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ON THE COMASS NORM OF A 3-COVECTOR

DOAN THE HIEU

1. Introdution

A form Φ on a Riemannian manifold which is closed and has comass one is called give in a *calibration*. An oriented submanifold to which Φ restricts to be exactly the volume form is called a Φ-submanifold. The fundamental lemma of the calibration theory says that each Φ-submanifold is homologically area minimizing.

The simplest but very important case is the case of constant coefficient (paralell) calibrations (i.e. covectors have comass one) on n dimensional euclidean vector space $\mathbf{E}^n$. This case is quite difficult and far from being understood. It serves as a model for local behavior of the general calibrated manifold (see [DHM]). Moreover, the investigation of biinvariant calibrations on symmetric spaces (see [M]) naturally leads to paralell calibrations on $\mathbf{E}^n$. And finally, many examples of calibrations that give rise to rich geometric structures of minimal manifolds (e.g. complex, special Lagrangian, associative, and Cayley) have constant coefficients (see [HL]).

Firstly, in order to use a k-covector Φ as a constant coefficient calibration, one needs to know the commas of Φ:

$$\|\Phi\|^* := \sup\{\Phi(\xi) | \xi \in G(k, \mathbf{E}^n)\},$$

where $G(k, \mathbf{E}^n) \subset \bigwedge^k \mathbf{E}^n$ is the Grassmannian of oriented unit k-plane in $\mathbf{E}^n$. But the computation of the comass of Φ is a primary obstacle to the use of Φ as a calibration. Secondly, one needs to know which planes ξ are calibrated by Φ, i.e., what is the face $G(\Phi)$ of the Grassmannian corresponding to Φ

$$G(\Phi) = \{\xi \in G(k, \mathbf{E}^n) \ / \ \Phi(\xi) = \|\Phi\|^*\}$$

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The last question is which k-submanifolds in $\mathbf{E}^n$ are Φ-submanifolds; i.e., the k-submanifolds whose tangent planes are calibrated by Φ (which are automatically homologically area minimizing). It is quite, often that, they are unions of k-planes, and even this case is of interest. Several beautiful examples of constant coefficient calibrations were given in [HL] and [H].

The purpose of this paper is to give another point of view on the comass of a covector in the simplest case (3-covector) and we use it to compute the comass of some classes of 3-covectors. In examples 3.4 and 3.5 we compute the comass of the fundamental 3-form τ on $so(4)$ and we define the contact set $G(\tau)$. Example 3.6 gives a contruction of a calibrated algebra structure on $\mathbf{R}^{4n-1}$ and hence a calibration of degree 3 on it. This is a generalization of the well known *associated calibration* on $\mathbf{R}^7$.

2. Calibrated algebra

Definition 2.1. Let $\mathbf{E}^n$ be an euclidean vector space of dimension n with inner product $<,>$. Suppose that there exists a mapping $\overline{\Phi}$ from $\mathbf{E}^n \times \mathbf{E}^n$ into $\mathbf{E}^n$ which satifies the following conditions:

(1) Φ is bilinear,

(2) Φ is antisymetry, i.e.

$$\overline{\Phi}(x, x) = 0 \quad \text{for all} \quad x \in \mathbf{E}^n,$$

(3) $\overline{\Phi}(x, y)$ is orthogonal to x and y for all $x, y \in \mathbf{E}^n$.

Then $(\mathbf{E}^n, \overline{\Phi})$ is called a calibrated algebra.

It is easy to prove that the equality $\overline{\Phi}(x, y) = -\overline{\Phi}(y, x)$ holds for all $x, y \in \mathbf{E}^n$, and the trilinear form $\langle x, \overline{\Phi}(y, z)\rangle$ is alternating, for all $x, y, z \in \mathbf{E}^n$. Thus each calibrated algebra defines a 3-covector on it, defined by

$$\Phi(x, y, z) = \langle x, \overline{\Phi}(y, z)\rangle.$$

Conversely, each 3-covector Φ on an Euclidean vector space $\mathbf{E}^n$ defines a calibrated algebra structure on it. Indeed, a 3-covector Φ is considered as an alternating and trilinear mapping from $\mathbf{E}^n \times \mathbf{E}^n \times \mathbf{E}^n$ to $\mathbf{R}$. For each $x \in \mathbf{E}^n$, let Φ_x be the mapping from $\mathbf{E}^n \times \mathbf{E}^n$ to $\mathbf{R}$ defined by

$$\Phi_x(y, z) = \Phi(x, y, z).$$

It is obvious that each Φ_x is bilinear, alternating and the mapping $x \longmapsto \Phi_x$ is linear, i.e.,

$$\begin{aligned} \Phi_{x+x'} &= \Phi_x + \Phi'_x, \\ \Phi_{\lambda x} &= \lambda \Phi_x, \quad \text{for any } \lambda \in \mathbf{R}. \end{aligned}$$

