ACTA MATHEMATICA VIETNAMICA
Volume 22, Number 1, 1997, pp. 147–157

REDUCTION OF MONOTONE LINEAR COMPLEMENTARITY PROBLEMS OVER CONES TO LINEAR PROGRAMS OVER CONES

M. KOJIMA, M. SHIDA AND S. SHINDOH

Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. This short note presents a constructive way of reducing monotone LCPs (linear complementarity problems) over cones to LPs (linear programs) over cones. In particular, the monotone semidefinite linear complementarity problem (SDLCP) in symmetric matrices, which was recently proposed by Kojima, Shindoh and Hara, is reducible to an SDP (semidefinite program). This gives a negative answer to their question whether the monotone SDLCP in symmetric matrices is an essential generalization of the SDP.

1. Introduction

Let R^m denote the m-dimensional Euclidean space. We use the notation $\boldsymbol{x}\cdot\boldsymbol{y}$ for the inner product of $\boldsymbol{x}$, $\boldsymbol{y}\in R^m$, *i.e.*, $\boldsymbol{x}\cdot\boldsymbol{y}=\sum_{j=1}^{m} x_j y_j$.

There have been proposed and studied various kinds of extensions and generalizations of the standard form LCP (linear complementarity problem) (see, *e.g.*, Cottle-Pang-Stone [3]): Given an $m\times m$ matrix $\boldsymbol{M}$ and a $\boldsymbol{q}\in R^m$,

$$\text{(1)}\qquad \text{find } (\boldsymbol{x},\boldsymbol{y})\in R^{2m};\ \boldsymbol{y}=\boldsymbol{M}\boldsymbol{x}+\boldsymbol{q},\ \boldsymbol{x}\geq\boldsymbol{0},\ \boldsymbol{y}\geq\boldsymbol{0} \text{ and } \boldsymbol{x}\cdot\boldsymbol{y}=0.$$

Among others, we are concerned with *an LCP over cones* or *a GLCP (generalized linear complementarity problem)*: Given an affine subspace F of R^{2m} and a closed convex cone $K\subset R^m$,

$$\text{(2)}\qquad \text{find } (\boldsymbol{x},\boldsymbol{y})\in R^{2m};\ (\boldsymbol{x},\boldsymbol{y})\in F,\ \boldsymbol{x}\in K,\ \boldsymbol{y}\in K^* \text{ and } \boldsymbol{x}\cdot\boldsymbol{y}=0,$$

Received October 16, 1996
Key words and phrases. Linear program, monotone linear complementarity problem, monotone semidefinite linear complementarity problem, semidefinite programming, interior point method

where K^* denotes the dual cone of K, *i.e.*,

$$K^* = \{\boldsymbol{y} \in R^m \ : \ \boldsymbol{x} \cdot \boldsymbol{y} \geq 0 \text{ for every } \boldsymbol{x} \in K\}.$$

There are numerous literatures ([3-8, 12, 14, 16], etc.) on LCPs over cones and related problems. The standard form LCP (1) is a special case of the LCP (2) over cones where we take

$$\begin{aligned} &F = \{(\boldsymbol{x}, \boldsymbol{y}) \in R^{2m} : \boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}\}, \\ &K = R^m_+ \ (\textit{i.e.}, \text{the nonnegative orthant of } R^m). \end{aligned}$$

We call $(\boldsymbol{x}, \boldsymbol{y}) \in F \cap (K \times K^*)$ *a feasible solution* of the LCP (2), and $(\boldsymbol{x}, \boldsymbol{y}) \in F \cap \text{int}(K \times K^*)$ *an interior feasible solution* of the LCP (2). Here int C denotes the interior of a set C in the Euclidean space. We say that an affine subspace F of R^{2m} is *monotone* if

$$(\boldsymbol{x} - \boldsymbol{x}') \cdot (\boldsymbol{y} - \boldsymbol{y}') \geq 0 \text{ for every } (\boldsymbol{x}, \boldsymbol{y}), \ (\boldsymbol{x}', \boldsymbol{y}') \in F, \tag{3}$$

and that the LCP (2) over cones is monotone if the affine subspace F associated with the problem is monotone.

