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RECOGNIZING FACET DEFINING INEQUALITIES

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Dedicated to Hoang Tuy on the occasion of his seventieth birthday

Abstract. We discuss a method for determining whether a valid inequality for a 0-1 polytope is facet defining. The method is based on a new procedure for generating a sequence of 0-1 points on a face of the polytope, guaranteed to be linearly independent. The sequence moves along k-dimensional surfaces of the face, and whenever this is possible for small k, the procedure becomes particularly simple. As a result, we give a new sufficient condition for a valid inequality to be facet defining, that generalizes several earlier conditions. We also give a necessary and sufficient condition. None of these conditions is always verifiable in polynomial time. yet in many situations their use has led to the discovery of new classes of facets. We illustrate this on the case of the vertex packing and set covering polyhedra.

1. Introduction

We consider 0-1 polytopes, i.e. bounded polyhedra of the form

$$P := \text{conv}\ \{x \in \{0,1\}^n : Ax \leq b\},$$

where A is an arbitrary real $m \times n$ matrix and b is an arbitrary nonzero m-vector.

We denote $M := \{1,\dots,m\}$, $N := \{1,\dots,n\}$. The i-th row and j-th column of A are a^i and a_j, respectively. The *dimension* of P, denoted $\dim P$, is one less than the maximum number of affinely independent points $x \in P$. The *equality set* of P is the set of inequalities $a^i x \leq b_i$ satisfied with equality by all $x \in P$; its *rank* is the maximum number of its linearly independent members. If this rank is r, then $\dim P = n - r$.

An inequality $\alpha x \leq \alpha_0$, with $\alpha_0 \neq 0$, is *valid* for P if it is satisfied by

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every $x \in P$. The *face* of P defined by $\alpha x \leq \alpha_0$ is $F := \{x \in P : \alpha x = \alpha_0\}$. F is proper if $\emptyset \neq F \neq P$, improper otherwise. A *facet* of P is a maximum-dimensional proper face; i.e. the face F is a facet if and only if $\dim F = \dim P - 1$.

Given a polytope P and an inequality $\alpha x \leq \alpha_0$ satisfied by all $x \in P$ and such that $\alpha x = \alpha_0$ for some (but not all) $x \in P$, it is of interest to know whether the face F defined by the inequality is a facet. There are two well-known methods for establishing this. The first one, sometimes called the direct method, consists of exhibiting $\dim P$ affinely independent points $x \in F$, which proves that F is a facet; or showing that no such set exists, which proves the opposite. The second, or indirect, method consists in showing that any inequality $\beta x \leq \beta_0$ satisfied by all $x \in P$ and such that $\beta x = \beta_0$ whenever $\alpha x = \alpha_0$, is a linear combination of $\alpha x \leq \alpha_0$ (with positive weight) and the members of the equality set of P, in which case F is a facet; or exhibiting some inequality with this property that is not such a combination, in which case F is not a facet. For a discussion of these proof methods see any of Bachem and Grötschel (1982), Pulleyblank (1983), Schrijver (1986) or Nemhauser and Wolsey (1988). The difficulty with the direct method does not consist in finding $\dim P$ points in F, which is typically easy, but in finding $\dim P$ affinely independent points or showing that no such set of points exists, which is often hard.

In this paper we discuss a variant of the direct method, based on a procedure for generating a sequence of 0-1 points in F that are guaranteed to be linearly independent. The procedure comes in two versions. The first one, called 2-reduction, is straightforward but may stop short of generating the required number of points. Nevertheless, it yields a sufficient condition for an inequality to be facet defining, that generalizes some well known sufficient conditions from the literature and makes it easy to prove the facet inducing property of large classes of inequalities. The second, more general version, called r-reduction, is always applicable, but computationally more cumbersome. It provides a weaker sufficient condition, as well as a necessary and sufficient condition for an inequality to be facet inducing.

The problem examined here was addressed earlier in a more general context by Edmonds, Lovász and Pulleyblank (1982), who gave algorithm for determining the dimension of an arbitrary polyhedron P^*, that is polynomial if the problem of maximizing a linear function over P^* is polynomially solvable. The algorithm constructs $\dim P^* + 1$ affinely independent points in P^* and n-$\dim P^*$ affinely independent equations satisfied by every point in P^*. By contrast, our procedure is restricted to 0-1 polytopes and does not enable its user to identify the equality set of the face F that it examines. Its focus in trying to find linearly independent points in F is to move along k-dimensional surfaces for $k = 1, 2$ etc. Whenever this is possible for a small k, the procedure is particularly simple and therein lies its main advantage. These results were first presented at the IPCO meeting in Waterloo (see Balas (1990)).

In recent joint work with Matteo Firschetti (Balas and Fischetti (1992)), the approach discussed here was successfully used to answer a long-standing open question concerning the conditions under which a facet inducing inequality for the monotone completion of the (symmetric or asymmetric) traveling salesman polytope is also facet inducing for TS polytope itself.

Our paper is organized as follows. The next two sections discuss 2-reduction and r-reduction, respectively; Section 4 takes up the case of positive 0-1 polytopes; Sections 5 and 6 discuss application to vertex packing and set covering, respectively.

2. 2-REDUCTION

Let $\alpha x \le \alpha_0$ be a valid inequality for P, with $\alpha_0 \ne 0$, $\alpha_j \ne 0$, $j \in N$, and let $F := \{x \in P : \alpha x = \alpha_0\}$, $F_I := F \cap \{0,1\}^n$. We will consider certain partitions of the index set N. A *partition* $\pi := \{N_1, \dots, N_s\}$ of N is a collection of disjoint, nonempty subsets N_k of N whose union is N. If the partition has s members (subsets), we call it an s-partition. The *trivial* partition is the n-partition $\pi = \{\{1\}, \dots, \{n\}\}$. The *improper* partition is the 1-partition $\pi = \{N\}$.

With any partition $\pi = \{N_1, \dots, N_n\}$ and any $x \in F_I$ we associate a vector $\pi(x) \in \mathbf{R}^s$, $\pi(x) = \{\pi_1(x), \dots, \pi_s(x)\}$, defined by

$$\pi_k(x) = \sum (\alpha_j x_j : j \in N_k) \quad k = 1, \dots, s,$$

and called the π-*pattern* of x with respect to $\alpha x \le \alpha_0$. We will say that the partition $\pi = \{N_1, \dots, N_n\}$ of N is $2-$*reducible* with respect to $\alpha x \le \alpha_0$ if there exists a pair $i, j \in \{1, \dots, s\}$, and a pair $x, y \in F_I$, such that

$$\pi_k(x) \begin{cases} \ne \pi_k(y) & k = i, j, \\ = \pi_k(y) & k \in \{1, \dots, s\} \setminus \{i, j\}. \end{cases} \tag{2}$$

Note that when π is the trivial partition of N, then (2) becomes

$$\alpha_k x_k \begin{cases} \ne \alpha_k y_k & k = i, j, \\ = \alpha_k y_k & k \in \{1, \dots, s\} \setminus \{i, j\}. \end{cases} \tag{3}$$

If π is 2-reducible and i, j is the pair of which relation (2) holds, we call 2-*reduction* the operation that replaces π with

$$\pi' := (\pi \setminus \{N_i, N_j\}) \cup N_{ij}, \tag{4}$$

here $N_{ij} := N_i \cup N_j$.