Suppose that $\{e_i\}_{i=1,2,\dots,n}$ is an orthonomal basis on $\mathbf{E}^n$, and $x = \sum\limits_i x_i e_i$. We have

$$\begin{aligned} \Phi(x, y, z) &= \Phi(\sum_i x_i e_i, y, z) \\ &= \sum_i x_i \Phi(e_i, y, z) \\ &= \sum_i x_i \Phi_{e_i}(y, z) \\ &= \langle x, \overline{\Phi}(y, z) \rangle, \end{aligned}$$

where

$$\overline{\Phi}(y, z) = \sum_i \Phi_{e_i}(y, z) e_i,$$

Obviously, $\overline{\Phi}$ is a bilinear mapping from $\mathbf{E}^n \times \mathbf{E}^n$ to $\mathbf{E}^n$ which is alternating and satifies the condition (3). Hence, $(\mathbf{E}^n, \overline{\Phi})$ is a calibrated algebra and is then called calibrated algebra assocated with Φ.

The norm $\|\overline{\Phi}\|$ of $\overline{\Phi}$ is defined by

$$\|\overline{\Phi}\| = \sup_{\|x\|=\|y\|=1} \|\overline{\Phi}(x, y)\|,$$

and the contact set $G(\overline{\Phi})$ of $\overline{\Phi}$ is defined by

$$\begin{aligned} G(\overline{\Phi}) &= \{(x, y) \mid \|x\| = \|y\| = 1; \|\overline{\Phi}(x, y)\| = \|\overline{\Phi}\|\} \\ &= \{\eta \in G(2, \mathbf{E}^n) \mid \|\overline{\Phi}(\eta)\| = \|\overline{\Phi}\|\}. \end{aligned}$$

We have the following lemma.

Lemma 2.2.

1. $\|\Phi\|^* = \|\overline{\Phi}\|$,
2. $G(\Phi) = \{\overline{\Phi}(\eta) \wedge \eta \mid \eta \in G(\overline{\Phi}\}$.

Proof. Let $\xi = x \wedge y \wedge z$ be a unit simple 3-vector. We can assume that x, y, z are orthonormal vectors. Then we have

$$\begin{aligned}\Phi(x,y,z) = &\ \langle x, \overline{\Phi}(y,z)\rangle \\ &\leq \|x\|.\|\overline{\Phi}(y,z)\| \leq \|\overline{\Phi}\|.\end{aligned}$$

Thus,

$$\|\Phi\|^* \leq \|\overline{\Phi}\|.$$

Now suppose that $\|\overline{\Phi}(y_0,z_0)\| = \|\overline{\Phi}\|$. Let $x_0 = \dfrac{\overline{\Phi}(y_0,z_0)}{\|\overline{\Phi}(y_0,z_0)\|}$, then we have

$$\begin{aligned}\Phi(x_0,y_0,z_0) = &\ \langle x_0, \overline{\Phi}(y_0,z_0)\rangle \\ = &\ \|x_0\|.\|\overline{\Phi}(y_0,z_0)\| \\ = &\ \|\overline{\Phi}(y_0,z_0)\| \\ = &\ \|\overline{\Phi}\|.\end{aligned}$$

Thus,

$$\|\Phi\|^* = \|\overline{\Phi}\|.$$

The proof of the second part is clear.

Definition 2.3. Let $(\mathbf{E}^n, \overline{\Phi})$ is a calibrated algebra associated with 3-covector Φ. The subspace $M \subset \mathbf{E}^n$ is called an invariant subspace of $\overline{\Phi}$ (or $\overline{\Phi}$ - invariant) if and only if $\overline{\Phi}(x,y) \in M$, for all $x, y \in M$.

Lemma 2.4. *Suppose that $\xi \in G(\Phi)$, then span(ξ) is 3-dimensionnal invariant subspace of $\overline{\Phi}$.*

Proof. Let $\{x,y,z\}$ be the orthonormal basis of span(ξ). The inequality

$$\begin{aligned}\Phi(x,y,z) &= \langle x, \overline{\Phi}(y,z)\rangle \\ &\leq \|x\|.\|\overline{\Phi}(y,z)\| = \|\overline{\Phi}(y,z)\|,\end{aligned}$$

implies that x and $\overline{\Phi}(y,z)$ are linearly dependent vectors, hence

$$\overline{\Phi}(y,z) = ax, \quad a \in \mathbf{R}.$$

Analogously,

$$\overline{\Phi}(z,x) = ay,$$

$$\overline{\Phi}(x,y) = az,$$

where

$$a = \|\Phi\|^*.$$

The proof of the lemma is completed.

It is easy to prove the following lemma

Lemma 2.5. *Let ξ be a unit simple 3-covector and span(ξ) is an invariant 3-dimension subspace of $\overline{\Phi}$. If $\{x, y, z\}$ is an orthonormal basis of span(ξ), then we have*

$$\overline{\Phi}(x, y) = az,$$
$$\overline{\Phi}(y, z) = ax,$$
$$\overline{\Phi}(z, x) = ay.$$

The number a is then called an eigenvalue of $\overline{\Phi}$ corresponding to the invariant subspace span(ξ).