The monotone LCP (2) over cones is known as a unified mathematical model for various problems including LPs (linear programs), convex QPs (quadratic programs) and SDPs (semidefinite programs). In particular, the monotone LCP (2) over cones may be regarded as a direct extension of the primal-dual pair of linear programs using a conic formulation (Nesterov-Nemirovskii [13]):

$$\begin{cases} \mathcal{P}: & \text{minimize} \quad \boldsymbol{c} \cdot \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{x} \in L + \boldsymbol{d}, \qquad \boldsymbol{x} \in K, \\ \mathcal{D}: & \text{minimize} \quad \boldsymbol{d} \cdot \boldsymbol{y} \quad \text{subject to} \quad \boldsymbol{y} \in L^\perp + \boldsymbol{c}, \quad \boldsymbol{y} \in K^*. \end{cases} \tag{4}$$

Here $\boldsymbol{c}$ and $\boldsymbol{d}$ are given constant vectors in R^m, L a given linear subspace of R^m, and $L^\perp \subset R^m$ the orthogonal complement of L in R^m. Let

$$F = (L + \boldsymbol{d}) \times (L^\perp + \boldsymbol{c}). \tag{5}$$

Then the m dimensional affine subspace F enjoys *the self-orthogonality*, *i.e.*,

$$(\boldsymbol{x}' - \boldsymbol{x}) \cdot (\boldsymbol{y}' - \boldsymbol{y}) = 0 \text{ for every } (\boldsymbol{x}', \boldsymbol{y}'), \ (\boldsymbol{x}, \boldsymbol{y}) \in F,$$

which is a special case of the monotonicity (3). By the construction, we obviously see that $\boldsymbol{x}$ is a feasible solution of the primal problem $\mathcal{P}$ and $\boldsymbol{y}$ is a feasible solution of the dual problem $\mathcal{D}$ if and only if the pair $(\boldsymbol{x}, \boldsymbol{y})$ is a feasible solution of the monotone LCP (2) with F of the form (5). Assume that

$$\exists(\boldsymbol{x}, \boldsymbol{y});\ \boldsymbol{x} \in (L + \boldsymbol{d}) \cap (\text{int } K) \text{ and } \boldsymbol{y} \in (L^{\perp} + \boldsymbol{c}) \cap (\text{int } K^*), \tag{6}$$

or equivalently that the monotone LCP (2) has an interior feasible solution $(\boldsymbol{x}, \boldsymbol{y})$. Then $\boldsymbol{x}$ is a minimum solution of the primal problem $\mathcal{P}$ and $\boldsymbol{y}$ is a minimum solution of the dual problem $\mathcal{D}$ if and only if the pair $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of the LCP (2). See Chapter 4 of [13] for more details.

The aim of this short note is to present a constructive way of reducing the monotone LCP (2) over cones to an LP over cones, which is motivated by a recent paper [11] by Kojima, Shindoh and Hara. They introduced the monotone semidefinite linear complementarity problem (SDLCP) in symmetric matrices for a mathematical model on which they established a theoretical foundation of interior-point methods. We can specialize or adapt their methods to primal-dual interior-point methods for solving SDPs (semidefinite programs). See also [9, 10]. In concluding remarks of their paper [11], Kojima, Shindoh and Hara pointed out that a certain convex quadratic program in symmetric matrices is reducible not only to a monotone SDLCP in symmetric matrices, which is a special case of LCPs over cones, but also to an SDP, which is a special case of LPs over cones, and raised a question whether the monotone SDLCP in symmetric matrices is an essential generalization of the SDP.

Section 2 is devoted to the main result, a constructive way of reducing the monotone LCP (2) over cones to an LP over cones. As a corollary of the main result, we will see in Section 3 that the monotone SDLCP in symmetric matrices is reducible to an SDP, which gives a negative answer to the question above.

2. Main result

It is well-known that the LCP (2) over cones can be reformulated as the following minimization problem with a bilinear objective function:

$$\begin{cases} \text{minimize} & \boldsymbol{x} \cdot \boldsymbol{y} \\ \text{subject to} & (\boldsymbol{x}, \boldsymbol{y}) \in F,\ \boldsymbol{x} \in K \text{ and } \boldsymbol{y} \in K^*. \end{cases} \tag{7}$$

Between the two problems, we have:

- $(\boldsymbol{x}, \boldsymbol{y})$ is a feasible solution of the LCP (2) if and only if it is a feasible solution of the problem (7),

- $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of the LCP (2) if and only if it is a minimum solution of the problem (7) with the objective value $\boldsymbol{x} \cdot \boldsymbol{y} = 0$.

