer labeling. This is done by converting an n-dimensional problem into a $(k(n+1)+k)$-dimensional one (as in [8]), where k is the number of points of the grid (or the number of vertices of a triangulation). We emphasize again that this approach is artificial: the actual calculations are based on n-dimensional pivoting using vector labeling.

In our approach an increase in dimension is caused by increasing the number of orderings used. However the pivots leading from one argreement to another, are performed within the same space all the time.

In cases where the sets $F(x)$ have a specific polyhedral form (e.g. $\{y \mid Ay \leq b(x)\}$) increase of dimension is even unnecessary.

It also appears that the approximation, described in 4.3, only depends on the convergence of the sequence x_m. There are no "weights" of convex combinations which have to converge, as in the case of vector labeling.

4.5. In the specific case of 1.4, where the functionals represent the barycentric coordinates, there is a slight intersection of the ideas of Boolean vector labeling, as described here and the $(2^{n+1} - 2)$-ray algorithm of [1].

Both ideas are based on the full exploration of the assigned $0, 1$-vectors. In the latter case, however, these vectors are used to modify the underlying simplicial subdivision and do not affect the pivot steps directly. It should be mentioned again that the above resemblance is only present when the number of functionals involved equals $\dim X + 1$.

4.6. This last remark poses an interesting question. The Boolean vector labeling algorithm might operate on a k dimensional space using an $n \times n$ LP pivot scheme, in case n orderings are selected. The algorithms based on simplicial subdivision are only possible in case $k + 1 = n$.

In order to avoid the use of modified primitive sets it seems natural to transform such a so called (k, n)-pivoting algorithm, which is only possible using agreements, into a $(n-1, n)$-pivoting algorithm, which might also be performed using triangulations.

Sofar, the author was not successful in establishing such a transformation.

References

1. T. M. Doup, G. van der Laan and A. J. J. Talman, *The* $(2^{n+1} - 2)$*-ray algorithm: a new simplicial algorithm to compute economic equilibria*, Mathematical Programming **39** (1987), 241-252.
2. K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519-537.
3. H. van Maaren, *Generalized pivoting and coalitions.* In: The Computation and Modelling of Economic Equilibria. A. J. J. Talman and G. van der Laan (eds.), North-Holland, Amsterdam, (1987), 155-176.
4. H. van Maaren, *Fixed point theory in multiply ordered spaces*, In: Fixed Point Theory and Applications, eds. M. A. Thera and J. B. Baillon, Pitman Research Notes in Mathematics Series. Longman, Harlow, Essex, UK. (1991), 407-423.

5. H. E. Scarf, *The Computation of Economic Equilibria*, Yale University Press, New Haven, 1973.

6. H. E. Scarf, *Production sets with indivisibilities*, Econometrica **49**(1) (1981), 1-32.

7. H. Tuy, *Pivotal methods for computing equilibrium points; unified approach and new restart algorithm*, Mathematical Programming **16** (1979), 210-227.

8. L. van der Heyden, *On a theorem of Scarf*, In: The Computation and Modelling of Economic Equilibria. A. J. J. Talman and G. van der Laan (eds.). North-Holland, Ansterdam, (1987), 177-192.

Delft University of Technology,
Applied Mathematics and Computer Science,
Mekelweg 4, 2628 CD Delft, The Netherlands