By virtue of above two lemmas, we can define the comass of Φ as follows:

$$\|\Phi\|^* = \sup\{\Phi(\xi) \mid \text{span}(\xi) \text{ is a } \overline{\Phi} - \text{ invariant subspace }\},$$

and

$$\|\Phi\|^* = \sup\{a | a \text{ is an eigenvalue of } \overline{\Phi}\}.$$

3. Examples

We can now construct an algorithm for computing the comass of 3-covectors on low dimensional Euclidean vector spaces.

Suppose $\mathbf{E}^n$ is a calibrated algebra associated with 3-covector Φ. For each $x \in \mathbf{E}^n$, let $\overline{\Phi}_x$ be the mapping from $\mathbf{E}^n$ to $\mathbf{E}^n$ defined by

$$\overline{\Phi}_x(y) = \overline{\Phi}(x, y).$$

It is easy to see that $\overline{\Phi}_x$ is linear and has the following properties:

1. $\overline{\Phi}_x(x) = 0$,
2. $\overline{\Phi}_x(y)$ is orthogonal to x and y.

Thus $\overline{\Phi}_x$ can be considered as a linear mapping from $x^\perp$ to $x^\perp$ where $x^\perp$ denotes the subspace of $\mathbf{E}^n$ containing all vectors orthogonal to x.

Since $\overline{\Phi}_x(y) \perp y$, $\overline{\Phi}_x$ has no eigenvector belonging to eigenvalue $\lambda \neq 0$, for all $x \in \mathbf{E}^n$. Thus, if $\overline{\Phi}_x \neq 0$, then $\overline{\Phi}_x$ has the invariant subspace of 2-dimension $M = \operatorname{span}(y, z)$ (y, z is the orthonormal basis of M) belonging to eigenvalue $a + ib$ and

$$\overline{\Phi}_x(y) = ay - bz,$$

$$\overline{\Phi}_x(z) = az + by.$$

Since $\overline{\Phi}_x(y) \perp y$ and $\overline{\Phi}_x(z) \perp z$, it implies that $a = 0$, and we have

Lemma 3.1. *The roots of characteristic polynomial of* $\overline{\Phi}_x$ *are* $\lambda = 0$ *(if* λ *is real) or pure imaginary* $\lambda = ib$.

By virtue of Lemma 3.1, we have

$$\det(\overline{\Phi}_x - \lambda I) = \lambda^k(\lambda^2 + f_1(x)) \dots (\lambda^2 + f_m(x)),$$

where $f_j(x) > 0$ and $\pm i\sqrt{f_j(x)}$ $(j = 1, 2, \dots, m)$ are pure imaginary roots of characteristic polynomial of $\overline{\Phi}_x$.

Let $A = \max_i \{ \sup_{\|x\|=1} f_i(x)\}$, and $\mathcal{A}$ is the set of all 3-vector $\xi = x \wedge y \wedge z$ (x, y, z are orthonormal vectors). There exists an index j, such that the equality $f_j(x_i) = A$ holds and $\operatorname{span}(y_i \wedge z_i)$ is a 2-dimensional invariant subspace of $\overline{\Phi}_{x_i}$. The following lemma gives a relationship between $\|\Phi\|^*$ and A and also one between $G(\Phi)$ and $\mathcal{A}$.

Lemma 3.2.

1. $\|\Phi\|^* = \sqrt{A}$,
2. $G(\Phi) = \mathcal{A}$.

Proof. 1. Suppose that $\Phi(x, y, z) = \|\Phi\|^*$, then $\operatorname{span}(y \wedge z)$ is an invariant subspace of $\overline{\Phi}_x$ and

$$\overline{\Phi}_x(y) = \|\Phi\|^* z,$$
$$\overline{\Phi}_x(z) = \|\Phi\|^* y,$$

and hence,

$$\|\Phi\|^* \leq \sqrt{A}.$$

Conversely, assume that there exists index j, such that $f_j(x_i) = \sqrt{A}$, and

$$\overline{\Phi}_x(y) = \sqrt{A} z,$$

$$\overline{\Phi}_x(z) = \sqrt{A}y.$$

It implies

$$\Phi(x, y, z) = \langle \overline{\Phi}_x(y), z \rangle = \sqrt{A}.$$

Thus,

$$\|\Phi\|^* = \sqrt{A}.$$

2- The proof of the second part is clear.

By using Lemma 3.2 we can compute the comass of some 3-covectors in low dimensional euclidean vector spaces.