In the remainder of this section, we assume that F is monotone, and reduce the problem (7) to an LP (17) over cones; starting from the problem (7) above, we derive a series of minimization problems (11), (12) and (14), which lead us to the final goal, the LP (17) over cones.

We begin with a simple special case where the affine subspace F of R^{2m} associated with the LCP (2) over cones (and also associated with the problem (7) above) is of the following *standard form* as in the standard form LCP (1):

$$F = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}\}. \tag{8}$$

Here $\boldsymbol{M}$ denotes an $m \times m$ matrix and $\boldsymbol{q} \in R^m$. Obviously, $\dim F = m$. Also it is well-known and easily verified from the monotonicity of the m dimensional affine subspace F of the standard form (8) that $\boldsymbol{M}$ is positive semidefinite; hence the $m \times m$ symmetric matrix $\boldsymbol{Q} \equiv (\boldsymbol{M} + \boldsymbol{M}^T)/2$ is positive semidefinite. Therefore we can rewrite the problem (7) as a minimization problem with a convex quadratic objective function:

$$\left\{ \begin{array}{ll} \text{minimize} & \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}, \ \boldsymbol{x} \in K \text{ and } \boldsymbol{y} \in K^*. \end{array} \right. \tag{9}$$

Remark. It was shown in the papers [1, 5, 15] that any monotone affine subspace of R^{2m} is at most m dimensional. When the monotone affine subspace associated with a linear complementarity problem over cones is m dimensional, the problem is reducible to a monotone LCP with F of the standard form (8) (see [1, 5, 15]). Hence we can reduce such a problem to a minimization problem, which is similar to the problem (9) above, with a convex quadratic objective function.

Now we show how to reduce the problem (7) with a general monotone affine subspace F of R^{2m} to a minimization problem with a convex quadratic objective function. Let $k = \dim F$. Take an $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in F$ and a $2m \times k$ matrix $\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix}$ whose columns form a basis of the k dimensional linear subspace

$$\left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} - \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} : \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in F \right\}$$

of R^{2m} to represent F such that

$$F = \left\{ \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix} \boldsymbol{u} + \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} : \boldsymbol{u} \in R^k \right\}. \tag{10}$$

Then we have

$$\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} + \boldsymbol{q}^T\boldsymbol{u} + r \text{ if } \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} = \begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix},$$

where $\boldsymbol{Q} = \frac{1}{2}(\boldsymbol{A}^T\boldsymbol{B} + \boldsymbol{B}^T\boldsymbol{A})$, $\boldsymbol{q} = \boldsymbol{A}^T\boldsymbol{b} + \boldsymbol{B}^T\boldsymbol{a}$ and $r = \boldsymbol{a}^T\boldsymbol{b}$. Under the assumption that F is monotone, the lemma below ensures that the $k \times k$ symmetric matrix $\boldsymbol{Q}$ is positive semidefinite.

Lemma 2.1. *An affine subspace F of the form* (10) *is monotone if and only if $\boldsymbol{A}^T\boldsymbol{B}$ is positive semidefinite, i.e., $\boldsymbol{u}\boldsymbol{A}^T\boldsymbol{B}\boldsymbol{u} \geq 0$ for every $\boldsymbol{u} \in R^k$.*

Proof. For every pair of

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} = \begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}, \quad \begin{pmatrix}\boldsymbol{x}'\\ \boldsymbol{y}'\end{pmatrix} = \begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix}\boldsymbol{u}' + \begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} \in F,$$

we see that

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{x}')\cdot(\boldsymbol{y}-\boldsymbol{y}') &= (\boldsymbol{A}\boldsymbol{u}-\boldsymbol{A}\boldsymbol{u}')\cdot(\boldsymbol{B}\boldsymbol{u}-\boldsymbol{B}\boldsymbol{u}')\\ &= (\boldsymbol{u}-\boldsymbol{u}')^T\boldsymbol{A}^T\boldsymbol{B}(\boldsymbol{u}-\boldsymbol{u}').\end{aligned}$$

Hence the desired result follows.