Note that while $\pi(x) \neq \pi(y)$, as a result of the 2-reduction we obtain $\pi'_k(x) = \pi'_k(y)$ for all subset indices k pertaining to the partition π'.

Consider now the following

2-reduction procedure (with respect to $\alpha x \leq \alpha_0$).

Initialization. Let π^1 be the trivial partition of N, let $x^1 \in F_I$, and $t := 1$.

Iterative Step. Given $\pi^t := \{N_1, \ldots, N_s\}$ and $x^t \in F_I$, find $x^{t+1} \in F_I$ such that

$$\pi^t_k(x^{t+1}) \begin{cases} \neq \pi^t_k(x^t) & k = i, j, \\ = \pi^t_k(x^t) & k \in \{1, \ldots, s\} \setminus \{i, j\}, \end{cases} \tag{5}$$

for some pair $i, j \in \{1, \ldots, s\}$. If no such x^{t+1} exists, stop. Otherwise let $N_{ij} := N_1 \cup N_j$,

$$\pi^{t+1} := \{\pi^t \setminus \{N_i, N_j\}) \cup N_{ij}, \tag{6}$$

$t := t + 1$, and repeat.

Lemma 1. *If $\pi^p = \{N_1, \ldots, N_s\}$ is the current partition of N at some iteration p of the 2-reduction procedure, then the π^p-patterns of all the vectors $x^t \in F_I$, $t = 1, \ldots, p$, used in the procedure up to iteration p, are the same; i.e.*

$$\pi^p(x^t) = (C_1, \ldots, C_s), \quad t = 1, \ldots, p,$$

where $C_1, \ldots, C_s$ are constants independent of t.

Proof. We use induction on p. For $p = 1$ the statement is trivially true. Suppose it is true for $p = 1, \ldots, q$, and let $p = q + 1 \geq 2$. If $\pi^q = \{N_1, \ldots, N_s\}$, by hypothesis we have

$$\pi^q_k(x^t) = C_k \quad \text{for } t = 1, \ldots, q,$$

for all $k \in \{1, \ldots, s\}$. At the $(q+1)$-*st* application of the 2-reduction step, we have for some $i, j \in \{1, \ldots, s\}$

$$\pi^q_k(x^{q+1}) = C_k \text{ for } k \in \{1, \ldots, s\} \setminus \{i, j\},$$

and

$$\begin{aligned}\pi_i^q(x^{q+1}) + \pi_j^q(x^{q+1}) &= \alpha_0 - \sum(\pi_k^q(x^{q+1}) : k \in \{1,\dots,s\} \setminus \{i,j\}) \\ &= \alpha_0 - \sum(C_k : k \in \{1,\dots,s\} \setminus \{i,j\}) \\ &= C_i + C_j.\end{aligned}$$

Hence for the partition $\pi^{q+1} = \{N'_1, \dots, N'_{s-1}\}$, where the subsets $N'_1, \dots, N'_{s-1}$ come from renumbering the subsets N_k, $k \in \{1,\dots,s\} \setminus \{i,j\}$ and $N_{ij} = N_i \cup N_j$, we have

$$\pi_k^{q+1}(x^t) = C_k \text{ for } t = 1,\dots,q+1 \text{ and } k = 1,\dots,s-1,$$

with $C_{k_*} = C_i + C_j$ for the new subset N'_{k_*} corresponding to N_{ij}. This completes the induction. □

Theorem 2. *If the 2-reduction procedure stops at iteration p, then π^p is an s-partition with $s = n - p + 1$, and the p vectors $x^t \in F_I$, $t = 1,\dots,p$, used in the procedure, are linearly independent.*

Proof. Since π^1 is an n-partition and at every iteration the number of subsets in the partition decreases by one, π^p is an s-partition with $s = n - p + 1$.

Now suppose the vectors $x^t \in F_I$, $t = 1,\dots,p$, are linearly dependent. Then there exist scalars $\lambda_1,\dots,\lambda_{p-1}$ such that $x^p = \sum(x^t\lambda_t : t = 1,\dots,p-1)$. Let π^{p-1} be the $(s+1)$-partition from which the s-partition π^p was obtained. Then for $k = 1,\dots,s+1$,

$$\begin{aligned}\pi_k^{p-1}(x^p) &= \sum(\pi_k^{p-1}(x^t)\lambda_t : t = 1,\dots,p-1) \\ &= C_k \sum(\lambda_t : t = 1,\dots,p-1),\end{aligned}$$

where the second equation follows from Lemma 1.

On the other hand, by the rules of the 2-reduction procedure and from Lemma 1,

$$\pi_k^{p-1}(x^p)\begin{cases} \neq C_k & k \in \{i,j\}, \\ = C_k & k \in \{1,\dots,s+1\} \setminus \{i,j\}, \end{cases}$$

for some $i,j \in \{1,\dots,s+1\}$; and so we obtain on the one hand $\sum(\lambda_t : t = 1,\dots,p-1) \neq 1$ and on the other $\sum(\lambda_t : t = 1,\dots,p-1) = 1$, a contradiction. □

The following corollaries are immediate consequences of Theorem 2.

Corollary 3. $\dim F \geq n - s$ $(= p - 1)$.

Corollary 4. *If the 2-reduction procedure stops with the improper partition $\pi^n = \{N\}$, then F is a facet of P.*

Next we address the question of the complexity of 2-reduction.

Proposition 5. *Given a partition $\pi = \{N_1, \ldots, N_s\}$ of N, a pair $i, j \in \{1, \ldots, s\}$, and a vector $x^t \in F$, let $f(n)$ be the complexity function of finding $x^{t+1} \in F_I$ that satisfies (5) or showing that no such x^{t+1} exists. Then the complexity of the 2-reduction procedure in $O(n^3 f(n))$.*

Proof. At each iteration, the problem stated in the theorem has to be solved at most $O(n^2)$ times, once for each pair $i, j \in \{1, \ldots, s\}$. Since there are at most n iterations, the statement follows. $\square$

Example 1. Let P be the convex hull of those $x \in \{0,1\}^9$ satisfying

$$\begin{aligned}
&5x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_7 + x_8 + x_9 \leq 8,\\
&4x_1 + 6x_2 + 4x_3 + 4x_4 + 4x_5 + 2x_6 + 2x_7 + 2x_8 + 2x_9 \leq 12,
\end{aligned}$$

and consider the inequality

$$3x_1 + 3x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_7 + x_8 + x_9 \leq 6,$$

obtained by adding the two inequalities that define P, dividing the resulting inequality by 3, and rounding down the right side to the nearest integer. This is a special case of Chvátal's (1975) procedure, and the resulting inequality is obviously valid for P. We want to know whether it is facet defining.