Example 3.3

Let $\{e_1, e_2, \ldots, e_6\}$ be an orthonormal basis of $\mathbf{R}^6$, and $\{w_1, w_2, \ldots, w_6\}$ be the dual basis of $\{e_1, e_2, \ldots, e_6\}$. We shall use the notation w_{pqr} for $w_p \wedge w_q \wedge w_r$. Consider the 3-covector

$$\Phi = w_{123} + w_{456}.$$

A direct computation shows that

$$\overline{\Phi} = (\Phi_{e_1}, \Phi_{e_2}, \ldots, \Phi_{e_6}),$$

where

$$\Phi_{e_1} = w_{23}, \qquad \Phi_{e_2} = -w_{13}, \qquad \Phi_{e_3} = w_{12},$$
$$\Phi_{e_4} = w_{56}, \qquad \Phi_{e_5} = -w_{46}, \qquad \Phi_{e_6} = w_{45}.$$

Let $x = (a_1, a_2, \ldots, a_6) \in \mathbf{R}^6$ be an arbitrary point of $\mathbf{R}^6$. It is easy to see that the matrix of $\overline{\Phi}_x$ corresponding to $\{e_1, e_2, \ldots, e_6\}$ is

$$\begin{pmatrix} 0 & a_3 & -a_2 & 0 & 0 & 0 \\ -a_3 & 0 & a_1 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & -a_5 \\ 0 & 0 & 0 & -a_6 & 0 & a_4 \\ 0 & 0 & 0 & a_5 & -a_4 & 0 \end{pmatrix}$$

and

$$\det(\overline{\Phi}_x - \lambda I) = \lambda^2(\lambda^2 + f_1(x))(\lambda^2 + f_2(x)),$$

where

$$f_1(x) = a_1^2 + a_2^2 + a_3^2,$$
$$f_2(x) = a_4^2 + a_5^2 + a_6^2.$$

Thus,

$$\|\Phi\|^* = \sqrt{A} = \max\{\sup f_1(x), \sup f_2(x)\} = 1,$$

and

$$G(\Phi) = \{e_1 \wedge e_2 \wedge e_3 \ ; \quad e_4 \wedge e_5 \wedge e_6\}.$$

Example 3.4. Let $so(n)$ be the Lie algebra of all skew symmetric $(n \times n)$ matrices with the product

$$[X, Y] = XY - YX.$$

The inner product on $so(n)$ is defined by

$$\langle X, Y\rangle = \mathrm{tr} XY^t = -\mathrm{tr} XY.$$

The fundamental 3-form τ on $so(n)$ is defined by

$$\tau(X, Y, Z) = \frac{1}{2}\langle X, [Y, Z]\rangle = \langle X, YZ\rangle = -\mathrm{tr} XYZ.$$

By using Lemma 3.2, we compute the comass of τ in the simplest case, when $n = 4$, i.e. $so(4) \cong \mathbf{R}^6$. The computation of τ in the general case (on $so(n)$), will be given later.

An orthonormal basis of $so(4)$ is provided by the matrices $E_{ij} \ \ (i < j)$ with $\dfrac{1}{\sqrt{2}}$ in (i, j)-position and $-\dfrac{1}{\sqrt{2}}$ in (j, i)-position and zeros elsewhere. Let X be arbitrary matrix in $so(4)$ and $\|X\| = 1$

$$X = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix},$$

$$X = \sqrt{2}(a_1E_{12} + a_2E_{13} + a_3E_{14} + a_4E_{23} + a_5E_{24} + a_6E_{34}).$$

A computation shows that, the matrix of $\overline{\tau}_X$ corresponding to the basis $\{E_{ij}\}_{i<j}$ is

$$\frac{1}{2}\begin{pmatrix} 0 & -a_4 & -a_5 & a_2 & a_3 & 0 \\ a_4 & 0 & -a_6 & -a_1 & 0 & a_3 \\ a_5 & a_6 & 0 & 0 & -a_1 & -a_2 \\ -a_2 & a_1 & 0 & 0 & -a_6 & a_5 \\ -a_3 & 0 & a_1 & a_6 & 0 & -a_4 \\ 0 & -a_3 & a_2 & -a_5 & a_4 & 0 \end{pmatrix},$$

and

$$\det(\overline{\tau}_X - \lambda I) = \lambda^2(\lambda^2 + f_1(X))(\lambda^2 + f_2(X)),$$

where

$$\begin{aligned} f_1(X) &= \frac{1}{4}\sum_i a_i^2 + \frac{1}{2}(a_1a_6 + a_3a_4 - a_2a_5) \\ &\leq \frac{1}{2}\sum_i a_i^2 = \frac{1}{4}. \end{aligned}$$

The equality holds if and only if X has the form

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix}.$$

Also

$$\begin{aligned} f_2(X) &= \frac{1}{4}\sum_i a_i^2 - \frac{1}{2}(a_1a_6 + a_3a_4 - a_2a_5) \\ &\leq \frac{1}{2}\sum_i a_i^2 = \frac{1}{4}, \end{aligned}$$

and the equality holds if and only if X has the form

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{pmatrix}.$$

Thus,

$$\|\tau\|^* = \frac{1}{2},$$

and

$$G(\tau) = \{L, R\},$$

where

$$L = \operatorname{span}(L_1, L_2, L_3) \quad ; \quad R = \operatorname{span}(R_1, R_2, R_3),$$

$$L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad R_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$L_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$L_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad R_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, by using lemma 2.3, we shall compute the comass of τ on $so(n)$ (example 3.5) and construct multi-associative calibrations.