Remark. It was shown in the paper [12] that an affine subspace F of R^{2m} is monotone if and only if the bilinear form $\boldsymbol{x}\cdot\boldsymbol{y}$ is convex on F. The lemma above is equivalent to this fact.

In view of the discussions above and Lemma 2.1, we can rewrite the problem (7) as a minimization problem with a convex quadratic objective function:

$$(11)\qquad \begin{cases}\text{minimize} & \boldsymbol{u}^T\boldsymbol{Q}\boldsymbol{u} + \boldsymbol{q}^T\boldsymbol{u} + r,\\ \text{subject to} & \boldsymbol{x} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{a},\ \boldsymbol{y} = \boldsymbol{B}\boldsymbol{u} + \boldsymbol{b},\\ & \boldsymbol{x} \in K \text{ and } \boldsymbol{y} \in K^*.\end{cases}$$

If we take $\boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{B} = \boldsymbol{M}$, $\boldsymbol{b} = \boldsymbol{q}$ and $r = 0$ in (11), we have the minimization problem (9). Therefore we will restrict ourselves to the minimization problem (11).

In Chapter 4 of their book [13], Nesterov and Nemirovskii showed that a convex program is reducible to an LP over cones. Since (11) is a convex program, we could apply their method to reduction of the problem (11) to an LP over cones. Although their method is quite general, it is not fit for reduction of a monotone SDLCP in symmetric matrices to an SDP which we discuss in the next section. So we employ another way of reducing the problem (11) to an LP over cones.

We further rewrite the problem (11) as

$$\text{(12)} \qquad \begin{cases} \text{minimize} & \alpha + \boldsymbol{q}^T\boldsymbol{u} + r, \\ \text{subject to} & \boldsymbol{u}^T\boldsymbol{Q}\,\boldsymbol{u} \le \alpha, \\ & \boldsymbol{x} = \boldsymbol{A}\,\boldsymbol{u} + \boldsymbol{a},\ \boldsymbol{y} = \boldsymbol{B}\,\boldsymbol{u} + \boldsymbol{b}, \\ & \boldsymbol{x} \in K \text{ and } \boldsymbol{y} \in K^*. \end{cases}$$

This problem is not an LP over cones yet because it involves the quadratic inequality constraint $\boldsymbol{u}^T\boldsymbol{Q}\,\boldsymbol{u} \le \alpha$. We will utilize a popular technique (based on Schur complements) in the field of linear matrix inequalities (see, *e.g.*, [2]) to replace the constraint by "a semidefinite inequality constraint" (13).

In our succeeding discussions, the symbol $\mathcal{S}(n)$ stands for the set of $n \times n$ real symmetric matrices, and the symbol $\mathcal{S}_+(n)$ for the set of $n \times n$ real symmetric positive semidefinite matrices. We regard $\mathcal{S}(n)$ as an $n(n+1)/2$ dimensional real vector space with the usual addition $\boldsymbol{X}+\boldsymbol{Y}$ of two matrices $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathcal{C}(n)$, the scalar multiple $\alpha\boldsymbol{X}$ of $\boldsymbol{X} \in \mathcal{S}(n)$ by $\alpha \in R$ and the inner product $\boldsymbol{X} \bullet \boldsymbol{Y} = \mathrm{Tr}\ \boldsymbol{X}^T\boldsymbol{Y}$ (*i.e.*, the trace of the product of $\boldsymbol{X}^T\boldsymbol{Y}$).

Since $\boldsymbol{Q}$ is a $k \times k$ symmetric positive semidefinite matrix, it can be factorized as $\boldsymbol{Q} = \boldsymbol{L}^T\boldsymbol{L}$ for some $k \times k$ matrix $\boldsymbol{L}$ (*e.g.,* apply the Cholesky factorization to $\boldsymbol{Q}$). Then $(\boldsymbol{u}, \alpha)$ satisfies the inequality constraint $\boldsymbol{u}^T\boldsymbol{Q}\,\boldsymbol{u} \le \alpha$ if and only if

$$\text{(13)} \qquad \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{L}\,\boldsymbol{u} \\ \boldsymbol{u}^T\boldsymbol{L}^T & \alpha \end{pmatrix} \in \mathcal{S}_+(k+1),$$