We start with the trivial partition $\pi^1 := \{\{1\}, \ldots, \{9\}\}$ and with $x^1 = (1,0,1,0,0,1,0,0,0)$. At each iteration we take $m := t$ (which is an option). At the first iteration we use $x^2 = (1,0,1,0,0,0,1,0,0)$ to obtain $\pi^2 := \{\{1\}, \ldots, \{5\}, \{6,7\}, \{8\}, \{9\}\}$; etc. The sequence of vectors $x^t \in F$ and partitions π^t of N is listed below:

$$\begin{aligned}
x^1 &= (1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0) & \pi^1 &= \{\{1\}, \ldots, \{9\}\}\\
x^2 &= (1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0) & \pi^2 &= \{\{1\}, \ldots, \{5\}, \{6,7\}, \{8\}, \{9\}\}\\
x^3 &= (1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0) & \pi^3 &= \{\{1\}, \ldots, \{5\}, \{6,7,8\}, \{9\}\}\\
x^4 &= (1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1) & \pi^4 &= \{\{1\}, \ldots, \{5\}, \{6,7,8,9\}\}
\end{aligned}$$

$$\begin{aligned}
x^5 &= (1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1) & \pi^5 &= \{\{1\},\{2\},\{3,4\},\{5\},\{6,\dots,9\}\}\\
x^6 &= (1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1) & \pi^6 &= \{\{1\},\{2\},\{3,4,5\},\{6,\dots,9\}\}\\
x^7 &= (0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1) & \pi^7 &= \{\{1,2\},\{3,4,5\},\{6,\dots,9\}\}\\
x^8 &= (0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1) & \pi^8 &= \{\{1,2\},\{3,\dots,9\}\}\\
x^9 &= (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0) & \pi^9 &= \{1,\dots,9\}.
\end{aligned}$$

Since π^9 is the improper partition, the inequality under examination defines a facet of P.

3. r-REDUCTION

The 2-reduction procedure can be generalized to an r-reduction procedure that starts with 2-reduction and continues beyond the point where the latter fails. While the 2-reduction procedure provides a sufficient condition for F to be a facet of P, this more general reduction procedure provides a necessary and sufficient condition.

Given a partition $\pi = \{N_1,\dots,N_n\}$ of N, with $\{1,\dots,s\} =: S$, we will say that π is *r-reducible* with respect to $\alpha x \le \alpha_0$ if there exists $R \subseteq S$, $|R| \ge 2$, and a collection of points $x^t \in F_I$, $t = 1,\dots,r = |R|$, whose π-patterns $\pi(x^1),\dots,\pi(x^r)$ are linearly independent and satisfy

$$\pi_k(x^t) = C_k, \quad t = 1,\dots,r;\ k \in S \setminus R$$

for some constants C_k.

If π is r-reducible and R is the subset of S for which (7) holds, we call *r-reduction* the operation that replaces π with

$$\pi' := (\pi \setminus \{N_k : k \in R\}) \cup N_R, \tag{8}$$

where R is the subset of S used in (7) and $N_R := \cup(N_k : k \in R)$. Clearly, r-reduction generalizes the operation of 2-reduction introduced in the previous section.

As in the case of 2-reduction, it is worth noting that the partition π' obtained by (8) satisfies $\pi'_k(x^1) = \cdots = \pi'_k(x^r)$ for all subset indices k pertaining to the partition π'.

We will call a partition of N *valid* with respect to $\alpha x \le \alpha_0$ if it was obtained from the trivial partition by a sequence of r-reduction steps (where r may differ from one step to another).

Reduction procedure (with respect to $\alpha x \leq \alpha_0$)

Initialization. Let π^1 be the trivial partition of N, let $x^1 \in F_I$ and $t := 1$.

Iterative Step. Given $\pi^t = \{N_1, \ldots, N_s\}$ and $x^v \in F_I$, find $r-1$ vectors $x^i \in F_I$, $i = v+1, \ldots, v+r-1$, $2 \leq r \leq s$, such that the r vectors $\pi^t(x^i)$, $i = v, v+1, \ldots, v+r-1$, are linearly independent and satisfy

$$\pi_k^t(x^{v+h}) = \pi_k^t(x^v), \quad h = 1, \ldots, r-1;\ k \in S \setminus R, \tag{9}$$

for some $R \subseteq S := \{1, \ldots, s\}$, $|R| = r$.

If no such set of vectors $x^i \in F_I$ is found, stop; otherwise let $N_R := \cup(N_k : k \in R)$,

$$\pi^{t+1} := \pi^t \setminus \{N_k : k \in R\}) \cup N_R, \tag{10}$$

$t := t+1$ and repeat.

We will use the following analog of Lemma 1.

Lemma 6. *If $\pi^p = \{N_1, \ldots, N_s\}$ is the current partition of N at some iteration p of the reduction procedure, and $x^t \in F_I$, $t = 1, \ldots, v$ are the vectors used to obtain π^p, then the π^p-patterns of all x^t, $t = 1, \ldots, v$ are the same, i.e.*

$$\pi^p(x^t) = (C_1, \ldots, C_s), \quad t = 1, \ldots, v,$$

where $C_1, \ldots, C_s$ are constants independent of t.

Proof. Parallels the proof of Lemma 1. □

We now prove for the general reduction procedure a property analogous to the one proved in Theorem 2 for the 2-reduction procedure.

Theorem 7. *Suppose the reduction procedure stops with the partition π^p after performing a sequence of $p-1$ r-reductions, with $r = r_1, \ldots, r_{p-1}$, respectively, and let $v := 1 + (r_1 - 1) + \cdots + (r_{p-1} - 1)$. Then π^p is an s-partition with $s = n - v + 1$; and the v vectors $x^t \in F_I$, $t = 1, \ldots, v = n - s + 1$, used in the procedure, are linearly independent.*

Proof. Since the trivial partition has n subsets and each r-reduction reduces the number of subsets by $r-1$, we have $s = n - ((r_1 - 1) + \cdots + (r_{p-1} - 1)) = n - v + 1$, as claimed. Suppose now that the vectors $x^t \in F_I$

used in the procedure are linearly dependent. Then there exist scalars λ_t, $t = 1, \ldots, v$, not all zero, such that $\sum(x^t\lambda_t : t = 1, \ldots, v) = 0$. Let $\pi^{p-1} = \{N_1, \ldots, N_m\}$ be the partition of N from which π^p was obtained as the result of an r_{p-1}-reduction. Then for $k = 1, \ldots, m$, $\sum(\pi_k^{p-1}(x^t)\lambda_t : t = 1, \ldots, v) = 0$. The $r_{p-1} - 1$ vectors used in the last reduction step are $x^t \in F$ for $t = v - r_{p-1} + 2, \ldots, v$ and from the last equation we have for $k = 1, \ldots, m$,

(11)
$$\begin{aligned} 0 &= \sum(\pi_k^{p-1}(x^t)\lambda_t : t = 1, \ldots, v - r_{p-1} + 1) \\ &\quad + \sum(\pi_k^{p-1}(x^t)\lambda_t : t = v - r_{p-1} + 2, \ldots, v) = \\ &= \pi_k^{p-1}(x^{v-r_{p-1}+1})\mu_1 + \sum(\pi_k^{p-1}(x^{v-r_{p-1}+j})\mu_j : j = 2, \ldots, r_{p-1}), \end{aligned}$$

where $\mu_1 := \sum(\lambda_t : t = 1, \ldots, v - r_{p-1} + 1)$, $\mu_j := \lambda_{p-r_{p-1}+j}$ for $j = 2, \ldots, r_{p-1}$. Here the second equation follows from the fact that, according to Lemma 6,

$$\pi_k^{p-1}(x^t) = C_k \text{ for } t = 1, \ldots, v - r_{p-1} + 1 \text{ and } k = 1, \ldots, m.$$

But (11) contradicts the linear independence of the π^{p-1}-patterns of the r_{p-1} vectors $x^t \in F_I$, $t = v - r_{p-1} + 1, \ldots, v$ used in the last reduction step, which proves statement. □

As in the case of the 2-reduction procedure, the following corollary is an immediate consequence of Theorem 7. Let p, s and v be as in Theorem 7.