Example 3.5

Lemma 3.5.1. *For all* $A \in \mathrm{Mat}(n, \mathbf{R})$, *we have the inequality*

$$\|A - A^t\| \leq 4\|A\|^2 - 4\sum_i a_{ii}^2,$$

where $A = (a_{ij})$ *and* A^t *is the transpose of* A.

Proof.

$$\begin{aligned} \|A - A^t\|^2 &= \sum_{i \neq j}(a_{ij} - a_{ji})^2 \\ &= 2\sum_{i \neq j} a_{ij}^2 - 2\sum_{i \neq j} a_{ij}a_{ji} \\ &\leq 4\sum_{i \neq j} a_{ij}^2 \\ &= 4\|A\|^2 - 4\sum_{i \neq j} a_{ii}^2. \end{aligned}$$

Lemma 3.5.2. *For all* $A, B \in so(n)$, $\|A\| = \|B\| = 1$, *we have*

$$\|[A, B]\| \leq 1.$$

Proof. First, we note that the group $O(n)$ acts on $so(n)$ on both sides, preserving the inner product. Moreover, for all $T \in O(n)$, for all $A, B \in so(n)$, we have

$$[Ad_T A, Ad_T B] = [TAT^{-1}, TBT^{-1}] = T[A, B]T^{-1},$$

and hence τ is a $O(n)$ - biinvariant.

Because of this, in order to prove the Lemma 3.3.2 we can assume that A is in the canonical form

$$A = \begin{pmatrix} 0 & a_1 & & & & & & & & \\ -a_1 & 0 & & & & & & & & \\ & & 0 & a_2 & & & O & & & \\ & & -a_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & a_k & & & \\ & & & & & -a_k & 0 & & & \\ & & & & & & & 0 & & \\ & & O & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix} \quad ; \quad (k \le n)$$

For simplicity, we denote $B = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_n \end{pmatrix}$, where $\vec{b}_i$ are vector rows of B.

We have

$$A.B = \begin{pmatrix} a_1\vec{b}_2 \\ -a_1\vec{b}_1 \\ \vdots \\ a_k\vec{b}_{2k} \\ -a_k\vec{b}_{2k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 b_{21} & 0 & \dots & a_1 b_{2n} \\ 0 & -a_1 b_{12} & \dots & -a_1 b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_k b_{2k,1} & a_k b_{2k,2} & \dots & a_k b_{2k,n} \\ -a_k b_{2k-1,1} & -a_k b_{2k-1,2} & \dots & -a_k b_{2k-1,n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

hence

$$\begin{aligned} 2\|AB\|^2 &= 2a_1^2(\vec{b}_1^2 + \vec{b}_2^2) + \ldots + 2a_k^2(\vec{b}_{2k-1}^2 + \vec{b}_{2k}^2) \\ &= a_1^2(\sum_i \vec{b}_i^2) - a_1^2\|C_{12}\|^2 + 2a_1^2 b_{12}^2 \\ &+ a_2^2(\sum_i \vec{b}_i^2) - a_2^2\|C_{34}\|^2 + 2a_2^2 b_{34}^2 \end{aligned}$$

$$\begin{aligned}
&\cdots\cdots\cdots \\
&+ a_k^2(\sum_i \vec{b}_i^2) - a_k^2\|C_{2k-1,2k}\|^2 + 2a_k^2 b_{2k-1,2k}^2 \\
&= (\sum_i a_i^2)(\sum_i \vec{b}_i^2) - \sum_i a_i^2\|C_{2i-1,2i}\|^2 + 2\sum_i a_i^2 b_{2i-1,2i}^2,
\end{aligned}$$

where C_{ij} denotes the $(n-2, n-2)$ matrix received from AB by effacing the ith rows and jth columns. We have

$$\begin{aligned}
\|[A,B]\|^2 = \|AB - (AB)^t\|^2 &\leq 4\|AB\|^2 - 4\sum_i a_i^2 b_{2i-1,2i}^2 \\
&= 2(\sum_i a_i^2)(\sum_i \vec{b}_i^2) - 2\sum_i a_i^2\|C_{2i-1,2i}\|^2 \\
&\leq 2.\frac{1}{2}.1 = 1.
\end{aligned}$$

The proof is completed.