where $\boldsymbol{I}$ denotes the $k \times k$ identity matrix. Hence we obtain the problem

$$\text{(14)} \qquad \begin{cases} \text{minimize} & \alpha + \boldsymbol{q}^T\boldsymbol{u} + r, \\ \text{subject to} & \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{L}\,\boldsymbol{u} \\ \boldsymbol{u}^T\boldsymbol{L}^T & \alpha \end{pmatrix} \in \mathcal{S}_+(k+1), \\ & \boldsymbol{x} = \boldsymbol{A}\,\boldsymbol{u} + \boldsymbol{a},\ \boldsymbol{y} = \boldsymbol{B}\,\boldsymbol{u} + \boldsymbol{b}, \\ & \boldsymbol{x} \in K \text{ and } \boldsymbol{y} \in K^*, \end{cases}$$

which is equivalent to the minimization problem (12).

The problem (14) is an LP over cones. To see this more concretely, we need to identify the space $\mathcal{S}(n)$ of $n \times n$ real symmetric matrices with the $n(n+1)/2$ dimensional Euclidean space $R^{n(n+1)/2}$ through an isomorphism between these two spaces (a one-to-one linear mapping from one space onto the other). Here we employ an isomorphism $\boldsymbol{\sigma}(\cdot; n) : \mathcal{S}(n) \to R^{n(n+1)/2}$ of the following form:

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{Z}; n) = (\ & z_{11}, && \sqrt{2}z_{12}, && \sqrt{2}z_{13}, && \ldots, && \sqrt{2}z_{1(n-1)}, && \sqrt{2}z_{1n}, \\ & && z_{22}, && \sqrt{2}z_{23}, && \ldots, && \sqrt{2}z_{2(n-1)}, && \sqrt{2}z_{2n}, \\ & && && && \ldots, && \ldots, && \ldots, \\ & && && && && z_{(n-1)(n-1)}, && \sqrt{2}z_{(n-1)n}, \\ & && && && && && z_{nn}\ )\end{aligned}$$

$$\text{for every} \quad \boldsymbol{Z} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix} \in \mathcal{S}(n). \tag{15}$$

Note that this isomorphism $\boldsymbol{\sigma}(\cdot; n) : \mathcal{S}(n) \to R^{n(n+1)/2}$ preserves values of the inner products in both spaces;

$$\boldsymbol{X} \bullet \boldsymbol{Y} = \boldsymbol{\sigma}(\boldsymbol{X}; n) \cdot \boldsymbol{\sigma}(\boldsymbol{Y}; n) \text{ for every } \boldsymbol{X},\ \boldsymbol{Y} \in \mathcal{S}(n). \tag{16}$$

(This property of the isomorphism $\boldsymbol{\sigma}(\cdot; n)$ is not relevant here but is used in the next section). Now define

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{u}, \alpha) &= \boldsymbol{\sigma}\left(\begin{pmatrix} \boldsymbol{I} & \boldsymbol{L}\boldsymbol{u} \\ \boldsymbol{u}^T\boldsymbol{L}^T & \alpha \end{pmatrix}; k+1 \right) \text{ for every } (\boldsymbol{u}, \alpha) \in R^{k+1}, \\ T_+(k+1) &= \boldsymbol{\sigma}\left(\mathcal{S}_+(k+1); k+1 \right), \\ \ell &= (k+1)(k+2)/2. \end{aligned}$$

Then $\boldsymbol{g}$ is an affine mapping from R^{k+1} into R^ℓ, so that we can find an $\ell \times (k+1)$ matrix $\boldsymbol{H}$ and an $\boldsymbol{h} \in R^\ell$ satisfying

$$\boldsymbol{g}(\boldsymbol{u}, \alpha) = \boldsymbol{H} \begin{pmatrix} \boldsymbol{u} \\ \alpha \end{pmatrix} + \boldsymbol{h} \text{ for every } (\boldsymbol{u}, \alpha) \in R^{k+1}.$$