Corollary 8. $\dim F \geq n - s$ $(= v - 1)$.

We now state a necessary and sufficient condition for F to be facet of P.

Theorem 9. *Let π be any valid s-partition of N with respect to $\alpha x \leq \alpha_0$. Then F is a facet of P if and only if there exist $s + \dim P - n$ vectors $x \in F$ whose π-patterns are linearly independent.*

Proof. Let $x^t \in F_I$, $t = 1, \ldots, n - s + 1$, be the vectors used to obtain π from the trivial partition. Recall that the π-patterns of all vectors x^t are the same (Lemma 6), say $\pi(x^t) = (C_1, \ldots, C_s)$, $t = 1, \ldots, n - s + 1$. We claim that $s \geq n - \dim P$. This is obviously true of the trivial partition (for which $s = n$), as 0 is certainly a lower bound on $\dim P$; and every r-reduction step, while reducing the size of s by $r - 1$, also adds $r - 1$ new vectors $x \in F_I$ to the current linearly independent set, thereby increasing the lower bound on $\dim P$ by the same amount. This proves the claim.

Sufficiency. Now suppose there exist $p := s + \dim P - n$ vectors $x \in F_I$ whose π-patterns are linearly independent. Without loss of generality (w.l.o.g.) we may assume that at least one of these vectors belongs to the set of those x^t, $t = 1, \ldots, n-s+1$, used in the reduction procedure; on the other hand, at most one of them can belong to that set, since the π-patterns of all vectors x^t, $t = 1, \ldots, n-s+1$, are equal to $(C_1, \ldots, C_s)$. So let $x^t \in F$, $t = n-s+2, n-s+3, \ldots, n-s+p$ be the $p-1$ vectors whose π-patterns are different from $(C_1, \ldots, C_s)$. By the same argument as in the proof of Theorem 7, we can show that the $n-s+p = \dim P$ vectors $x^t \in F$, $t = 1, \ldots, n-s+p$, are linearly independent. For suppose not. Then there exist scalars λ_t, $t = 1, \ldots, n-s+p$, not all zero, such that $\sum(x^t \lambda_t : t = 1, \ldots, n-s+p) = 0$ and hence

$$\sum(\pi_k(x^t)\lambda_t : t = 1, \ldots, n-s+p) = 0 \text{ for } k = 1, \ldots, s.$$

This last equation yields for $k = 1, \ldots, s$,

$$\begin{aligned} 0 &= \sum(\pi_k(x^t)\lambda_t : t = 1, \ldots, n-s+1) \\ &\quad + \sum(\pi_k(x^t)\lambda_t : t = n-s+2, \ldots, n-s+p) \\ &= \pi_k(x^{n-s+1})\mu_1 + \sum(\pi_k(x^{n-s+j})\mu_j : j = 2, \ldots, p), \end{aligned} \tag{12}$$

where $\mu_1 := \sum(\lambda_t : t = 1, \ldots, n-s+1)$, $\mu_j := \lambda_{n-s+j}$ for $j = 2, \ldots, p$. However, equation (12) contradicts the linear independence of the π-patters of the p vectors $x^t \in F_I$, $t = n-s+1, \ldots, n-s+p$, which proves the sufficiency of the condition in the theorem.

Necessity. Suppose F is a facet P. Then there exists a set T of $\dim P$ linearly independent vectors $x \in F_I$. Further, without loss of generality. We may assume that the vectors x^t, $t = 1, \ldots, n-s+1$, used to obtain π from the trivial partition, belong to T. Let the remaining $p-1$ vectors of T be x^t, $t = n-s+2, \ldots, n-s+p$ $(= \dim P)$, and let X be the $(n-s+p) \times n$ matrix whole t-th row is x^t, $t = 1, \ldots, n-s+p$. We claim that the π-patterns of the p vectors x^t, $t = n-s+1, \ldots, n-s+p$, are linearly independent. For consider the $(n-s+p) \times s$ matrix Π whose t-th row is the π-pattern $\pi(x^t)$ of x^t for $t = 1, \ldots, n-s+p$. Since the first $n-s+1$ rows of Π are equal, the rank of Π is determined by its last p rows. Now suppose for the sake of contradiction that $\text{rank}(\Pi) < p$. Then for every subset V of $S := \{1, \dots, s\}$ of cardinality $p = s + \dim P - n$, there exist scalars λ_k, $k = 1, \dots, p$ not all zero, such that $\sum(\Pi_k \lambda_k : k \in V) = 0$ where Π_k is the k-th column of Π.

On the other hand, since $\text{rank}(X) = \dim P = n - s + p$, there exists a subset W of N of cardinality $n - s + p$, such that for any $W' \subseteq W$, $\sum(X_j \mu_j : j \in W') = 0$ implies $\mu_j = 0$, $j \in W'$. We claim that w.l.o.g. we may assume $W \supseteq N_k$ for at least p indices $k \in S$ (here N_k is the k-th subset of the partition π). For if not, i.e. if $N_k \setminus W \neq \emptyset$ for more than $s - p$ indices $k \in S$, then $|W| < n - s + p$, contrary to our assumption on $|W|$. Now let V^* be the set of those $k \in S$ such that $N_k \subseteq W$. Since $|V^*| \geq p$, there exists scalars λ_k, $k \in V^*$, not all zero, such that $\sum(\Pi_k \lambda_k : k \in V^*) = 0$, or, using the definition of Π,

$$\begin{aligned} 0 &= \sum \big(\sum(\alpha_j X_j : j \in N_k) \lambda_k : k \in V^* \big) \\ &= \sum(X_j \mu_j : j \in W'), \end{aligned}$$

where $\mu_j = \alpha_j \lambda_k$ for $j \in N_k$ and $k \in V^*$, and where $W' := \cup(N_k : k \in V^*)$.

As mentioned above, the last equation implies $\mu_j = 0$, $j \in W'$, which in turn, together with $\alpha_j \neq 0$, $j \in N$, implies $\lambda_k = 0$, $k \in V^*$, a contradiction. This proves that the π-patterns of the $p = s + \dim P - n$ vectors x^t, $t = n - s + 1, \dots, n - s + p$, are linearly independent. □

We notice that for an arbitrary 0-1 polytope P, even if F is a facet of P, the reduction procedure may stop with an s-partition π of N such that $s > n - \dim P + 1$. This can happen if at the last iteration there exists no set of r vectors $x^t \in F_I$ whose π-patterns are linearly independent and satisfy (9), for any $r \in [2, s + \dim P - n]$, but there exists a set of $s + \dim P - n$ vectors in F_I whose π-patterns are linearly independent without satisfying (9). For a full dimensional polytope, however, this cannot happen, as the next corollary shows.