Let $\mathcal{L} = \operatorname{span}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$, and $\mathcal{R} = \operatorname{span}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$, where

$$\mathcal{L}_1 = \begin{pmatrix} L_1 & O \\ O & O \end{pmatrix} \quad ; \quad \mathcal{R}_1 = \begin{pmatrix} R_1 & O \\ O & O \end{pmatrix}$$

$$\mathcal{L}_2 = \begin{pmatrix} L_2 & O \\ O & O \end{pmatrix} \quad ; \quad \mathcal{R}_2 = \begin{pmatrix} R_2 & O \\ O & O \end{pmatrix}$$

$$\mathcal{L}_3 = \begin{pmatrix} L_3 & O \\ O & O \end{pmatrix} \quad ; \quad \mathcal{R}_3 = \begin{pmatrix} R_3 & O \\ O & O \end{pmatrix}$$

Lemma 3.5.3

1) $\|\tau\|^* = \frac{1}{2}$,

2) $G(\tau) = \{\operatorname{Ad}_T(\mathcal{L}); \operatorname{Ad}_T(\mathcal{R}) \mid T \in O(n)\}$.

Proof. 1) Let $A, B, C \in so(n), \|A\| = \|B\| = \|C\| = 1$, and A is in the canonical form. Then

$$\begin{aligned}
\tau(A,B,C) = \frac{1}{2}\langle [A,B], C\rangle &\leq \frac{1}{2}\|[A,B]\|.\|C\| \\
&\leq \frac{1}{2}.1.1 = \frac{1}{2},
\end{aligned}$$

and the equality holds if and only if

a) $[A, B] = C$,

b) $AB = -BA$,

c) $\sum_i a_i^2 \|C_{2i-1,2i}\|^2 = 0$

(see the proof of Lemma 3.5.1 and 3.5.2).

– If there exists $a_i \neq 0$ and $a_j = 0$ ($\forall j \neq i$), then

$$AB - BA = 0.$$

– If there exist $a_i \neq 0;\ a_j \neq 0;\ a_k \neq 0$, (where without loss of generality we may assume that $a_1 \neq 0;\ a_2 \neq 0;\ a_3 \neq 0$) then the equality holds if

$$\|C_{12}\|^2 = \|C_{34}\|^2 = \|C_{56}\|^2 = 0$$

and we imply that $A = 0$. This contradicts the hypothesis.

Thus, the equality holds only in the following case: there exist $a_i \neq 0;\ a_j \neq 0;$ and $a_k = 0$ for all $k \neq i, j$. We can assume that $a_1 \neq 0$ and $a_2 \neq 0$.

From the computation of the comass of τ and the contact set $G(\tau)$ on $so(4)$ (example 3.4), together with the condition $AB = -BA$, we deduce that

$$|a_1| = |a_2| = \frac{1}{2}.$$

Case 1. If $a_1 = a_2 = \pm\frac{1}{2}$, then B and C must be in the form

$$\begin{pmatrix} \begin{matrix} 0 & 0 & b & c \\ 0 & 0 & c & -b \\ -b & -c & 0 & 0 \\ -c & b & 0 & 0 \end{matrix} & O \\ O & O \end{pmatrix}$$

then $\text{span}(A, B, C) = \mathcal{L}$.

Case 2. If $a_1 = -a_2 = \pm\frac{1}{2}$, then B and C must be in the form

$$\begin{pmatrix} \begin{matrix} 0 & 0 & b & c \\ 0 & 0 & -c & b \\ -b & c & 0 & 0 \\ -c & -b & 0 & 0 \end{matrix} & O \\ O & O \end{pmatrix},$$

then $\text{span}(A, B, C) = \mathcal{R}$.

2) The proof of second part is obvious.

Remark. The comass of τ was computed by D.T.Thi in 1977 to show that S^3 (resp. $SU(2)$) is homologically volume minimizing in $SO(n)$ (resp. $SU(n)$). But the computation here is given quite different and the description of $G(\tau)$ is new. It shows that only S^3 (resp. $SU(2)$) is calibrated by τ up to isometry, and the isometry can be defined explicitely.

Example 3.6.

Consider $\mathbf{H}^n$, the set of column of height n of quaternions. For each pair $X = (x_1, x_2, \dots, x_n) \in \mathbf{H}^n$; $Y = (y_1, y_2, \dots, y_n) \in \mathbf{H}^n$, we define the product XY as below

$$XY = Z = (z_1, z_2, \dots, z_n),$$

where

$$\begin{aligned} z_1 &= x_1 y_1 - \bar{y}_2 x_2 - \bar{y}_3 x_3 - \dots - \bar{y}_n x_n, \\ z_2 &= y_2 x_1 + x_2 \bar{y}_1, \\ z_3 &= y_3 x_1 + x_3 \bar{y}_1, \\ &\dots\dots \\ z_n &= y_n x_1 + x_n \bar{y}_1, \end{aligned}$$

Denote $1 = (1, 0, \dots, 0)$ be the unit element of $\mathbf{H}^n$, let $\text{Re}\mathbf{H}^n$ be the span of $1 \in \mathbf{H}^n$, and $\text{Im}\mathbf{H}^n$ be the orthogonal complement of $\text{Re}\mathbf{H}^n$. Then each $X \in \mathbf{H}^n$ has a unique orthogonal decomposition

$$X = X_1 + X' \qquad X_1 \in \text{Re}\mathbf{H}^n \qquad X' \in \text{Im}\mathbf{H}^n.$$