We also see that $T_+(k+1)$ forms a closed convex cone in R^ℓ. Consequently we obtain an LP over cones

$$\begin{cases} \text{minimize} & \alpha + \boldsymbol{q}^T\boldsymbol{u} + r, \\ \text{subject to} & \boldsymbol{z} = \boldsymbol{H}\binom{\boldsymbol{u}}{\alpha} + \boldsymbol{h} \in T_+(k+1), \\ & \boldsymbol{x} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{a},\ \boldsymbol{y} = \boldsymbol{B}\boldsymbol{u} + \boldsymbol{b}, \\ & \boldsymbol{x} \in K \text{ and } \boldsymbol{y} \in K^*. \end{cases} \tag{17}$$

This LP over cones is equivalent to the LCP (2) over cones in the sense that

- $(\boldsymbol{x}, \boldsymbol{y})$ is a feasible solution of the LCP (2) over cones if and only if $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \alpha, \boldsymbol{z})$ is a feasible solution of the LP (17) over cones for some $(\boldsymbol{u}, \alpha, \boldsymbol{z})$,
- $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of the LCP (2) over cones if and only if $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \alpha, \boldsymbol{z})$ is a minimum solution of the LP (17) over cones with the objective value $\alpha + \boldsymbol{q}^T \boldsymbol{u} + r = 0$ for some $(\boldsymbol{u}, \alpha, \boldsymbol{z})$.

3. Reduction of the monotone SDLCP in symmetric matrices to an SDP

Let $\mathcal{F}$ be an $n(n+1)/2$ dimensional affine subspace of $\mathcal{S}(n)^2$. We assume that $\mathcal{F}$ is monotone, *i.e.*,

$$(\boldsymbol{X} - \boldsymbol{X}') \bullet (\boldsymbol{Y} - \boldsymbol{Y}') \geq 0 \text{ for every} (\boldsymbol{X}, \boldsymbol{Y}),\ (\boldsymbol{X}', \boldsymbol{Y}') \in \mathcal{F}.$$

The monotone SDLCP in symmetric matrices is the problem:

(18)

$$\text{find } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}(n)^2;\ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F},\ \boldsymbol{X} \in \mathcal{S}_+(n),\ \boldsymbol{Y} \in \mathcal{S}_+(n) \text{ and } \boldsymbol{X} \bullet \boldsymbol{Y} = 0.$$

This problem is a special case of the LCP (2) over cones. In fact, $(\boldsymbol{X}, \boldsymbol{Y})$ is a solution of the monotone SDLCP (18) in symmetric matrices if and only if $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{\sigma}(\boldsymbol{X}; n), \boldsymbol{\sigma}(\boldsymbol{Y}; n))$ is a solution of the monotone LCP over cones:

$$\text{(19)} \quad \text{find } (\boldsymbol{x}, \boldsymbol{y}) \in R^{2m};\ (\boldsymbol{x}, \boldsymbol{y}) \in F,\ \boldsymbol{x} \in T_+(n),\ \boldsymbol{y} \in T_+(n) \text{ and } \boldsymbol{x} \cdot \boldsymbol{y} = 0,$$

where

$$\begin{aligned} m &= n(n+1)/2, \\ F &= \{(\boldsymbol{\sigma}(\boldsymbol{X}; n), \boldsymbol{\sigma}(\boldsymbol{Y}; n)) \ :\ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}\}, \\ T_+(n) &= \boldsymbol{\sigma}(\mathcal{S}_+(n); n). \end{aligned}$$

(Recall that $\boldsymbol{\sigma}(\cdot; n)$ is the isomorphism from $\mathcal{S}(n)$ onto R^m defined by (15) and that it satisfies (16)). We remark here that

$$\{\boldsymbol{Y} \in \mathcal{S}(n) : \boldsymbol{X} \bullet \boldsymbol{Y} \geq 0 \text{ for every } \boldsymbol{X} \in \mathcal{S}_+(n)\} = \mathcal{S}_+(n).$$

This implies that the dual of $T_+(n)$ coincides with $T_+(n)$ itself.