Corollary 10. *Let P be full-dimensional. Then F is a facet of P if and only if the improper partition is valid with respect to $\alpha x \leq \alpha_0$.*

Proof. Sufficiency follows from Corollary 8. Now suppose F is a facet of P, and let π be the s-partition with which the reduction procedure stops. If $s > 1$, from Theorem 9 there exist $s + \dim P - n = s$ vectors $x \in F_I$ whose π-patterns are linearly independent. But since P is full-dimensional, these vectors form a set for which condition (9) of the iterative step is trivially satisfied as it becomes vacuous (since $R = S$), hence a set that can be used for another iteration of the reduction procedure, contrary to the assumption that the procedure has stopped with π. Thus $s = 1$ and the improper partition is valid with respect to $\alpha x \leq \alpha_0$. $\square$

Example 2. Consider the same polytope P and face F as in Example 1, but start with a different $x \in F$, say $x^1 = (0, 0, 1, 1, 1, 0, 0, 0, 0)$. Then 2-reduction is not applicable to the trivial partition π^1, since there is no vector $x^2 \in F_I$ to satisfy condition (9) for $r = 2$ (with x^1 in the role of x^t). The smallest r for which (9) can be satisfied is $r = 4$, with x^2, x^3, x^4 and the resulting π^2 shown below:

$$
\begin{array}{ll}
 & x^1 = (0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0)\ \pi^1 = \{\{1\}, \dots, \{9\}\} \\
 & x^2 = (0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0) \\
r = 4: & \\
 & x^3 = (0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0) \\
 & x^4 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0)\ \pi^2 = \{\{1\}, \{2\}, \{3\}, \{5\}, \{4, 6, 7, 8\}, \{9\}\}.
\end{array}
$$

At all the remaining iterations, 2-reductions are possible:

$$
\begin{array}{lll}
r = 2: & x^5 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1) & \pi^3 = \{\{1\}, \{2\}, \{3\}, \{5\}, \{4, 6, 7, 8, 9\}\} \\
r = 2: & x^6 = (0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1) & \pi^4 = \{\{1\}, \{2\}, \{5\}, \{3, 4, 6, 7, 8, 9\}\} \\
r = 2: & x^7 = (0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1) & \pi^5 = \{\{1\}, \{2\}, \{3, \dots, 9\}\} \\
r = 2: & x^8 = (1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1) & \pi^6 = \{\{2\}, \{1, \dots, 9\}\} \\
r = 2: & x^9 = (0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1) & \pi^7 = \{1, \dots, 9\}.
\end{array}
$$

4. The case of positive 0-1 polytopes

In all the results so far we have assumed that the valid inequality $\alpha x \leq \alpha_0$ defining F has no zero coefficients. While this is a genuine restriction for a general 0-1 polytope, it is no restriction at all for positive 0-1 polytopes, as we will presently show.

By a *positive 0-1 polytope* we mean the convex hull of 0-1 points x satisfying a system like $Ax \leq b$ or $Ax \geq b$, with $A \geq 0$ and $b > 0$. Let A be $m \times n$, and let

$$
\begin{aligned}
P^{<}(A, b) &:= \text{conv}\ \{x \in \{0, 1\}^n : Ax \leq b\}, \\
P^{>}(A, b) &:= \text{conv}\ \{x \in \{0, 1\}^n : Ax \geq b\}.
\end{aligned}
$$

Whenever it does not create confusion, we write $P^{<}$ and $P^{>}$ for $P^{<}(A, b)$ and $P^{>}(A, b)$, respectively.

It is not hard to see that $P^<$ is full dimensional if and only if $a_j \leq b$ for all $j \in N$, whereas $P^>$ is full dimensional if and only if $a_j \leq Ae - b$ for all $j \in N$. For the rest of this section we assume that $P^<$ and $P^>$ are full dimensional.

An inequality $\alpha x \leq \alpha_0$, valid for $P^<$, is *maximal* if there exists no $k \in N$ and $\alpha'_k > \alpha_k$, such that

$$\alpha'_k x_k + \sum(\alpha_j x_j : j \in N \setminus \{k\}) \leq \alpha_0$$

is valid for $P^<$. Similarly, an inequality $\alpha x \geq \alpha_0$, with $\alpha_0 > 0$, valid for $P^>$, is *minimal* if there exists no $k \in N$ and $\alpha''_k < \alpha_k$ such that

$$\alpha''_k x_k + \sum(\alpha_j x_j : j \in N \setminus \{k\}) \geq \alpha_0$$

is valid for $P^>$.

We denote by $F^< := \{x \in P^< : \alpha x = \alpha_0\}$ and $F^> := \{x \in P^> : \alpha x = \alpha_0\}$ the faces of $P^<$ and $P^>$ defined by $\alpha x \leq \alpha_0$ and $\alpha x \geq \alpha_0$, respectively. Further, we denote $F_I^< := F^< \cap \{0,1\}^n$ and $F_I^> := F^> \cap \{0,1\}^n$.

Proposition 11A. *A valid inequality $\alpha x \leq \alpha_0$ for $P^<$, with $\alpha_0 > 0$, is maximal if and only if for every $j \in N$ there exists $x \in F_I^<$ such that $x_j = 1$.*

Proposition 11B. *A valid inequality $\alpha x \geq \alpha_0$ for $P^>$, with $\alpha_0 > 0$, is minimal if and only if for every $j \in N$ there exists $x \in F_I^>$ such that $x_j = 1$.*

Corollary 12A. *If $\alpha x \leq \alpha_0$, with $\alpha_0 > 0$, is a maximal valid inequality for $P^<$, then $0 \leq \alpha_j \leq \alpha_0$ for all $j \in N$.*

The property stated in Corollary 12A does not have its exact analog for minimal valid inequalities for $P^>$. Thus for $P^>$ defined by the system

$$x_1 + x_2 + x_3 \geq 2, \quad x_1, x_2, x_3 = 0 \text{ or } 1,$$

the inequality

$$-x_1 + 2x_2 + 2x_3 \geq 1$$

is both valid and minimal, although $\alpha_1 = -1 < 0$ and $\alpha_2 = \alpha_3 = 2 > \alpha_0$. However, the following weaker property holds.

Corollary 12B. *If $\alpha x \geq \alpha_0$, with $\alpha_0 > 0$, is a minimal valid inequality for $P^>$ with $\alpha_j \geq 0$, $j \in N$, then $\alpha_j \leq \alpha_0$ for all $j \in N$.*

This situation suggests that we next examine the connection between $P^<(A,b)$ and the associated polytope

$$P^>(A,\tilde{b}) := \text{conv}\,\{y \in \{0,1\}^n \mid Ay \geq \tilde{b} := Ae - b\}$$

obtained by complementing the variables x_j, $j \in N$, in the definition of $P^<(A,b)$. We first note that if the inequality $\alpha x \leq \alpha_0$, valid for $P^<(A,b)$, is maximal, it does not follow that the inequality $\alpha y \geq \alpha e - \alpha_0$, obviously valid for $P^>(A,\tilde{b})$, is minimal. Indeed, $2x_1 + x_2 + x_3 \leq 3$ is a maximal valid inequality for the polytope $P^<$ defined by

$$x_1 + x_2 + x_3 \leq 2, \quad x_1, x_2, x_3 = 0 \text{ or } 1,$$

but the corresponding inequality $2y_1 + y_2 + y_3 \geq 1$ is not minimal for $P^>(A,\tilde{b})$, which is defined by

$$y_1 + y_2 + y_3 \geq 1, \quad y_1, y_2, y_3 = 0 \text{ or } 1.$$

These considerations prompt us to introduce the following stronger notion of maximality and minimality of inequalities. A valid inequality $\alpha x \leq \alpha_0$ for $P^<(A,b)$ will be called *strongly maximal* if it is maximal, and the inequality $\alpha y \geq \alpha e - \alpha_0$ is minimal for $P^>(A,\tilde{b})$. Similarly, a valid inequality $\alpha x \geq \alpha_0$ for $P^>(A,b)$ will be called *strongly minimal* if it is minimal, and the inequality $\alpha y \leq \alpha e - \alpha_0$ is maximal for $P^<(A,\tilde{b})$.