The conjugation is defined by

$$\overline{X} = X_1 - X',$$

thus,

$$X_1 = \frac{1}{2}(X + \overline{X}) \qquad X' = \frac{1}{2}(X - \overline{X}).$$

Elementary facts concerning conjugation are

$$\overline{\overline{X}} = X \qquad \overline{XY} = \bar{Y}\bar{X} \qquad X\overline{X} = \overline{X}X = |X|^2,$$

$$\langle X, Y \rangle = \text{Re}X\overline{Y} = \frac{1}{2}(X\overline{Y} + Y\overline{X}).$$

Theorem 3.6.1. *For all* $X = (x_1, x_2, \dots, x_n)$; $Y = (y_1, y_2, \dots, y_n)$, *we have*

1) $|XY| \leq |X|.|Y|$,

2) *the equality holds if and only if*

   i) $\overline{y}_2 x_2 \uparrow\uparrow \overline{y}_3 x_3 \uparrow\uparrow \dots \uparrow\uparrow \overline{y}_n x_n$,

   ii) $|x_i y_j| = |x_j y_i| \quad i, j \geq 2, \quad i \neq j.$

(The notation $a \uparrow\uparrow b$ *means that* $a = kb \quad (k \in \mathbf{R},\ k \geq 0)$*).*

To prove the theorem, first we prove the following two lemmas

Lemma 3.6.2. *For all* $a, b, c, d \in \mathbf{H}$, *we have*

$$\langle ac, \overline{d}b \rangle \ = \ \langle da, b\overline{c} \rangle.$$

Proof. Let $x = (a, b)$; $\ y = (c, d) \in \mathbf{O}$ (where $O = H \oplus H$ is the set of all Caley numbers, see [HL]). We have

$$\begin{aligned} |x.y|^2 &= (ac - \overline{d}b)^2 + (da + b\overline{c})^2 \\ &= a^2c^2 + d^2b^2 - 2\langle ac, \overline{d}b \rangle + d^2a^2 + b^2c^2 + 2\langle da, b\overline{c} \rangle, \end{aligned}$$

and

$$|x|^2|y|^2 = (a^2 + b^2)(c^2 + d^2),$$

Because $\mathbf{O}$ is a norm algebra, i.e.

$$|x.y| = |x|.|y|,$$

we have

$$\langle ac, \overline{d}b \rangle \ = \ \langle da, b\overline{c} \rangle.$$

Lemma 3.6.3. *For all* $a, b, c, d \in \mathbf{H}$, *we have*

$$2\langle \overline{c}a, \overline{d}b \rangle \ \leq \ a^2d^2 + b^2c^2;$$

the equality holds if and only if $\overline{c}a \uparrow\uparrow \overline{d}b$, *and* $|ad| = |cb|$.

Proof.

$$\begin{aligned} 2\langle \overline{c}a, \overline{d}b \rangle \ &\leq \ 2|\overline{c}a|.|\overline{d}b| = 2|c|.|a|.|d|.|b| \\ &= 2|ad|.|cb| \leq |ad|^2 + |cb|^2, \end{aligned}$$

the equality holds if and only if

$$\overline{c}a \uparrow\uparrow \overline{d}b \ \text{ and } \ |ad| = |cb|.$$

Proof of Theorem 3.6.1

$$\begin{aligned}
|X.Y|^2 &= z_1^2 + z_2^2 + \ldots + z_n^2 \\
&= \sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{j=2}^{n} (x_1^2 y_j^2 + x_j^2 y_1^2) - 2\sum_{i=2}^{n} \langle x_1 y_1, \overline{y}_i x_i \rangle \\
&\quad + 2\sum_{i=1}^{n} \langle x_i \overline{y}_1, y_i x_1 \rangle + 2\sum_{\substack{i,j=2 \\ i\neq j}}^{n} \langle \overline{y}_i x_i, \overline{y}_j x_j \rangle \\
&\leq \Big(\sum_{i=1}^{n} x_i^2\Big)\Big(\sum_{i=1}^{n} y_i^2\Big) = |X|^2 |Y|^2
\end{aligned}$$

The proof of the second part is omitted.

Definition 3.6.4. Let $\overline{\Phi}(X,Y) = -\dfrac{1}{2}(\overline{X}Y - \overline{Y}X) = \mathrm{Im}\overline{Y}X$, for all $X, Y \in \mathbf{H}^n$. This product will be called the quasi cross product on $\mathbf{H}^n$.

Remarks.

1) If $X, Y \in \mathbf{H}^n,\ X \perp Y$, then

$$\overline{\Phi}(X,Y) = \overline{Y}X. \tag{3.6.5}$$

2) For all $X, Y \in \mathrm{Im}\mathbf{H}^n$, we have

$$\overline{\Phi}(X,Y) \in \mathrm{Im}\mathbf{H}^n. \tag{3.6.6}$$

Indeed, since $\langle X, Y\rangle = \mathrm{Re}\overline{Y}X = 0$, we imply that $\overline{Y}X = \mathrm{Im}\overline{Y}X$. This proves 1). The second part is obvious.