Applying the way of reducing the LCP (2) to the minimization problem (14) in the previous section to the LCP (19) over cones, we obtain an SDP which is equivalent to the monotone SDLCP (18) in symmetric matrices:

$$\tag{20} \begin{cases} \text{minimize} & \alpha + \boldsymbol{q}^T\boldsymbol{u} + r, \\ \text{subject to} & \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{L}\boldsymbol{u} \\ \boldsymbol{u}^T\boldsymbol{L}^T & \alpha \end{pmatrix} \in \mathcal{S}_+(m+1), \\ & \boldsymbol{x} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{a},\ \boldsymbol{y} = \boldsymbol{B}\boldsymbol{u} + \boldsymbol{b}, \\ & \boldsymbol{X} = \boldsymbol{G}(\boldsymbol{x}) \in \mathcal{S}_+(n) \text{ and } \boldsymbol{Y} = \boldsymbol{G}(\boldsymbol{y}) \in \mathcal{S}_+(n). \end{cases}$$

Here

$$\begin{aligned} r &\in R, \\ \boldsymbol{q},\ \boldsymbol{a},\ \boldsymbol{b} &\in R^m, \\ \boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{L} &: m \times m \text{ matrices}, \\ \boldsymbol{G}(\cdot) &= \boldsymbol{\sigma}^{-1}(\cdot; n) \ :\ \text{an isomorphism from } R^m \text{ onto } \mathcal{S}(n) \\ &\quad (\text{see (15) for the definition of } \boldsymbol{\sigma}). \end{aligned}$$

4. Concluding remarks

We have shown in this short note that the monotone LCP (2) over cones is reducible to an LP over cones, which is obviously a special case of convex programs. So the readers might think that LPs over cones and/or monotone LCPs over cones are less general than convex programs. It should be noted, however, that a convex program is reducible to an LP over cones (see Chapter 4 of [13]). On the other hand, we know under the assumption (6) that a necessary and sufficient optimality condition for the primal-dual pair (4) of LPs over cones can be stated in terms of a monotone LCP over cones. Therefore we may say that convex programs, LPs over cones and monotone LCPs over cones are of similar generality.

References

1. J. F. Bonnans and C. C. Gonzaga, *Convergence of interior point algorithms for the monotone linear complementarity problem*, Technical Report 2074, INRIA, Rocquencourt, France, 1993.

2. S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

3. R. W. Cottle, J. S. Pang and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, New York, 1992.

4. S. M. Gowda and T. I. Seidman, *Generalized linear complementarity problems*, Mathematical Programming **46** (1990), 329-340.

5. O. Güler, *Generalized linear complementarity problems*, Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21228 April 1992 (Revised September 1993).

6. G. J. Habetler and A. L. Price, *Existence theory for generalized nonlinear complementarity problems*, Journal of Optimization Theory and Applications **7** (1971), 223-239.

7. S. Karamardian, *Generalized complementarity problem*, Journal of Optimization Theory and Applications **8** (1971), 161-168.

8. S. Karamardian, *Complementarity problem over cones with monotone and pseudomonotone maps*, Journal of Optimization Theory and Applications **18** (1976), 445-454.

9. M. Kojima, *A primitive interior point algorithm for semidefinite programs in Mathematica*, Research Report #B-293, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo 152, Japan, December 1994.

10. M. Kojima, S. Kojima and S. Hara, *Linear algebra for semidefinite programming*, Research Report #B-290, Dept. of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo 152, Japan, October 1994, revised July 1996, RIMS Kokyuroku to appear.

11. M. Kojima, S. Shindoh and S. Hara, *Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices*, SIAM Journal on Optimization **7** (1997), 86-125.

12. O. L. Mangasarian and J. S. Pang, *The extended linear complementarity problem*, SIAM Journal on Matrix Analysis (1995) to appear.

13. Yu. E. Nesterov and A. S. Nemirovskii, *Interior Point Polynomial Methods in Convex Programming: Theory and Applications*, SIAM, 1993.

14. R. Sznajder and M. S. Gowda, *Generalizations of P_0- and P- properties; extended vertical and horizontal LCPs*, Linear Algebra and Its Applications (to appear).

15. R. H. Tütüncü and M. J. Todd, *Reducing horizontal linear complementarity problems*, Technical Report, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York 14853, USA, 1994.

16. Y. Ye, *A fully polynomial-time approximation algorithm for computing a stationary point of the general linear complementarity problem*, Mathematics of Operations Research **18** (1993), 334-345.

Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152, Japan

Department of Mathematics, Faculty of Engineering
Kanagawa University
Rokkakubashi, Kanagawa-ku, Yokohama 221, Japan,

Department of Mathematics and Physics
The National Defense Academy
Hashirimizu 1-10-20, Yokosuka, 239, Japan