Proposition 13A. *A valid inequality $\alpha x \leq \alpha_0$ for $P^<$ is strongly maximal if and only if for every $j \in N$ there exists $x \in F_I^<$ such that $x_j = 1$ and $x' \in F_I^<$ such that $x'_j = 0$.*

Proof. Follows from applying the definition and setting $x'_j = 1 - y_j$, $j \in N$. □

Proposition 13B. *A valid inequality $\alpha x \geq \alpha_0$ for $P^>$ is strongly minimal if and only if for every $j \in N$ there exists $x \in F_I^>$ such that $x_j = 1$ and $x' \in F_I^>$ such that $x'_j = 0$.*

Proof. Same as for Proposition 13A. □

Corollary 14A. *If $\alpha x \leq \alpha_0$ is a strongly maximal valid inequality for $P^<$, then $0 \leq \alpha_j \leq \min\{\alpha_0, \alpha e - \alpha_0\}$ for all $j \in N$.*

Proof. If $\alpha x \leq \alpha_0$ is maximal for $P^<$ then from Corollary 12A, $0 \leq \alpha_j \leq \alpha_0$. If, in addition, $\alpha y \geq \alpha e - \alpha_0$ is minimal for $P^>(A,\tilde{b})$, then from Corollary 12B, $\alpha_j \leq \alpha e - \alpha_0$. □

Corollary 14B. *$\alpha x \geq \alpha_0$ is a strongly minimal valid inequality for $P^>$, then $0 \leq \alpha_j \leq \min\{\alpha_0, \alpha e - \alpha_0\}$ for all $j \in N$.*

Proof. If $\alpha y \leq \alpha e - \alpha_0$ is maximal for $P^<(A, \tilde{b})$, then $0 \leq \alpha_j \leq \alpha e - \alpha_0$ for all $j \in N$. If $\alpha x \geq \alpha_0$ is minimal for $P^>(A, b)$ with $\alpha_j \geq 0$, $j \in N$, then $\alpha_j \leq \alpha_0$ for all $j \in N$. $\square$

Proposition 15A. *The inequality $\alpha x \leq \alpha_0$ defines a facet of $P^<(A, b)$ if and only if $\alpha y \geq \alpha e - \alpha_0$ defines a facet of $P^>(A, \tilde{b})$.*

Proof. Follows from the fact that $x \in P^<(A, b)$ if and only if $y \in P^>(A, \tilde{b})$ for $y = e - x$, and a collection of points $x^i \in P^<(A, b)$, $i = 1, \ldots, t$ is affinely independent if and only if the collection of point $y^i := e - x^i$, $i = 1, \ldots, t$, is affinely independent. $\square$

Proposition 15B. *The inequality $\alpha x \geq \alpha_0$ defines a facet of $P^>(A, b)$ if and only if $\alpha y \leq \alpha e - \alpha_0$ defines a facet of $P^<(A, b)$.*

Proof. Same as for Proposition 15A. $\square$

It follows directly from Propositions 15A and 15B, that every nontrivial facet defining inequality for $P^<$ (for $P^>$) is strongly maximal (strongly minimal).

For $S \subseteq N$, we denote by A^S the submatrix of A consisting of the columns indexed by S.

Theorem 16A. *Let $\alpha x \leq \alpha_0$ be a maximal valid inequality for $P^<(A, b)$, and let $V := \{j \in N : \alpha_j \neq 0\}$. Then $\alpha x \leq \alpha_0$ defines a facet of $P^<(A, b)$ if and only if $\alpha^V x^V \leq \alpha_0$ defines a facet of $P^<(A^V, b)$.*

Proof. Sufficiency. Suppose $\alpha^V x^V \leq \alpha_0$ defines a facet of $P^<(A^V, b)$. Then there exists a set of $|V|$ linearly independent vectors $x^V_{(i)} \in P^<(A^V, b)$ such that $\alpha^V x^V_{(i)} = \alpha_0$, $i = 1, \ldots, |V|$. Extending each $x^V_{(i)}$ to a vector $x^i \in \mathbf{R}^n$ by setting $x^i_j = 0$, $j \in N \setminus V$ (and $x^i_j = x^V_{(i)j}$, $j \in V$) yields $|V|$ linearly independent vectors $x^i \in \mathbf{R}^n$ such that $\alpha x^i = \alpha_0$, $i = 1, \ldots, |V|$. Further, since $\alpha x \leq \alpha_0$ is maximal, for every $k \in N \setminus V$ there exists some $x \in P^<(A, b)$ such that $x_k = 1$, $x_j = 0$, $j \in N \setminus (V \cup \{k\})$, and $\alpha x = \alpha_0$. It is easy to see that the $n \times n$ matrix Z whose rows are these $|V| + |N \setminus V|$ vectors x^i is of the form

$$X = \begin{pmatrix} X_1 & 0 \\ X_2 & I \end{pmatrix}$$

with X_1 nonsingular (of order $|V|$) and I the identity of order $|N \setminus V|$.

Thus the rows of X are linearly independent and $\alpha x \le \alpha_0$ defines a facet of $P^<(A, b)$.

Necessity. Suppose $\alpha^V x^V \le \alpha_0$ does not define a facet of $P^<(A^V, b)$. Then there exists $\beta^V \in \mathbf{R}_+^{|V|}$, $\beta_0 \in \mathbf{R}_+$, such that every $x^V \in P^<(A^V, b)$ satisfies $\beta^V x^V \le \beta_0$, every $x^V \in P^<(A^V, b)$ such that $\alpha^V x^V = \alpha_0$ satisfies $\beta^V x^V = \beta_0$, and $(\beta^V, \beta_0) \ne \lambda(\alpha^V, \alpha_0)$ for all $\lambda \ne 0$. But then setting $\beta = (\beta^V, 0) \in \mathbf{R}^n$, we obtain an inequality $\beta x \le \beta_0$ satisfied by all $x \in P^<(A, b)$ and satisfied with equality by all x such that $\alpha x = \alpha_0$, with $(\beta, \beta_0) \ne \lambda(\alpha, \alpha_0)$ for all $\lambda \ne 0$. Thus $\alpha x \le \alpha_0$ does not define a facet of $P^<(A, b)$.

Theorem 16B. *Let $\alpha x \ge \alpha_0$ be a minimal valid inequality for $P^>(A, b)$. Then $\alpha x \ge \alpha_0$ defines a facet of $P^>(A, b)$ if and only if $\alpha^V x^V \ge \alpha_0$ defines a facet of $P^>(A^V, b - A^{N\setminus V} e^{N\setminus V})$.*

Proof. From Proposition 15B, $\alpha x \ge \alpha_0$ defines a facet of $P^>(A, b)$ if and only if $\alpha y \le \alpha e - \alpha_0$ defines a facet of $P^<(A, \tilde{b})$; and $\alpha^V x^V \ge \alpha_0$ defines a facet of $P^>(A^V, b' := b - A^{N\setminus V} e^{N\setminus V})$ if and only if $\alpha^V y^V \le \alpha^V e^V - \alpha_0$ defines a facet $P^<(A^V, \tilde{b}')$, where $\tilde{b}' := A^V e^V - b' = Ae - b = \tilde{b}$. Thus all we have to show is that $\alpha y \le \alpha e - \alpha_0$ defines a facet of $P^<(A, \tilde{b})$ if and only if $\alpha^V y^V \le \alpha^V e^V - \alpha_0$ defines a facet of $P^<(A^V, \tilde{b})$. For the case when $\alpha y \le \alpha e - \alpha_0$ is maximal, this was established in Proposition 16A. If $\alpha y \le \alpha e - \alpha_0$ is not maximal, then (since $\alpha x \ge \alpha_0$ is minimal) $\alpha^V y^V \le \alpha^V e^V - \alpha_0$ is also not maximal, and so neither of them is facet defining. $\square$

In view of Theorems 16A and 16B, in examining the conditions under which a valid inequality is facet defining for $P^<$ or for $P^>$ we may restrict ourselves to inequalities with strictly positive coefficients. Thus the results of sections 2 and 3 are of general validity for $P^<$ and $P^>$ (i.e. not affected by the restriction to inequalities with nonzero coefficients).