Theorem 3.6.7. $(\mathrm{Im}\mathbf{H}^n, \overline{\Phi})$ *is a calibrated algebra.*

Proof. We first observe that, the quasi cross product on $\mathrm{Im}\mathbf{H}^n$ is alternating since $\overline{\Phi}(X,X) = 0$, for all $X \in \mathrm{Im}\mathbf{H}^n$.

We show that

$$\langle Z, \overline{\Phi}(X,Y)\rangle \;=\; \langle Z, XY\rangle \;\text{ for all }\; X, Y, Z \in \mathbf{H}^n.$$

Indeed, we have

$$\begin{aligned}
\langle Z, \overline{\Phi}(X,Y)\rangle &= -\langle Z, \overline{\Phi}(Y,X)\rangle \\
&= -\langle Z, \mathrm{Im}\overline{X}Y\rangle \\
&= -\langle Z, \overline{X}Y\rangle \\
&= \langle Z, XY\rangle.
\end{aligned}$$

And by virtue of the Lemma 3.6.2, it is easy to prove that

1) If $a, b \in \mathrm{Im}\mathbf{H}$, then
$$\langle a, ab\rangle = 0.$$

2) If $a \in \mathrm{Im}\mathbf{H}$ and $b \in \mathbf{H}$, then
$$\langle b, ba\rangle \ = \ 0.$$

Finally, by using the definition of the product XY, direct computation shows that
$$\langle\overline{\Phi}(X, Y), X\rangle \ = \ \langle XY, X\rangle \ = \ 0,$$
and
$$\langle\overline{\Phi}(X, Y), Y\rangle \ = \ \langle XY, Y\rangle \ = \ 0.$$
The theorem is proved.

Consider the trilinear form defined by
$$\Phi(X, Y, Z) = \langle X, \overline{\Phi}(Y, Z)\rangle.$$

Obviously, Φ is alternating and by virtue of Lemma 3.6.1, Remark 3.6.5 and Theorem 3.6.7, the following lemma is immediate.

Lemma 3.6.8.

1) $\|\Phi\|^* = 1$,

2) $G(\Phi) = \{\overline{\Phi}(X, Y) \wedge X \wedge Y \mid (X, Y) \in G(\overline{\Phi})\}$.

More explicitly, $G(\Phi)$ is the set of all 3-covectors $\xi \in \bigwedge^3(\mathrm{Im}\mathbf{H}^n)$ of the form $\xi = \overline{\Phi}(X, Y) \wedge X \wedge Y$, in which X, Y satisfy the following conditions

1) $\sum\limits_i x_i y_i = 0$,

2) $\overline{y}_2 x_2 \uparrow\uparrow \overline{y}_3 x_3 \uparrow\uparrow \ldots \uparrow\uparrow \overline{y}_n x_n$.

3) $|x_i y_j| = |x_j y_i| \quad i, j \geq 2, \quad i \neq j$.

Thus, Φ is a calibration and is called multi-associated calibration.

Remark. 1- In [M2] F. Morgan showed that the comass of *Double Slag* and *Double Assoc* is one. But the comass of *Triple Slag* and *Triple Assoc* is not knowed (also for the Multiple Slag and the Multiple Assoc). The information about *SLAG-ASSOC* calibrations of type (k, l) here is new. All of them belong to $F^*(SLAG) = \{\Phi \ / \ G(\Phi) \supseteq G(\Phi_{SLAG})\}$.

2- Let $V = \mathrm{Im}\mathbf{H} \times \underbrace{\mathrm{Im}\mathbf{H} \times \ldots \times \mathrm{Im}\mathbf{H}}_{k} \times \underbrace{\mathbf{H} \times \ldots \times \mathbf{H}}_{l} \times \{0\} \times \ldots \times \{0\} \cong \mathbf{R}^{3(k+1)+4\ell}$, then $\Phi_{|V}$ is a calibration belonging to $F^*(SLAG)$ and is called *SLAG-ASSOC* calibration of type (k, ℓ).

Examples. Double-Slag calibration is *SLAG-ASSOC* calibration of type $(2, 0)$

Double-Assoc calibration is *SLAG-ASSOC* calibration of type $(0, 2)$.

Triple-Assoc calibration is *SLAG-ASSOC* calibration of type $(0, 3)$.

Triple-Slag calibration is *SLAG-ASSOC* calibration of type $(3, 0)$

SLAG-ASSOC calibtation of type $(1, 1)$ is a calibration on $\mathrm{Im}\mathbf{H} \times \mathrm{Im}\mathbf{H} \times \mathbf{H} \cong \mathbf{R}^{10} \ldots$

These calibrations belong to $F^*(SLAG)$. Their faces contain many *ASSOC* , *SLAG*, and CP^k faces.

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Hue Pedagogical Institute, 32 Le Loi, Hue, Vietnam