5. Application to vertex packing

Let $P^<(G) := P^<(A, e)$ be the *vertex packing polytope* defined on the (undirected) graph $G = (V, E)$ whose edge-vertex incidence matrix is A. In other words, $P^<(G)$ is the convex hull of incidence vectors of vertex packings (independent sets, stable sets) of G. For $S, T \subseteq V$, we denote by (S, T) the set of edges (i, j) such that $i \in S$, $j \in T$, and we write $\alpha(G)$ for the cardinality of a maximum vertex packing (i.e. the independence or stability number) of G. A natural question to ask in connection with

$\alpha(G)$ is under what conditions does the (obviously valid) inequality

$$\sum(x_j : j \in V) \leq \alpha(G) \tag{13}$$

define a facet of $P^{<}(G)$? Chvátal (1975) gave the following well known partial answer to this question. Call an edge u of G *α-critical* if $\alpha(G - u) = \alpha(G) + 1$, and let E^* be the set of α-critical edges of G. Also, let $G^* := (V, E^*)$.

Theorem 17 (Chvátal 1975). *If G^* is connected, then the inequality* (13) *defines a facet of $P^{<}(G)$.*

This sufficient condition can be weakened by using the concept of an *α-critical cutset*, defined as a cutset $(W, V \setminus W)$ such that $\alpha(G(W)) + \alpha(G(V \setminus W)) \geq \alpha(G) + 1$, where $G(W)$ is the subgraph of G induced by W.

Theorem 18 (Balas and Zemel, 1977). *If G has an α-critical cutset $(W, V \setminus W)$ such that for $T := W$ and $T := V \setminus W$*

(i) *the inequality $\sum(x_j : j \in T) \leq \alpha(G(T))$ defines a facet of $G(T)$; and*

(ii) *every maximum packing of $G(T)$ is contained a maximum packing of G;*

then the inequality (13) *defines a facet of $P^{<}(G)$.*

The sufficient condition of Theorem 18 is not necessary for (13) to be facet defining, as the example below shows.

Using the results of Section 2, we will now give a sufficient condition for the inequality (13) to define a facet of $P^{<}(G)$, which is a direct generalization of Chvátal's above mentioned result, and is weaker than the condition of Theorem 18. The key to this generalization is the following.

Remark. An edge $u = (i, j)$ is α-critical if and only if G has two maximum vertex packings S and T, such that $T = (S \setminus \{i\}) \cup \{j\}$.

Proof. If the condition holds, then $S \cup \{j\}$ $(= T \cup \{i\})$ is a vertex packing of $G - u$, hence u is α-critical. Conversely, if u is α-critical, then $G - u$ has a vertex packing Y such that $|Y| = \alpha(G) + 1$ and $i, j \in Y$. Then $S := Y \setminus \{j\}$ and $T := Y \setminus \{i\}$ are both maximum vertex packings in G, with $T = (S \setminus \{i\}) \cup \{j\}$. $\square$

It is now not hard to see that the presence of an α-critical edge (i, j) implies the existence of a pair $x, y \in F$ satisfying $x_i = 1 - y_i = y_j = 1 - x_j$, and $x_k = y_k$, $k \in V \setminus \{i,j\}$ which is equivalent to condition (3); hence 2-reduction generalizes Chvátal's procedure of constructing the α-critical graph G^* from the case where the subsets N_k of π are singletons, to the more general case where π is a nontrivial partition.

To be specific, let $G^\# := (V, E^\#)$ be the graph constructed as follows:

Initialization. Let S^1 be a maximum vertex packing of G, let $G^1 := (V, E^1)$, $E^1 = \emptyset$, and $t := 1$.

Iterative Step. Given $G^t = (V, E^t)$ with connected components induced by $V_1, \dots, V_s \subseteq V$, find a maximum vertex packing S^{t+1} of G such that

$$|S^{t+1} \cap V_k| \begin{cases} \neq |S^t \cap V_k| & k = i, j, \\ = |S^t \cap V_k| & k \in \{1, \dots, s\} \setminus \{i,j\}, \end{cases} \tag{14}$$

for some pair $i, j \in \{1, \dots, s\}$. if no such S^{t+1} exists, let $G^\# := G^t$ and stop. Otherwise insert into G^t an edge joining the two components V_i and V_j, i.e. set $E^{t+1} := E^t \cup (u,v)$ for some $u \in V_i$, $v \in V_j$, $G^{t+1} := (V, E^{t+1})$. let $t := t + 1$ and repeat.

Theorem 19. *If $G^\#$ is connected, then $\sum(x_j : j \in V) \leq \alpha(G)$ defines a facet of $P^<(G)$.*

Proof. The above procedure amounts to 2-reduction, and $G^\#$ is connected if and only if the procedure stops with the improper partition. Hence the theorem is a specialization to the vertex packing polytope of Corollary 4. □

Example 3. Consider the graph G of Figure 1, obtained from the complete 3-partite graph with 5 vertices in each part, $K_{5,5,5}$, by inserting into each of the three parts the edges of a 5-cycle, and deleting the edge sets $(V_1, \{6\})$, $(V_2, \{11\})$, $(V_3, \{1\})$. A maximum vertex packing of G contains two vertices of V_i and one V_{i+1}, for some $i \in \{1,2,3\}$, or else just one vertex of each V_i, namely 1, 6 and 11. So the inequality

$$\sum(x_j : j = 1, \dots, 15) \leq 3$$

is certainly valid. We want to know whether it defines a facet of $P^<(G)$.

Chvátal's sufficient condition is not satisfied, since the only critical edges are those of the three 5-cycles and thus $G^\#$ is disconnected. Balas and Zemel's sufficient condition is also not satisfied, since G has no critical cutset satisfying (i): although if W stands for the vertex set of a triangle or a pentagon of G the inequality $\sum(x_j : j \in W) \leq \alpha(G(W))$ defines a facet of $P^{<}(G(W))$, in each of these cases the inequality $\sum(x_j : j \in V \setminus W) \leq \alpha(G(V \setminus W))$ is not facet defining for $P^{<}(G(V \setminus W))$.

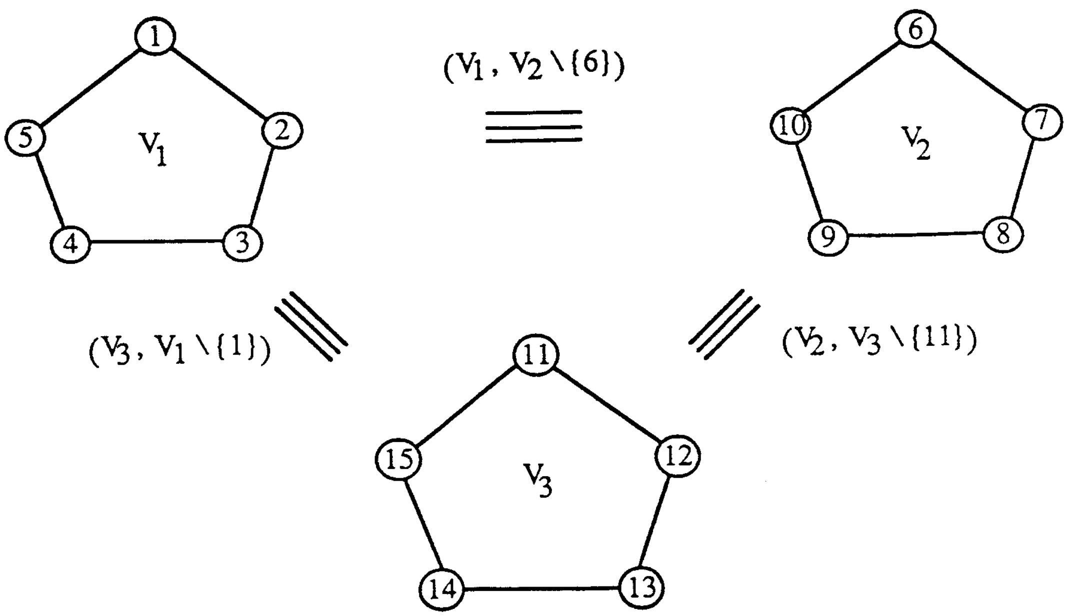

Figure 1

We now apply the procedure that generates $G^{\#}$. The first four iterations yield the component with vertex set $\{1, \ldots, 5\}$ as a path of length 4. The next eight iterations yield two more components (paths of length 4), with vertex sets $\{6, \ldots, 10\}$ and $\{11, \ldots, 15\}$. Thus G^{13} has three components with vertex sets V_1, V_2 and V_3.

Next consider the maximum vertex packing $S^{13} := \{6, 8, 11\}$ and $S^{14} := \{1, 6, 11\}$. We have

$$\begin{aligned} 0 = &|S^{13} \cap V_1| \neq |S^{14} \cap V_1| = 1, \\ 2 = &|S^{13} \cap V_2| \neq |S^{14} \cap V_2| = 1, \\ &|S^{13} \cap V_3| = |S^{14} \cap V_3| = 1, \end{aligned}$$

hence condition (14) is satisfied and we add to G^{13} an edge joining a vertex of V_1 to one of V_2 in order to obtain G^{14}. Let the two components of G^{14} have vertex sets $V_1' = \{1, \ldots, 10\}$ and $V_2' = \{11, \ldots, 15\}$, and consider the maximum vertex packings $S^{14} = \{1, 6, 11\}$ and $S^{15} = \{11, 13, 1\}$. Then

$$2 = |S^{14} \cap V_1'| \neq |S^{15} \cap V_1'| = 1,$$

and

$$1 = |S^{14} \cap V_2'| \neq S^{15} \cap V_2'| = 2,$$

hence $G^{15} = G^{\#}$ is obtained from G^{14} by adding an edge joining a vertex of V_1' to one of V_2'. Since $G^{\#}$ is connected, the inequality $\sum(x_j : j \in V) \leq 3$ defines a facet of $P^{<}(G)$. □

It is perhaps worth mentioning that the Chvátal rank of the inequality of Example 3 is quite high: the rank 1 Chvátal inequality with the same lefthand side has a righthand side of 7 instead of 3.

6. Application to set covering

Let A be an $m \times n$ 0-1 matrix and $b = e$. Then $P^{>}(A, e)$ is the set *covering polytope*, i.e. the convex hull of 0-1 solutions to $Ax \geq e$. A set $S \subseteq N$ is called a cover for A if $\sum(a_j : j \in S) \geq e$. The incidence vector of a cover will also be called a cover. If $\beta(A)$ denotes the minimum cardinality of a cover for A, it is of interest to know when the inequality

$$\sum(x_j : j \in N) \geq \beta(A), \tag{15}$$

obviously valid, defines a facet of $P^{>}(A, e)$. A sufficient condition for this, analogous to Chvátal's condition for the vertex packing polytope, can be stated in the following terms. For $i, j \in N$, $i \neq j$, denote by A^{ij} the matrix obtained from A by removing all rows $k \in M$ such that $a_{ki} = a_{kj} = 1$. Call a pair $i, j \in N$ *β-critical* if $\beta(A^{ij}) = \beta(A) - 1$, and let the graph $G^* = (V, E^*)$ have vertex for every $i \in N$ and an edge for every β-critical pair $i, j \in N$.

Theorem 20 (Sassano 1985). *If G^* is connected, the inequality* (15) *defines a facet of $P^{>}(A, e)$.*

Again, the sufficient condition of our Corollary 4 weakens the condition of Theorem 20. To see this, notice the following:

Remark. A pair $i, j \in N$ is β-critical if and only if there exist minimum covers S and T for A such that $T = (S \setminus \{i\}) \cup \{j\}$.

Proof. If such covers exist, then $S \setminus \{i\}$ $(= T \setminus \{j\})$ is a cover for A^{ij}, hence the pair i, j is β-critical. Conversely, if the pair i, j is β-critical, then A^{ij}

has a minimum cover Y such that $|Y| = \beta(A) - 1$ and $Y \cap \{i,j\} = \emptyset$. Then $S := Y \cup \{i\}$ and $T := Y \cup \{j\}$ are minimum covers for A and $T = (S \setminus \{i\}) \cup \{j\}$. □

As in the case of the set packing polytope, we now generate a graph $G^{\#}$ by the following procedure.

Initialization. Let S^1 be a minimum cover for $P^{>}$, let $G^1 = (V, E^1)$, with $V := N$ and $E^1 := \emptyset$, and $t := 1$.

Iterative Step. Given G^t with connected components induced by $V_1, \ldots, V_p \subseteq V$, find a minimum cover S^{t+1} such that

$$(16) \qquad |S^{t+1} \cap V_k| \begin{cases} \neq |S^t \cap V_k| & \text{for } k = i, j, \\ = |S^t \cap V_k| & \text{for } k \in \{1, \ldots, p\} \setminus \{i, j\}, \end{cases}$$

for some pair $i, j \in \{1, \ldots, s\}$. If no such S^{t+1} exists, let $G^{\#} = G^t$ and stop. Otherwise insert into G^t an edge joining the two components V_i and V_j, i.e. set $E^{t+1} := E^t \cup (u, v)$ for some $u \in V_1$ and $v \in V_j$, $G^{t+1} := (V, E^{t+1})$, $t := t + 1$, and repeat.

Theorem 21. *If $G^{\#}$ is connected, then the inequality* (16) *defines a facet of $P^{>}(A, e)$.*

Clearly, Theorem 21 is a specialization of Corollary 4. On the other hand, the condition of Theorem 20 asserts that (16) is facet defining if the graph $G^{\#}$ obtained by restricting the Iterative Step of our procedure to pairs $i, j \in \{1, \ldots, p\}$ such that V_i and V_j are *singletons*, is connected. Hence the sufficient condition of Theorem 21 is a weakening of the one in Theorem 20.

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